# MARKOV-BERNSTEIN TYPE INEQUALITY FOR TRIGONOMETRIC POLYNOMIALS WITH RESPECT TO DOUBLING WEIGHTS ON $[-\omega,\omega]$

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Abstract. Various important weighted polynomial inequalities, such as Bernstein, Marcinkiewicz, Nikolskii, Schur, Remez, etc. inequalities, have been proved recently by Giuseppe Mastroianni and Vilmos Totik under minimal assumptions on the weights. In most of the cases this minimal assumption is the doubling condition. Here, based on a recently proved Bernstein-type inequality by D.S. Lubinsky, we establish Markov-Bernstein type inequalities for trigonometric polynomials with respect to doubling weights on $[-\omega,\omega]$. Namely, we show the theorem below.

**Theorem.** *Let $p \in [1,\infty)$ and $\omega \in (0,1/2]$. Suppose $W(\arcsin((\sin\omega)\cos t))$ is a doubling weight. Then there is a constant $C$ depending only on $p$ and the doubling constant $L$ so that*

$$\int_{-\omega}^{\omega} |T_n'(t)|^p W(t) \left(\omega/n + \sqrt{\omega^2-t^2}\right)^p \, dt \le Cn^p \int_{-\omega}^{\omega} |T_n(t)|^p W(t)\, dt$$

*holds for every $T_n \in \mathcal{T}_n$, where $\mathcal{T}_n$ denotes the class of all real trigonometric polynomials of degree at most $n$.*

## 1. The Weights

For Introduction we refer to Sections 1 and 2 of the Mastroianni-Totik paper [15] and the references therein. See [1] – [6], [8], [9], [11], [14], and [16]. See also [7] and [12]. Here we just formulate the original and some equivalent definitions that we shall use. We shall work with integrable periodic weight functions $W$ satisfying the so-called doubling condition:

$$W(2I) \le LW(I)$$

for intervals $I \subset \mathbb{R}$, where $L$ is a constant independent of $I$, $2I$ is the interval with length $2|I|$ ($|I|$ denotes the length of the interval $I$) and with midpoint at the midpoint of $I$, and

$$W(I) := \int_I W(u)\, du\,.$$

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In other words, $W$ has the doubling property if the measure of a twice enlarged interval is less than a constant times the measure of the original interval.

Associated with a periodic weight function $W$ on $\mathbb{R}$, let

$$W_n(t) := n \int_{t-1/n}^{t+1/n} W(\theta)\, d\theta \,. \tag{1.1}$$

Let $\omega \in [-1/2, 1/2]$. Associated with a weight function $W(x)$ on $[-\omega, \omega]$, we define

$$W_\omega(t) = W(\arcsin((\sin\omega)\cos t)) \tag{1.2}$$

and

$$W_{n,\omega}(t) := n \int_{t-1/n}^{t+1/n} W_\omega(\theta)\, d\theta \,. \tag{1.3}$$

The class of all real trigonometric polynomials of degree at most $n$ will be denoted by $\mathcal{T}_n$. Associated with a trigonometric polynomial $T_n \in \mathcal{T}_n$, we define

$$T_{n,\omega}(t) := T_n(\arcsin((\sin\omega)\cos t)). \tag{1.4}$$

**Lemma 1.1.** *Let $W$ be a periodic weight function on $\mathbb{R}$. Then $W$ is a doubling weight if and only if there are constants $s > 0$ and $K > 0$ depending only on $W$ such that*

$$W_n(t_2) \leq K(1 + n|t_2 - t_1|)^s W_n(t_1) \,.$$

*holds for all $n \in \mathbb{N}$ and $t_1, t_2 \in \mathbb{R}$. Here, if $L$ is the doubling constant, then with the choice $s = \log L$, $K$ depends only on $L$.*

Markov-Bernstein type inequalities play a basic role in proving inverse theorems of approximation. Mastroianni and Totik [15] proved the following Bernstein-type inequality recently.

