ON A GENERALIZATION OF THE BERNSTEIN-MARKOV INEQUALITY

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Abstract. We show that

$$\|P'Q\|_{L_p(I)} \le c^{1+1/p}(N+M)\log(\min(N,M+1)+1)\|PQ\|_{L_p(I)}$$

for all trigonometric polynomials P and Q of degree N and M, respectively, where $0 < p \le \infty,\ I := [-\pi,\pi]$ and $c > 0$ is a suitable absolute constant. We also show that

$$\|f'g\|_{L_p(J)} \le c^{1+1/p}(N+M)^2\|fg\|_{L_p(J)}$$

for all algebraic polynomials f and g of degree N and M, respectively, where $0 < p \le \infty,\ J := [-1,1]$ and $c > 0$ is a suitable absolute constant. Both of our trigonometric and algebraic results are sharp up to the factor $c^{1+1/p}$. In fact, we prove our results for much the wider classes of generalized trigonometric and algebraic polynomials.

1. Introduction

Let GTP_N denote the set of generalized trigonometric polynomials of degree at most N, i.e. 2π-periodic functions of the form

$$(1.1)\qquad |\omega|\prod_{j=1}^{s}|\sin((x-z_j)/2)|^{r_j}$$

with $0 < r_j \in \mathbf{R},\ z_j \in \mathbf{C}$, and $0 \ne \omega \in \mathbf{C}$. The positive real number $N = \frac{1}{2}\sum_{j=1}^{s} r_j$ is called the degree of the polynomial. If all the r_j's and N are integers, then the corresponding function is the absolute value of an ordinary trigonometric polynomial (of degree N). It is well-known that these polynomials satisfy the Bernstein-type inequality:*

* Here and in what follows, $c, c_1, c_2, \ldots$ will always denote suitable positive absolute constants, not necessarily the same at each occurrence.

Theorem A ([1], Theorem A.4.12 and Corollary A.4.13). *Let χ be a nonnegative, nondecreasing, convex function defined on $[0,\infty)$. Then*

$$\int_{-\pi}^{\pi} \chi(N^{-q}|P'(t)|^q)\,dt \le \int_{-\pi}^{\pi} \chi(cP(t)^q)\,dt \tag{1.2}$$

for every $P \in \mathrm{GTP}_N$ of the form (1.1) with each $r_j \ge 1$, and for every $0 < q \le 1$. In particular,

$$||P'||_{L_p(I)} \le c^{1+1/p} N ||P||_{L_p(I)} \qquad (0 < p \le \infty) \tag{1.3}$$

holds for every $P \in \mathrm{GTP}_N$ of the form (1.1) with each $r_j \ge 1$, where $I = [-\pi, \pi]$.

(1.3) can be generalized to the following:

Theorem B. *We have constant $c > 0$ such that*

$$||P'Q||_{L_p(I)} \le A_{N,M} ||PQ||_{L_p(I)} \tag{1.4}$$

where

$$A_{N,M} = c^{1+1/p}(N + M\min\{p,1\})(M+1) \qquad (0 < p \le \infty), \tag{1.5}$$

for every $P \in \mathrm{GTP}_N$ of the form (1.1) with each $r_j \ge 1$ and for every $Q \in \mathrm{GTP}_M$.

This is a special case of Theorem 10.4 in [4].

On the other hand, let GAP_N be the set of generalized algebraic polynomials of degree at most N, i.e. functions of the form

$$|\omega| \prod_{j=1}^{m} |x - z_j|^{r_j} \tag{1.6}$$

with $0 < r_j \in \mathbf{R}$, $z_j \in \mathbf{C}$, and $0 \ne \omega \in \mathbf{C}$. The positive real number $N = \sum_{j=1}^m r_j$ is called the degree of the polynomial. If all the r_j's are integers then the corresponding function is the absolute value of an ordinary algebraic polynomial (of degree N). These polynomials satisfy the following Bernstein and Markov type inequalities, respectively:

Theorem C. *We have*

$$||\sqrt{1-x^2} f'(x) g(x)||_{L_p(J)} \le A_{N+1, M+1/p} ||fg||_{L_p(J)} \qquad (0 < p \le \infty) \tag{1.7}$$

for every $f \in \mathrm{GAP}_N$ of the form (1.6) with each $r_j \ge 1$ and for every $g \in \mathrm{GAP}_M$, where $J = [-1, 1]$, and $A_{N,M}$ is defined in (1.5). In particular,

$$||\sqrt{1-x^2} f'(x)||_{L_p(J)} \le c^{1+1/p} N ||f||_{L_p(J)} \qquad (0 < p \le \infty) \tag{1.8}$$

for every $f \in \mathrm{GAP}_N$ of the form (1.6) with each $r_j \ge 1$.

