ON THE ZEROS OF COSINE POLYNOMIALS: SOLUTION TO A PROBLEM OF LITTLEWOOD

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Abstract. Littlewood in his 1968 monograph “Some Problems in Real and Complex Analysis” [12, problem 22] poses the following research problem, which appears to still be open:

Problem. “If the $n_j$ are integral and all different, what is the lower bound on the number of real zeros of $\sum_{j=1}^{N} \cos(n_j \theta)$? Possibly $N - 1$, or not much less.”

No progress appears to have been made on this in the last half-century. We show that this is false.

Theorem. There exists a cosine polynomial $\sum_{j=1}^{N} \cos(n_j \theta)$ with the $n_j$ integral and all different so that the number of its real zeros in the period is $O\left( N^{9/10} (\log N)^{1/5} \right)$.

1. Littlewood’s 22nd Problem

Problem. “If the $n_j$ are integral and all different, what is the lower bound on the number of real zeros of $\sum_{j=1}^{N} \cos(n_j \theta)$? Possibly $N - 1$, or not much less.”

Here “real zeros” means “zeros in a period”. Denote the number of zeros of a trigonometric polynomial $T$ in the period $[-\pi, \pi)$ by $N(T)$.

Note that if $T$ is a real trigonometric cosine polynomial of degree $n$, then it is of the form $T(t) = \exp(-int)P(\exp(it))$, $t \in \mathbb{R}$, where $P$ is a reciprocal algebraic polynomial of degree $2n$, and if $T$ has only real zeros, then $P$ has all its zeros on the unit circle. So in terms of reciprocal algebraic polynomials one is looking for a reciprocal algebraic polynomial with coefficients in $\{0, 1\}$, with $2N$ terms, and with $N - 1$ or fewer zeros on the unit circle. Even achieving $N - 1$ is fairly hard. An exhaustive search up to degree $2N = 32$ yields only 10 example achieving $N - 1$ and only one example with fewer.

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This first example disproving the “possibly $N - 1$” part of the conjecture is

$$\sum_{j=0, j \notin \{9, 10, 11, 14\}}^{14} (z^j + z^{28-j})$$

which has 8 roots of modulus 1 and corresponds to a cosine sum of 11 terms with 8 roots in the period.

It is hard to see how one might generate infinitely many such examples or indeed why Littlewood made his conjecture.

The following is a reciprocal polynomial with 32 terms and exactly 14 zeros of modulus 1:

$$\sum_{j=0, j \notin \{10, 11, 17, 19\}}^{19} (z^j + z^{38-j}).$$

So it corresponds to a cosine sum of 16 terms with 14 zeros in $[-\pi, \pi)$.

In other words the sharp version of Littlewood’s conjecture is false again, though barely. The following is a reciprocal polynomial with 280 terms and 52 zeros of modulus 1:

$$\sum_{j=0, j \notin \{124, 125, 126, 127, 128, 134, 135, 143, 145, 147, 148, 151, 152\}}^{152} (z^j + z^{304-j}).$$

So it corresponds to a cosine sum of 140 terms with 52 zeros in $[-\pi, \pi)$.

Once again the sharp version of Littlewood’s conjecture is false, though this time by a margin. It was found by a version of the greedy algorithm (and some guessing). There is no reason to believe it is a minimal example.

The interesting feature of this example is how close it is to the Dirichlet kernel $(1 + z + z^2 + \ldots + z^{304})$. This is not accidental and suggests the approach that leads to our main result.

Littlewood explored many problems concerning polynomials with various restrictions on the coefficients. See [9], [10], and [11], and in particular Littlewood’s delightful monograph [12]. Related problems and results may be found in [2] and [4], for example. One of these is Littlewood’s well-known conjecture of around 1948 asking for the minimum $L_1$ norm of polynomials of the form

$$p(z) := \sum_{j=0}^{n} a_j z^{k_j},$$

where the coefficients $a_j$ are complex numbers of modulus at least 1 and the exponents $k_j$ are distinct nonnegative integers. It states that such polynomials have $L_1$ norms on the unit circle that grow at least like $c \log n$. This was proved by S. Konyagin [7] and independently by McGehee, Pigno, and Smith [13] in 1981. A short proof is available in [5]. It is believed that the minimum, for polynomials of degree $n$ with complex coefficients of modulus at least 1 is attained by $1 + z + z^2 + \ldots + z^n$, but this is open.
2. Auxiliary Functions

The key is to construct $n$ term cosine sums that are large most of the time. This is the content of this section.