**Theorem 1.2.** *Let $W$ be a doubling weight and let $1 \leq p < \infty$ arbitrary. Then there is a constant $C$ depending only on the doubling constant $L$ so that*

$$\int_{-\pi}^{\pi} |T_n'(t)|^p W(t)\, dt \leq Cn^p \int_{-\pi}^{\pi} |T_n(t)|^p W(t)\, dt$$

*holds for every $T_n \in \mathcal{T}_n$.*

The above inequality is extended in [7] for all $p > 0$. The purpose of this paper is to establish the right analogue of Theorem 1.2 when the period $[-\pi, \pi)$ is replaced with a shorter interval $[-\omega, \omega]$. Our main result is the Markov-Bernstein type inequality below.

**Theorem 1.3.** *Let $p \in [1, \infty)$ and $\omega \in (0, 1/2]$. Suppose $W(\arcsin((\sin\omega)\cos t))$ is a doubling weight. Then there is a constant $C$ depending only on $p$ and the doubling constant $L$ so that*

$$\int_{-\omega}^{\omega} |T_n'(t)|^p W(t) \left(\omega/n + \sqrt{\omega^2 - t^2}\right)^p dt \le Cn^p \int_{-\omega}^{\omega} |T_n(t)|^p W(t)\, dt$$

*holds for every $T_n \in \mathcal{T}_n$.*

Our main tool is the following inequality proved recently by Lubinsky [12] (see also the paper [10] by Kobindarajah and Lubinsky).

**Theorem 1.4 (Lubinsky).** *Let $\omega \in (0, 1/2]$. We have*

$$\int_{-\omega}^{\omega} |T_n'(t)|^p \left(\omega/n + \sqrt{\omega^2 - t^2}\right)^p dt \le Cn^p \int_{-\omega}^{\omega} |T_n(t)|^p\, dt$$

*for all $T_n \in \mathcal{T}_n$.*

This may be viewed as the $L_p$ version of the Videnskii's inequality [17] below.

**Theorem 1.5 (Videnskii).** *Let $\omega \in (0, 1/2]$. We have*

$$\max_{t \in [-\omega, \omega]} |T_n'(t)| \left(\omega/n + \sqrt{\omega^2 - t^2}\right) \le Cn \max_{t \in [-\omega, \omega]} |T_n(t)|$$

*for all $T_n \in \mathcal{T}_n$.*

## 2. The Main Theorem

In this paper $C, C_1, C_2, \ldots$ always denote constants depending (possibly) only on the value of $p$ and the doubling constant $L$ in the doubling weight involved.

Using the notation of (1.1) – (1.4), we prove the following basic theorem.

**Theorem 2.1.** *Let $\omega \in (0, 1/2]$. Suppose $W$ is a weight function on $[-\omega, \omega]$ such that $W_\omega$ is a doubling weight. doubling weight. Let $1 \le p < \infty$ be arbitrary. Then there is a constant $C$ depending only on the doubling constant $L$ for $W_\omega$ such that for every $T_n \in \mathcal{T}_n$ we have*

$$\begin{aligned} C^{-1} \int_{-\pi}^{\pi} |T_{n,\omega}(t)|^p W_\omega(t)\, (\sin\omega) |\sin t|\, dt \le \\ \le \int_{-\pi}^{\pi} |T_{n,\omega}(t)|^p W_{n,\omega}(t)\, (\sin\omega) |\sin t|\, dt \le \\ \le C \int_{-\pi}^{\pi} |T_{n,\omega}|^p W_\omega(t) (\sin\omega) |\sin t|\, dt\,, \end{aligned} \tag{2.1}$$

*or equivalently*

$$\begin{aligned} C^{-1} \int_{-\omega}^{\omega} |T_n(t)|^p W(t)\, dt \le \int_{-\pi}^{\pi} |T_{n,\omega}(t)|^p W_{n,\omega}(t) (\sin\omega) |\sin t|\, dt \\ \le C \int_{-\omega}^{\omega} |T_n(t)|^p W(t)\, dt\,. \end{aligned} \tag{2.1}$$

As the next lemma shows, $W_{n,\omega}$ is very close to be a nonnegative trigonometric polynomial of degree at most $n$.