This can be obtained from Theorem B with a substitution

$$P(t) = f(\cos t) \in \mathrm{GTP}_N \quad \text{and} \quad Q(t) = g(\cos t)|\sin t|^{1/p} \in \mathrm{GTP}_{M+1/p} \tag{1.9}$$

(cf. E.10 a] on p. 409 of [1]).

Theorem D (cf. [4], Theorems 10.3 and 10.6, as well as [2], Theorem 1). *We have*

$$||f'g||_{L_p(J)} \le \begin{cases} A^2_{N,M}||fg||_{L_p(J)} & \text{if } 0 < p < \infty, \\ c(N+M)^2||fg||_{L_\infty(J)} & \text{if } p = \infty \end{cases} \tag{1.10}$$

for every $f \in \mathrm{GAP}_N$ *of the form* (1.6) *with each* $r_j \ge 1$ *and for every* $g \in \mathrm{GAP}_M$. *In particular,*

$$||f'||_{L_p(J)} \le c^{1+1/p}N^2||f||_{L_p(J)} \tag{1.11}$$

for every $f \in \mathrm{GAP}_N$ *of the form* (1.6) *with each* $r_j \ge 1$.

While the ordinary Bernstein and Markov inequalities (1.3), (1.8) and (1.11) are known to be sharp as for the order of magnitude, the same is not known for (1.4), (1.7) and (1.10). In fact, it is the purpose of the present paper to improve these inequalities under certain conditions.

2. The case of generalized trigonometric polynomials

Theorem 1. *We have*

$$||P'Q||_{L_p(I)} \le c^{1+1/p}(N+M)\log(\min(N, M+1)+1)||PQ||_{L_p(I)} \qquad (0 < p \le \infty) \tag{2.1}$$

for any two $P \in \mathrm{GTP}_N$ *and* $Q \in \mathrm{GTP}_M$ *such that the roots of* P *and* Q *have multiplicities at least* 1. *Moreover,* (2.1) *is sharp apart from the constant* $c^{1+1/p}$.

This improves considerably Theorem B (under the stronger condition that the multiplicities in Q are at least 1).

In fact, we will prove slightly more; this is formulated as a lemma, and it is a generalization of Theorem A.

Lemma 1. *Let* χ *be a nonnegative, nondecreasing, convex function defined on* $[0, \infty)$. *Then*

$$\int_{-\pi}^{\pi} \chi\left(\left(\frac{|P'(t)|Q(t)}{(N+M)\log(N+1)}\right)^q\right) dt \le \int_{-\pi}^{\pi} \chi(c(P(t)Q(t))^q)\, dt \tag{2.2}$$

for every $P \in \mathrm{GTP}_N$ *of the form* (1.1) *with each* $r_j \ge 1$, *for every* $Q \in \mathrm{GTP}_M$, *and for every* $0 < q \le 1$. *In particular,*

$$||P'Q||_{L_p(I)} \le c^{1+1/p}(N+M)\log(N+1)||PQ||_{L_p(I)} \qquad (0 < p \le \infty). \tag{2.3}$$

(2.3) follows from (2.2) with $q = \min(1,p)$ and $\chi(x) = x^{\max(1,p)}$. On the other hand, (2.3) implies (2.1). Indeed, we can apply (2.3) with the roles of P and Q interchanged to get

$$||Q'P||_{L_p(I)} \le c^{1+1/p}(N+M)\log(M+1)||PQ||_{L_p(I)} \qquad (0 < p \le \infty).$$

This coupled with the generalized Bernstein inequality

$$||(PQ)'||_{L_p(I)} \le c^{1+1/p}(N+M)||PQ||_{L_p(I)} \qquad (0 < p \le \infty)$$

(cf. (1.3)) yields the statement of Theorem 1.