**Lemma 1.** There is an absolute constant $c_1$ such that for all $n$ and $\alpha > 1$ there are coefficients $a_0, a_1, \ldots, a_n$ with each $a_j \in \{0,1\}$ such that

$$\text{meas}\{t \in [-\pi, \pi) : |P_n(t)| \leq \alpha\} \leq c_1 \alpha n^{-1/2},$$

where

$$P_n(t) = \sum_{j=0}^{n} a_j \cos(jt).$$

**Proof.** We will prove the stronger result that there is an absolute constant $c_1$ such that for all $\alpha > 0$ and all $n$

$$\lambda(\alpha) := 2^{-(n+1)} \sum_{\{a_0,a_1,\ldots,a_n\}} \text{meas}\{t \in [-\pi, \pi) : |P_n(t)| \leq \alpha\} \leq c_1 \alpha n^{-1/2}.$$

If $X_0, X_1, \ldots, X_n$ are independent Bernoulli random variables with

$$P(X_j = 0) = P(X_j = 1) = \frac{1}{2}, \quad j = 0,1,\ldots,n,$$

then the indicated average is an expected value. Let

$$R_n(t) = \sum_{j=0}^{n} X_j \cos(jt)$$

and note that

$$\lambda(\alpha) = \int_{-\pi}^{\pi} P(|R_n(t)| \leq \alpha) \, dt.$$

Define

$$D_n(t) := \sum_{j=0}^{n} \cos(jt).$$

The expected value of $R_n(t)$ is $\mu_n(t) := D_n(t)/2$; its variance is

$$\sigma_n^2(t) := \frac{1}{4} \sum_{0}^{n} \cos^2(jt) = \frac{1}{8}(n + 1 + D_n(2t)).$$

We now apply a uniform normal approximation to get the desired result. Define the cumulative normal distribution function by

$$\Phi(x) := \int_{-\infty}^{x} \frac{e^{-u^2/2}}{\sqrt{2\pi}} \, du.$$
Define
\[
\varrho_2 := \frac{1}{n+1} \sum_{j=0}^{n} \text{Var}(X_j \cos(jt)) = \\
= \frac{1}{4(n+1)} \sum_{j=0}^{n} \cos^2(jt) = \frac{1}{8} \left( 1 + \frac{D_n(2t)}{n+1} \right), \\
\varrho_3 := \frac{1}{n+1} \sum_{j=0}^{n} \mathbb{E} \left[ \left( X_j - \frac{1}{2} \right)^3 \cos(jt) \right].
\]

We suppress the dependence of each of these on \(n\) and \(u\). The Berry-Esseen bound in Bhattacharya and Ranga Rao [1, Theorem 12.4, page 104] is that
\[
P(R_n(t) \leq c) - \Phi \left( \frac{c - \mu_n(t)}{\sigma_n(t)} \right) \leq \frac{11\varrho_3}{4\sqrt{n} \varrho_2^{3/2}}.
\]

It is elementary that \(\varrho_3 \leq 1/8\). Moreover there is an absolute constant \(c_2 > 0\) such that \(\varrho_2 > c_2\) for all \(t \in \mathbb{R}\) and all \(n = 1, 2, \ldots\). Finally the function \(\Phi\) has derivative bounded by \((2\pi)^{-1/2}\) so
\[
|\Phi(x) - \Phi(y)| \leq (2\pi)^{-1/2}|x - y|, \quad x, y \in \mathbb{R}.
\]