**Theorem 2.2.** *Suppose $W_\omega$ satisfies the doubling condition. Then there is a constant $C > 0$ depending only on the doubling constant $L$, and for each $n \in \mathbb{N}$ there is a nonnegative even trigonometric polynomial $Q_n$ of degree at most $n(1+1/p)$ so that*

$$C^{-1}W_{n,\omega}(t) \le Q_n(t)^p \le CW_{n,\omega}(t) \tag{2.2}$$

*and*

$$|Q_n'(t)|^p \le Cn^p W_{n,\omega}(t) \tag{2.3}$$

*uniformly in $x \in \mathbb{R}$.*

*Proof of Theorem 2.2.* It is easy to see that $W_{n,\omega}(t) \sim W_{m,\omega}(t)$ uniformly in $x$ whenever $n \sim m$.[1] So it is sufficient to verify the existence of a nonnegative trigonometric polynomial $Q_n$ of degree $2mn$ with the stated properties for some fixed $m \in \mathbb{N}$ satisfying $2m \ge s/p + 2$, where $s$ is the number from Lemma 1.1. To be more specific, we define $2m$ as the smallest even number not less than $s/p + 2$. Let

$$S_n(\theta) = n^{-(2m-1)} \left( \frac{\sin((n+1/2)\theta)}{\sin(\theta/2)} \right)^{2m} \tag{2.4}$$

be the Jackson kernel. It is well known that

$$\int_{-\pi}^{\pi} |\theta|^l S_n(\theta)\, dt \sim n^{-l} \tag{2.5}$$

for each $0 \le l < 2m-1$. Indeed, the inequalities

$$\begin{aligned} S_n(\theta) &\le C^m n\,, & |\theta| &\le 1/n\,,\\ S_n(\theta) &\le C^m n^{-(2m-1)}\theta^{-2m}\,, & 1/n &\le |t| \le \pi\,, \end{aligned}$$

are easy to establish, from where

$$\int_{-\pi}^{\pi} |\theta|^l S_n(\theta)\, d\theta \le C_1 n^{-l}$$

is obvious. On the other hand, there is a constant $C_2 > 0$ depending only on $m$ so that

$$S_n(\theta) \ge C_2 n\,, \qquad |\theta| \le 1/n\,,$$

from where

$$\int_{-\pi}^{\pi} |\theta|^l S_n(\theta)\, d\theta \ge C_1 n^{-l}$$

[1] In what follows $A \sim B$ means that the ratio of the two sides is between two positive constants. Here the "similarity" constants in $W_{n,\omega}(t) \sim W_{m,\omega}(t)$ depend only on the doubling constant $L$ and the "similarity" constants in $n \sim m$.

for each $0 \leq l < 2m-1$ follows. By this (2.5) is completely shown. It clearly implies that

$$\int_{-\pi}^{\pi} (1+n|\theta|)^{s/p} S_n(\theta)\, d\theta \sim 1\,. \tag{2.6}$$

Now we define

$$Q_n(t) := \int_{-\pi}^{\pi} W_{n,\omega}(\theta)^{1/p} S_n(t-\theta)\, d\theta\,. \tag{2.7}$$

Then $Q_n \in \mathcal{T}_{2mn}$ and

$$Q_n'(t) = \int_{-\pi}^{\pi} W_{n,\omega}(\theta)^{1/p} S_n'(t-\theta)\, d\theta\,. \tag{2.8}$$

Applying Lemma 1.1 and (2.6), we obtain

$$\begin{aligned} Q_n(t) &= \int_{-\pi}^{\pi} W_{n,\omega}(t-\theta)^{1/p} S_n(\theta)\, d\theta \\ &\leq \int_{-\pi}^{\pi} W_{n,\omega}(\theta)^{1/p} K^{1/p} (1+n|\theta|)^{s/p} S_n(\theta)\, d\theta \leq C^{1+1/p} W_{n,\omega}(t)^{1/p}\,. \end{aligned}$$