In order to prove Lemma 1, we need several auxiliary statements.

Lemma 2. *Let $P \in \mathrm{GTP}_N$, Δ an arbitrary interval with midpoint t, and*

$$\mathcal{M}(P,\Delta) := \sum_{t_j \in \Delta} r_j,$$

where r_j is the multiplicity of the root t_j of P. Then

$$\mathcal{M}(P,\Delta) \le \left(\frac{e}{2}N|\Delta| + 1\right)\frac{||P||_{L_\infty(I)}}{P(t)} \qquad \textit{for all} \quad \Delta.$$

Proof. It follows from the proof of E.11 on pp. 236-237 of [1] that if p is an ordinary trigonometric polynomial of degree at most n then

$$\left(\frac{2\mathcal{M}(p,\Delta)}{e|\Delta|n}\right)^{\mathcal{M}(p,\Delta)} \le \frac{||p||_{L_\infty(I)}}{|p(t)|}. \tag{2.4}$$

First we assume that in the representation of $P \in \mathrm{GTP}_N$, each r_j is rational with a common denominator $q \in \mathbf{N}$, and apply (2.4) to the ordinary trigonometric polynomial $p = P^{2q}$ of degree at most $n = 2qN$. Since evidently $\mathcal{M}(p,\Delta) = 2q\mathcal{M}(P,\Delta)$, we get

$$\left(\frac{4q\mathcal{M}(P,\Delta)}{e|\Delta|2qN}\right)^{2q\mathcal{M}(P,\Delta)} \le \left(\frac{||P||_{L_\infty(I)}}{|P(t)|}\right)^{2q},$$

i.e.

$$\left(\frac{2\mathcal{M}(P,\Delta)}{e|\Delta|N}\right)^{\mathcal{M}(P,\Delta)} \le \frac{||P||_{L_\infty(I)}}{|P(t)|}. \tag{2.5}$$

We may assume that $\mathcal{M}(P,\Delta) \ge \frac{e|\Delta|N}{2} + 1$ (otherwise there is nothing to prove), and hence we may replace the left hand side exponent in (2.5) by 1. This proves the lemma when each r_j is rational. Since (2.5) is independent of the common denominator q of the rational exponents, an obvious limit procedure yields the result for arbitrary exponents.

Lemma 3 ([1], E.7 b] on p. 409; cf. also [3]). *The inequality*

$$m(\{t \in [-\pi, \pi) : P(t) \geq \lambda ||P||_{L_\infty(I)}\}) \geq \frac{\mu(\lambda)}{N+1}$$

holds for every $P \in \mathrm{GTP}_N$*, where* $0 < \lambda < 1$ *is arbitrary, and* $\mu(\lambda) > 0$ *depends only on* λ.

We use the notation $\mathbf{T} := \mathbf{R} \pmod{2\pi}$.

Lemma 4. *Suppose* $P \in \mathrm{GTP}_N$*. Let* $a \in [-\pi, \pi)$ *be such that*

$$P(a) = ||P||_{L_\infty(I)} . \tag{2.6}$$

Then

$$m\{t \in [a - 1/N, a + 1/N] : P(t) \geq c_2^{-1} ||P||_{L_\infty(I)}\} \geq \frac{c_1}{N} .$$

Proof of Lemma 4. With $\omega := \pi - 1/N$ and $n := \lfloor N \rfloor$, let

$$R_N(t) := \left| T_n \left(\frac{\sin((t - \pi - a)/2)}{\sin(\omega/2)} \right) \right| \in \mathrm{GTP}_N ,$$

where $T_n(x) = \cos(nt)$, $x = \cos t$, is the Chebyshev polynomial of degree n. Then there is an absolute constant $c_2 > 1$ such that

$$c_2 \leq |R_N(a)| = ||R_N||_{L_\infty(I)} \tag{2.7}$$

and

$$|R_N(t)| \leq 1 , \qquad t \in \mathbf{T} \setminus [a - 1/N, a + 1/N] . \tag{2.8}$$