It follows that there is an absolute constant \(c_1\) such that
\[
P(-\alpha \leq R_n(u) \leq \alpha) \leq c_1 \alpha n^{-1/2}.
\]

\[
3. \text{ The Main Theorem}
\]

**Theorem 1.** There exists a cosine polynomial \(\sum_{j=1}^{N} \cos(n_j \theta)\) with the \(n_j\) integral and all different so that the number of its real zeros in the period is
\[
O \left( N^{9/10} (\log N)^{1/5} \right).
\]

We note that we have not worked hard to replace the exponent 9/10 with a smaller one that we may call a “close to optimal” exponent in the result. One can hope to replace the exponent 9/10 in Theorem 1 by a slightly smaller one.

The proof of our main theorem above follows immediately from the following Lemma 2 stated below and Lemma 1. Namely, take \(m := N + 1\), \(n = m^{2/5}(\log m)^{-4/5}\), \(\alpha = n^{1/4}\) and \(\beta = c_1 \alpha n^{-1/2} = c_1 n^{-1/4}\).

**Lemma 2.** Let \(n \leq m\),
\[
D_m(t) := \sum_{j=0}^{m} \cos(jt), \\
P_n(t) := \sum_{j=0}^{n} a_j \cos(jt), \quad a_j \in \{0, 1\}.
\]
Suppose \( \alpha \geq 1 \) and
\[
\text{meas}\{t \in [-\pi, \pi) : |P_n(t)| \leq \alpha\} \leq \beta.
\]
Let \( S_m := D_m - P_n \). Then the number of zeros of \( S_m \) in \([−\pi, \pi)\) is at most
\[
\frac{c_3 m}{\alpha} + c_4 m \beta + c_5 nm^{1/2} \log m,
\]
where \( c_3, c_4, \) and \( c_5 \) are absolute constants.

To prove Lemma 2 we need the following consequence of the Erdős-Turan Theorem [15, p. 278]; see also [6].

**Lemma 3.** Let
\[
S_m(t) = \sum_{j=0}^{m} a_j \cos(jt), \quad a_j \in \{0, 1\},
\]
be not identically zero. Denote the number of zeros of \( S_m \) in an interval \( I \subset [-\pi, \pi) \) by \( \mathcal{N}(I) \). Then
\[
\mathcal{N}(I) \leq c_6 m |I| + c_6 \sqrt{m} \log m,
\]
where \( c_6 \) is an absolute constant and \( |I| \) denotes the length of \( I \).

Now we prove Lemma 2.

**Proof.** We write
\[
\{t \in [-\pi, \pi) : |P_n(t)| \leq \alpha\} = \bigcup_{j=1}^{k} I_j,
\]
where the intervals \( I_j \) are disjoint and \( k \leq 2n \). Let
\[
I_0 := \{t \in [-\pi, \pi) : |D_m(t)| \geq \alpha\}.
\]
Note that \( I_0 \subset [-c/\alpha, c/\alpha] \). Then \( S_m \) has all its zeros in \( \bigcup_{j=0}^{k} I_j \). By Lemma 3 we have
\[
\mathcal{N}(I_j) \leq c_6 m |I_j| + c_6 \sqrt{m} \log m, \quad j = 1, 2, \ldots, k,
\]
and
\[
\mathcal{N}(I_0) \leq c_6 m |I_0| + c_6 \sqrt{m} \log m \leq \frac{c_7 m}{\alpha} + c_7 \sqrt{m} \log m
\]
with an absolute constant \( c_7 \). So
\[
\mathcal{N}([-\pi, \pi)) \leq \sum_{j=0}^{k} \mathcal{N}(I_j)
\]
\[
\leq \frac{c_7 m}{\alpha} + c_7 \sqrt{m} \log m + c_6 \sum_{j=1}^{k} m |I_j| + kc_7 \sqrt{m} \log m
\]
\[
\leq \frac{c_7 m}{\alpha} + c_6 m \beta + 2nc_7 \sqrt{m} \log m
\]
and the proof is finished. \( \square \)
4. Average Number of Real Zeros

Why did Littlewood make this conjecture? He might have observed that the average number of zeros of a trigonometric polynomial of the form

\[ 0 \neq T(t) = \sum_{j=1}^{n} a_j \cos(jt), \quad a_j \in \{0, 1\}, \]

has in \([0, 2\pi]\) is at least \(cn\). This is what we elaborate in this section. Associated with a polynomial \(P\) of degree exactly \(n\) with real coefficients we introduce \(P^*(z) := z^n P(1/z)\).