The opposite inequality is simpler. For $|t| \leq 1/(2n)$, we have $W_{n,\omega}(t-\theta) \sim W_{n,\omega}(t)$ and $S_n(\theta) \sim n$, therefore

$$\begin{aligned} Q_n(t) &\geq \int_0^{1/(2n)} W_{n,\omega}(t-\theta)^{1/p} S_n(\theta)\, d\theta \\ &\geq C_1^{1/p} W_{n,\omega}(t)^{1/p} \int_0^{1/(2n)} n\, d\theta \geq C_2^{1+1/p} W_{n,\omega}(t)^{1/p}\,, \end{aligned}$$

and the proof of (2.2) is complete. To prove (2.3), observe that

$$\begin{aligned} S_n'(\theta) &\leq C^m n^2\,, && |\theta| \leq 1/n\,, \\ S_n'(\theta) &\leq C^m n^{-(2m-2)} \theta^{-2m}\,, && 1/n \leq |\theta| \leq \pi\,, \end{aligned}$$

which follows from direct differentiation and from Bernstein's inequality

$$\max_{-\pi \leq \theta \leq \pi} |S_n'(\theta)| \leq nm \max_{-\pi \leq \theta \leq \pi} |S_n(\theta)| \leq C^m mn^2\,,$$

since (2.4) implies

$$\max_{-\pi \leq \theta \leq \pi} |S_n(\theta)| \leq C^m n\,.$$

With this and (2.8) the proof of (2.3) is identical with the proof of the upper bound in (2.2). $\square$

By a routine application of the Mean Value Theorem and Theorem 1.5 (Videnskii) we obtain

**Corollary 2.3.** *Let $\omega \leq 1$. Let $T_{n,\omega}$ be defined by (1.4). Then*

$$\int_{-\pi}^{\pi} |T_{n,\omega}(t)|^p (\sin\omega) W_{n,\omega}(t)\, dt \leq C_1 n \int_{-\pi}^{\pi} |T_{n,\omega}(t)|^p (\sin\omega) |\sin t| W_{n,\omega}(t)\, dt$$

*for all $T_n \in \mathcal{T}_n$ with a constant $C_M$ depending only on $M$.*

*Proof.* Indeed, by Theorem 2.2, there is a nonnegative even trigonometric polynomial $Q_n$ of degree at most $n(1+1/p)$ such that (2.2) holds. Observe that

$$Q_n(t) = U_{n,\omega}(\arcsin((\sin\omega)\cos t))$$

with some $U_n \in \mathcal{T}_n$. Hence, with the notation

$$S_{n,\omega} = T_{n,\omega} U_{n,\omega} \in \mathcal{T}_{cn}$$

it is sufficient to prove that

$$\int_{-\pi}^{\pi} |S_{n,\omega}(t)|^p (\sin\omega) dt \leq C_1 n \int_{-\pi}^{\pi} |S_{n,\omega}(t)|^p (\sin\omega) |\sin t|\, dt$$

However, this follows easily from the fact that

$$m\left(\left\{t \in [-\pi,\pi) : |S_{n,\omega}(t)| \geq \frac{1}{2} \max_{-\pi \leq t \leq \pi} |S_{n,\omega}(t)|\right\}\right) \geq C/n$$

with a constant depending only on $p$. The above inequality can be shown by a routine combination of Theorem 1.5 (Videnskii) and the Mean Value Theorem.

*Proof of Theorem 2.1.* Let $p \geq 1$. We verify that there is a constant $C$ depending only on the doubling constant $L$ such that for every $T_n \in \mathcal{T}_n$ we have

(2.9)
$$\int_{-\pi}^{\pi} |T_n'(\arcsin((\sin\omega)\cos t)|^p (\sin\omega((1/n + |\sin t|))^p W_{n,\omega}(t)(\sin\omega)|\sin t|\, dt \leq$$
$$\leq Cn^p \int_{-\pi}^{\pi} |T_{n,\omega}(t)|^p W_{n,\omega}(t)(\sin\omega)|\sin t|\, dt\,.$$