Assume (2.6) holds. Let

$$A := \{t \in [a - 1/N, a + 1/N] : P(t) \leq c_2^{-1} ||P||_{L_\infty(I)}\} , \tag{2.9}$$

and $Q := PR_N \in \mathrm{GTP}_{2N}$. Then, using (2.6)–(2.9), we obtain

$$Q(t) \leq c_2^{-1} ||Q||_{L_\infty(I)} , \qquad t \in A \cup (\mathbf{T} \setminus [a - 1/N, a + 1/N]) . \tag{2.10}$$

Let

$$B := [a - 1/N, a + 1/N] \setminus A . \tag{2.11}$$

Observe that (2.10), (2.11), and Lemma 3 applied to $Q \in \mathrm{GTP}_{2N}$ imply that

$$m(B) \geq m\{t \in \mathbf{T} : Q(t) \geq c_2^{-1} ||Q||_{L_\infty(I)}\} \geq \frac{c_1}{N}.$$

Hence

$$m(B) = m\{t \in [a - 1/N, a + 1/N] : P(t) \geq c_2^{-1} ||P||_{L_\infty(I)}\} \geq \frac{c_1}{N}$$

and the lemma is proved.

Proof of Lemma 1. First we prove (2.3) for $p = \infty$. Without loss of generality we may assume that $|P'(\pi)|Q(\pi) = ||P'Q||_{L_\infty(I)}$. It is sufficient to handle the case when P has only real zeros. This can be seen by Lemmas 5.1–5.3 in [2].

We now apply Lemma 4 with $|P'|Q \in \mathrm{GTP}_{N+M}$ and $a = \pi$ to obtain

$$|P'(t)|Q(t) \geq \frac{1}{c_2} ||P'Q||_{L_\infty(I)} \qquad (t \in B), \tag{2.12}$$

where

$$B \subset K := \left[\pi - \frac{1}{N+M}, \pi + \frac{1}{N+M}\right], \quad |B| \geq \frac{c_1}{N+M}. \tag{2.13}$$

By Lemma 2 with $t = \pi$ we get $\mathcal{M}(|P'|Q, B) \leq \mathcal{M}(|P'|Q, K) \leq e + 1 < 4$. Thus $\mathcal{M}(|P'|, B) < 4$ which implies $\mathcal{M}(P, B) < 5$. Denote the different zeros of P in $(-\pi, \pi]$ by α_j with respective multiplicities $r_j \geq 1$, $j = 1, 2, \ldots, m$ (then, of course, $2N = \sum_{j=1}^m r_j$). Thus (2.13), $\mathcal{M}(P, B) < 5$ and $r_j \geq 1$, $j = 1, 2, \ldots, m$ yield that there exists a $t \in B$ such that

$$|t - \alpha_j| > \frac{c}{N+M} \qquad (j = 1, 2, \ldots, m).$$

Fixing this t, we introduce the following intervals:

$$I_k := \left[t - \frac{2^k c}{N+M}, t + \frac{2^k c}{N+M}\right) \qquad (k = 0, 1, \ldots, [\log_2 N]),$$

and let

$$I_{[\log_2 N]+1} := [t - \pi, t + \pi).$$

Using (2.12) for our t we can easily deduce that

$$\mathcal{M}(P, I_k) \leq \mathcal{M}(|P'|, I_k) + 1 \leq \mathcal{M}(|P'|Q, I_k) + 1 \leq$$

$$\leq \left(\frac{e}{2}(N+M)|I_k| + 1\right) \frac{||P'Q||_{L_\infty(I)}}{|P'(t)|Q(t)} + 1 = ecc_2 2^k + c_2 + 1 \leq c_3 2^k \qquad (k = 0, 1, \ldots, [\log_2 N]).$$

Also, because of the choice of t, P does not have a zero in I_0. Therefore we can estimate as follows:

$$\frac{|P'(\pi)|Q(\pi)}{||PQ||_{L_\infty(I)}} \leq c_2 \frac{|P'(t)Q(t)|}{|P(t)Q(t)|} = \frac{c_2}{2} \left| \sum_{j=1}^m r_j \cot \frac{t - \alpha_j}{2} \right| \leq$$

$$\leq \sum_{k=1}^{[\log_2 N]+1} \frac{c_2}{2} \sum_{j \in I_k \setminus I_{k-1}} r_j \left| \cot \frac{t - \alpha_j}{2} \right| \leq c_2 \sum_{k=1}^{[\log_2 N]+1} \frac{\mathcal{M}(P, I_k)}{c_4 |I_{k-1}|} \leq$$

$$\leq c_2 \sum_{k=1}^{c_3[\log_2 N]+1} \frac{24e(N+M)}{c_4} \leq c_5(N+M)\log(N+1).$$

This proves (2.3) for $p=\infty$.