**Theorem 2.** Let

\[ S(t) := \sum_{j=1}^{n} a_j \cos(jt) \quad \text{and} \quad \widetilde{S}(t) := \sum_{j=1}^{n} a_{n+1-j} \cos(jt), \]

where each of the coefficients \(a_j\) is real and \(a_1 a_n \neq 0\). Let \(w_1\) be the number of zeros of \(S\) in \([0, 2\pi]\), and let \(w_2\) be the number of zeros of \(\widetilde{S}\) in \([0, 2\pi]\). Then \(w_1 + w_2 \geq 2n\).

**Proof.** Let \(P(z) = \sum_{j=1}^{n} a_j z^j\). Without loss of generality we may assume that \(P\) does not have zeros on the unit circle; the general case follows by a simple limiting argument with the help of Rouche’s Theorem. Note that if \(P\) has exactly \(k\) zeros in the open unit disk then \(z P^*(z)\) has exactly \(n - k\) zeros in the open unit disk. Also,

\[ 2S(t) = \text{Re}(P(e^{it})) \quad \text{and} \quad 2\widetilde{S}(t) = \text{Re}(e^{it} P^*(e^{it})). \]

Hence the theorem follows from the Argument Principle. Note that if a continuous curve goes around the origin \(k\) times then it crosses the real axis at least \(2k\) times. \(\square\)

Theorem 2 has some interesting consequences. As an example we can state and easily see the following.

**Theorem 3.** The average number of zeros of trigonometric polynomials in the class

\[ \left\{ \sum_{j=1}^{n} a_j \cos(jt), \quad a_j \in \{-1, 1\} \right\} \]

in \([0, 2\pi]\) is at least \(n\). The average number of zeros of trigonometric polynomials in the class

\[ \left\{ 0 \neq \sum_{j=1}^{n} a_j \cos(jt), \quad a_j \in \{0, 1\} \right\} \]

in \([0, 2\pi]\) is at least \(n/4\).

**Proof.** Most of the cosine sums in both classes naturally break into pairs with a large combined total number of real zeros in the period. \(\square\)
5. Conclusion

Let $0 \leq n_1 < n_2 < \cdots < n_N$ be integers. A cosine polynomial of the
form $T_n(\theta) = \sum_{j=1}^{N} \cos(n_j \theta)$ must have at least one real zero in a period.
This is obvious if $n_1 \neq 0$, since then the integral of the sum on a period
is 0. The above statement is less obvious if $n_1 = 0$, but for sufficiently
large $N$ it follows from Littlewood’s Conjecture simply. Here we mean the
Littlewood’s Conjecture proved by S. Konyagin [7] and independently by
not difficult to prove the statement in general even in the case $n_1 = 0$. One
possible way is to use the identity

$$\sum_{j=1}^{n_N} T_n((2j-1)\pi/n_N) = 0.$$

See [8], for example. Another way is to use Theorem 2 of [14]. So there is
certainly no shortage of possible approaches to prove the starting observation
of our conclusion even in the case $n_1 = 0$. It seems likely that the number
of zeros of the above sums in a period must tend to infinity with $N$. This
does not appear to be easy. The case when the sequence $0 \leq n_0 < n_1 < \cdots$
is fixed will be handled in a forthcoming paper [3].

References

[11] J.E. Littlewood, On polynomials $\sum \pm z^m$ and $\sum e^{\alpha m} z^m$, $z = e^{\theta i}$, J. London Math. Soc. 41 (1966), 367–376.


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