Indeed, by Theorem 2.2, there is a nonnegative even trigonometric polynomial $Q_n$ of degree at most $n(1+1/p)$ such that (2.2) holds. We have

$$\int_{-\pi}^{\pi} |T_n'(\arcsin((\sin\omega)\cos t)|^p ((\sin\omega)(1/n + |\sin t|))^p W_{n,\omega}(t)(\sin\omega)|\sin t|\, dt \sim$$
$$\sim \int_{-\pi}^{\pi} |T_n'(\arcsin((\sin\omega)(\cos t))|^p ((\sin\omega)(1/n + |\sin t|))^p Q_n(t)^p (\sin\omega)|\sin t|\, dt\,.$$

Here

$$Q_n(t) = U_n(\arcsin(\omega\cos t)) \tag{2.10}$$

with some trigonometric polynomial $U_n$ of degree $n(2+1/p)$. Also

$$T_n' U_n = (T_n U_n)' - T_n U_n' \,,$$

therefore

$$\begin{aligned}
&\int_{-\pi}^{\pi} |T_n'(\arcsin((\sin\omega)(\cos t))|^p (\sin\omega(1/n+|\sin t|))^p W_{n,\omega}(t)\,(\sin\omega)|\sin t|\,dt \\
&\le C_1 \int_{-\pi}^{\pi} |(T_nU_n)'(\arcsin((\sin\omega)(\cos t))|^p(\sin\omega(1/n+|\sin t|))^p(\sin\omega)|\sin t|\,dt + \\
&+ C_1 \int_{-\pi}^{\pi} |(T_nU_n')(\arcsin((\sin\omega)(\cos t))|^p(\sin\omega(1/n+|\sin t|))^p(\sin\omega)|\sin t|\,dt \\
&\le C_1C_2(n(2+1/p))^p \int_{-\pi}^{\pi} |(T_nU_n)(\arcsin((\sin\omega)(\cos t)))|^p\,(\sin\omega)|\sin t|\,dt + \\
&+ C_3 n^p \int_{-\pi}^{\pi} |(T_nU_n)(\arcsin((\sin\omega)(\cos t))|^p\,(\sin\omega)|\sin t|\,dt \\
&\le Cn^p \int_{-\pi}^{\pi} |T_{n,\omega}(t)|^p W_n(t)\,(\sin\omega)|\sin t|\,dt\,,
\end{aligned}$$

where at the first inequality we used that $(A+B)^p \le C_1(A^p+B^p)$ for arbitrary $A,B \ge 0, p \ge 1$; at the second inequality, to estimate the first term, we used Lubinsky's inequality in $L_p, 0<p<\infty$ (see [1]) for trigonometric polynomials of degree at most $n(2+1/p)$; while to estimate the second term, the bound for $|Q_n'|$ given by Theorem 2.2 has been used; in the third inequality Theorem 2.2 has been used again. Thus the proof of (2.9) is complete.

Now let $M$ be a large positive integer to be chosen later, and set

$$I_k := \left[\frac{2k\pi}{Mn}, \frac{2(k+1)\pi}{Mn}\right], \qquad k = 0,1,\dots,Mn-1\,.$$

Let $\zeta_k \in I_k$ be the place where $|T_{n,\omega}(t)|^p(\sin\omega)|\sin t|$ attains its maximum on $I_k$, and let $\theta_k \in I_k$ be a place where $W_{n,\omega}(t)$ attains its maximum on $I_k$ (note that $W_{n,\omega}$ is positive continuous). Finally we define

$$R_n := \sum |T_{n,\omega}(\zeta_k)|^p(\sin\omega)|\sin\zeta_k| W_{n,\omega}(\theta_k)\,,$$

where, and in what follows, $\sum$ is the taken for $k=0,1,\dots,Mn-1$. Let $\xi_k \in I_k$ be arbitrary. Let $J_k$ be the interval with endpoints $\zeta_k$ and $\xi_k$. Using Hölder's inequality, we obtain