We now turn to the proof of (2.2). Applying (2.3) with $p=\infty$ to the generalized trigonometric polynomials P and QR, with

$$R(t) := \left|\frac{\sin(N+M)t}{\sin t}\right|^{2/q},$$

instead of P and Q, respectively, then using Nikolskii's inequality

$$||\chi(P)||_{L_p(I)} \leq (c(1+qN))^{1/q-1/p}||\chi(P)||_{L_p(I)} \qquad (P \in \mathrm{GTP}_N, 0<q<p\leq\infty)$$

(cf. [1], Theorem A.4.3) with $\chi(x)=x$, $p=\infty$ and with PQR instead of P, we obtain

$$||P'QR||^q_{L_\infty(I)} \leq c_1^q(N+M)^q \log^q(N+1)||PQR||^q_{L_\infty(I)} \leq$$

$$\leq c(N+M)^{q+1}\log^q(N+1)||PQR||^q_{L_q(I)}.$$

Since $R(0)^q=(N+M)^2$, the latter inequality implies

$$|P'(0)|^q Q(0)^q \leq c(N+M)^{q-1}\log^q(N+1)||PQR||^q_{L_q(I)}.$$

Using this with $P(\cdot+\tau)$ and $Q(\cdot+\tau)$ instead of $P(\cdot)$ and $Q(\cdot)$, respectively (τ is a fixed parameter), we obtain

$$\left(\frac{|P'(\tau)|Q(\tau)}{(N+M)\log(N+1)}\right)^q \leq \int_{-\pi}^{\pi} c(P(t)Q(t))^q \frac{R(t-\tau)^q}{||R^q||_{L_1}}\,dt,$$

since evidently $||R^q||_{L_1} \sim N+M$. Hence by Jensen's inequality

$$\chi\left(\int_a^b S(t)w(t)\,dt\right) \leq \int_a^b \chi(S(t))w(t)\,dt$$

(cf. [1], E.20 on p. 414) applied with $[a,b]=[-\pi,\pi]$, $S=c(PQ)^q$, and

$$w(t) = \frac{R(t)^q}{||R^q||_{L_1(I)}}$$

we obtain

$$\chi\left(\left(\frac{|P'(\tau)|Q(\tau)}{(N+M)\log(N+1)}\right)^q\right) \leq \int_{-\pi}^{\pi} \chi(c(P(t)Q(t))^q)\frac{R(t-\tau)^q}{||R^q||_{L_1(I)}}\,dt.$$

Integrating with respect to τ and using Fubini's theorem yields the desired inequality.

It remains to show that (2.1) is sharp. To see this, we may assume that N and M are positive integers. Let $q = \min(1,p)$ and

$$S_{N+M}(t) := \left| \frac{\sin((N+M+1)t)}{\sin t} \right|^{2/q} = C_{N+M} \left| \prod_{k=1}^{N+M} \sin \frac{t-\alpha_k}{2} \prod_{k=1}^{N+M} \sin \frac{t+\alpha_k}{2} \right|^{2/q},$$

where

$$\alpha_k := \frac{k\pi}{N+M+1}, \qquad k = 1, 2, \dots, N+M.$$

Let P_N and Q_M be generalized trigonometric polynomials of degree $2N/q$ and $2M/q$, respectively, defined by

$$P_N(t) := \begin{cases} \left| \prod_{k=1}^{2N} \sin \frac{t-\alpha_k}{2} \right|^{2/q} & \text{if } N \le M, \\ \left| \prod_{k=1}^{N+M} \sin \frac{t-\alpha_k}{2} \prod_{k=2M+1}^{N+M} \sin \frac{t+\alpha_k}{2} \right|^{2/q} & \text{if } N > M, \end{cases}$$

and

$$Q_M(t) := \frac{S_{N+M}(t)}{P_N(t)},$$

respectively. It is easy to see that

$$S_{N+M}(0) = \max_{t \in (-\pi,\pi]} S_{N+M}(t).$$

Hence

$$\frac{\max_{t\in(-\pi,\pi]} |P_N'(t)| Q_M(t)}{\max_{t\in(-\pi,\pi]} P_N(t) Q_M(t)} \ge \frac{|P_N'(0)| Q_M(0)}{P_N(0) Q_M(0)} =$$