$$
\begin{aligned}
R_n &- \sum |T_{n,\omega}(\xi_k)|^p (\sin\omega)|\sin\xi_k| W_{n,\omega}(\theta_k) \\
&= \sum \left( |T_{n,\omega}(\zeta_k)|^p (\sin\omega)|\sin\zeta_k| - |T_{n,\omega}(\xi_k)|^p (\sin\omega)|\sin\xi_k| \right) W_{n,\omega}(\theta_k) \\
&\le \sum \int_{J_k} p|T'_{n,\omega}(t)||T_{n,\omega}(t)|^{p-1}(\sin\omega)|\sin t| W_{n,\omega}(\theta_k)\,dt \\
&+ \sum \int_{J_k} |T_{n,\omega}(t)|^p (\sin\omega)|\cos t| W_{n,\omega}(\theta_k)\,dt \\
&\le p\left( \sum \int_{J_k} |T'_{n,\omega}(t)|^p (\sin\omega)|\sin t| W_{n,\omega}(\theta_k)\,dt \right)^{1/p} \times \\
&\times \left( \sum \int_{J_k} (|T_{n,\omega}(t)|^{(p-1)p/(p-1)} (\sin\omega)|\sin t| W_{n,\omega}(\theta_k)\,dt \right)^{(p-1)/p} \\
&+ \sum \int_{J_k} |T_{n,\omega}(t)|^p (\sin\omega) W_{n,\omega}(\theta_k)\,dt
\end{aligned}
$$

Using the fact that for $u, v \in I_k$ we have $W_{n,\omega}(u) \sim W_{n,\omega}(v)$ uniformly, then applying (2.9) and Corollary 2.3 we can continue

$$
\begin{aligned}
&\le p\left( \int_{-\pi}^{\pi} |T'_{n,\omega}(t)|^p (\sin\omega)|\sin t| W_{n,\omega}(t)\,dt \right)^{1/p} \times \\
&\times \left( \int_{-\pi}^{\pi} |T_{n,\omega}(t)|^p (\sin\omega)|\sin t| W_{n,\omega}(t)\,dt \right)^{(p-1)/p} + \\
&+ \int_{-\pi}^{\pi} |T_{n,\omega}(t)|^p (\sin\omega) W_{n,\omega}(t)\,dt \\
&\le \left( pC_1 n^p \int_{-\pi}^{\pi} |T_{n,\omega}(t)|^p (\sin\omega)|\sin t| W_{n,\omega}(t)\,dt \right)^{1/p} \times \\
&\times \left( \int_{-\pi}^{\pi} |T_{n,\omega}(t)|^p (\sin\omega)|\sin t| W_{n,\omega}(t)\,dt \right)^{(p-1)/p} + \\
&+ C_1 n \int_{-\pi}^{\pi} |T_{n,\omega}(t)|^p (\sin\omega)|\sin t| W_{n,\omega}(t)\,dt = \\
&= C_2 n \int_{-\pi}^{\pi} |T_{n,\omega}(t)|^p (\sin\omega)|\sin t| W_{n,\omega}(t)\,dt \le \\
&\le C_2 n \frac{2\pi}{Mn} \sum |T_n(\zeta_k)|^p (\sin\omega)|\sin\zeta_k| W_{n,\omega}(\theta_k) = \\
&= \frac{C_2 n(2\pi)}{Mn} R_n\,.
\end{aligned}
$$

So we have proven

$$R_n - \sum |T_{n,\omega}(\xi_k)|^p (\sin\omega)|\sin\xi_k| W_{n,\omega}(\theta_k) \le \frac{C_2(2\pi)}{M} R_n\,,$$

from which it follows that

$$R_n - \sum |T_{n,\omega}(\xi_k)|^p (\sin\omega)|\sin\xi_k| W_{n,\omega}(\theta_k) \le \frac{1}{2} R_n\,, \tag{2.11}$$

provided $M \geq 4\pi C_2$.

Using also that $W_{n,\omega}(\theta_k) \sim W_{n,\omega}(\eta_k)$ uniformly whenever $\eta_k \in I_k$, we obtain that there is a constant $C$ such that for any $\xi_k, \eta_k \in I_k$ we have

$$\sum |T_{n,\omega}(\xi_k)|^p (\sin\omega)|\sin\xi_k| W_{n,\omega}(\eta_k) \geq \frac{1}{C} R_n\,.$$

In particular, this is true for the points $\xi_k$ and $\eta_k$ where $|T_{n,\omega}(t)|(\sin\omega)|\sin t|$ and $W_{n,\omega}(t)$, respectively, attain their minimum on $I_k$, from which we obtain that all possible sums

$$\sum |T_{n,\omega}(u_k)|^p (\sin\omega)|\sin u_k| W_{n,\omega}(v_k)\,, \qquad u_k,\, v_k \in I_k\,,$$

are uniformly of the same size ($\sim R_n$). If we also observe that

$$W_{n,\omega}(v_k) \sim n \int_{I_k} W_\omega(t)\,dt\,,$$

it follows, with some constant $C > 0$, that

$$\begin{aligned} &\frac{n}{C} \sum \int_{I_k} \left( \max_{v \in I_k} |T_{n,\omega}(v)|^p (\sin\omega)|\sin v| \right) W_\omega(u)\,du \leq \\ &\leq \sum |T_{n,\omega}(u_k)|^p (\sin\omega)|\sin u_k| W_{n,\omega}(v_k) \leq \\ &\leq Cn \sum \int_{I_k} \left( \min_{v \in I_k} |T_{n,\omega}(v)|^p (\sin\omega)|\sin v| \right) W_\omega(u)\,du \end{aligned}$$

whenever $u_k, v_k \in I_k$. Setting $u_k = v_k = 2k\pi/(Mn) + t$ and integrating this with respect to $t \in [0, 1/(Mn)]$, it follows that

$$\begin{aligned} &\frac{1}{C} \sum \int_{I_k} \left( \max_{v \in I_k} |T_{n,\omega}(v)|^p (\sin\omega)|\sin v| \right) W_\omega(u)\,du \leq \\ &\leq \sum \int_{I_k} |T_{n,\omega}(t)|^p (\sin\omega)|\sin t| W_{n,\omega}(t)\,dt \leq \\ &\leq C \sum \int_{I_k} \left( \min_{v \in I_k} |T_{n,\omega}(v)|^p (\sin\omega)|\sin v| \right) W_\omega(u)\,du\,. \end{aligned}$$

We now conclude that

$$\begin{aligned} \frac{1}{C} \sum \int_{I_k} |T_{n,\omega}(t)|^p (\sin\omega)|\sin t| W_\omega(t)\,dt &\leq \sum \int_{I_k} |T_{n,\omega}(t)|^p (\sin\omega)|\sin t| W_{n,\omega}(t)\,dt \\ &\leq C \sum \int_{I_k} |T_{n,\omega}(t)|^p (\sin\omega)|\sin t| W_\omega(t)\,dt\,, \end{aligned}$$

which we wanted to prove. $\square$

## 3. Proof of Theorem 1.3

*Proof of Theorem 1.3.* With the help of Theorem 2.1 and with a piece of its proof, the proof of the theorem is a triviality now. It is sufficient to prove that

$$\int_{-\pi}^{\pi} |T_n'(\arcsin(\sin\omega\cos t)|^p(\sin\omega((1/n)+|\sin t|))^p W_\omega(t)(\sin\omega)|\sin t|\,dt \leq$$
$$\leq Cn^p \int_{-\pi}^{\pi} |T_{n,\omega}(t)|^p W_\omega(t)(\sin\omega)|\sin t|\,dt\,.$$

holds for every $T_n \in \mathcal{T}_n$. We have already proven this with $W_\omega$ replaced by $W_{n,\omega}$, see (2.9). What remains to observe is that Theorem 2.1 allows us to replace $W_{n,\omega}$ by $W_\omega$. To this end we need to remark that if $W(\arcsin((\sin\omega)\cos t))$ is a doubling weight, then $W(\arcsin((\sin\omega)\cos t))(\sin\omega)(1/n+|\sin t|))^p$ with doubling constant independent of $n$.

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