$$= \begin{cases} \frac{1}{q} \sum_{k=1}^{2N} \cot \frac{\alpha_k}{2} \ge c(N+M) \log(N+1) & \text{if } N \le M, \\ \frac{1}{q} \sum_{k=1}^{2M} \cot \frac{\alpha_k}{2} \ge c(N+M) \log(M+1) & \text{if } N > M \end{cases}$$

with an absolute constant $c > 0$. This proves the sharpness for $p = \infty$.

Now let $0 < p < \infty$. We have

$$P_N'(t) Q_M(t) = \begin{cases} \frac{1}{q} S_{N+M}(t) \sum_{k=1}^{2N} \cot \frac{t-\alpha_k}{2} & \text{if } N \le M, \\ \frac{1}{q} S_{N+M}(t) \left(\sum_{k=1}^{N+M} \cot \frac{t-\alpha_k}{2} + \sum_{k=2M+1}^{N+M} \cot \frac{t+\alpha_k}{2} \right) & \text{if } N > M. \end{cases}$$

We shall give a lower estimate of the L_p norm of this polynomial over the interval $[0, \alpha_1/2]$. Evidently

$$S_{N+M}(t) \ge c(N+M)^{2/q} \qquad (0 \le t \le \alpha_1/2).$$

On the other hand, if $t \in [0, \alpha_1/2]$ and $N \le M$, then

$$\left| \sum_{k=1}^{2N} \cot \frac{t-\alpha_k}{2} \right| \ge \sum_{k=1}^{N} \cot \frac{\alpha_k - \alpha_1/2}{2} \ge c(N+M) \log(N+1),$$

while if $t \in [0, \alpha_1/2]$ and $N > M$, then

$$\left|\sum_{k=1}^{N+M} \cot \frac{t-\alpha_k}{2} + \sum_{k=2M+1}^{N+M} \cot \frac{t+\alpha_k}{2}\right| \geq$$

$$\geq \sum_{k=2j}^{M} \cot \frac{\alpha_k - \alpha_j}{2} - \sum_{k=2M+1}^{N+M} \left|\cot \frac{t-\alpha_k}{2} + \cot \frac{t+\alpha_k}{2}\right| \geq$$

$$\geq c_1(N+M)\log(M+1) - c_2 \sum_{k=2M+1}^{N+M} \frac{\sin t}{\sin^2 \frac{\alpha_k}{2}} \geq$$

$$\geq c_1(N+M)\log(M+1) - c_3 \frac{N+M}{M} \geq c_4(N+M)\log(M+1).$$

Thus we obtain

$$\int_0^{\alpha_1/2} |P_N'(t)Q_M(t)|^p \, dt \geq \frac{c}{N+M}((N+M)^{1+2/q}\log(\min(N,M)+1))^p \geq$$

$$\geq c(N+M)^{p-1+2p/q}\log^p(\min(N,M)+1).$$

This compared with

$$||P_N Q_M||_{L_p}^p = ||S_{N+M}||_{L_p}^p \sim (N+M)^{2p/q-1}$$

proves that

$$||P_N' Q_M||_{L_p} \geq c(N+M)\log(\min(N,M)+1)||P_N Q_M||_{L_p}.$$

3. The case of generalized algebraic polynomials

In this section we improve the estimates of the inequalities (1.7) an (1.10).

Theorem 2. *We have*

(3.1)
$$||\sqrt{1-x^2} f'(x) g(x)||_{L_p(J)} \leq c^{1+1/p}(N+M)\log(\min(N,M+1)+1)||fg||_{L_p(J)} \qquad (0 < p \leq \infty)$$

for any two $f \in \mathrm{GAP}_N$ and $g \in \mathrm{GAP}_M$ such that the roots of f and g have multiplicities at least 1*. Moreover,* (3.1) *is sharp apart from the constant $c^{1+1/p}$.*

Proof. Using the substitution (1.9) in (2.3) we obtain

$$||\sqrt{1-x^2} f'(x) g(x)||_{L_p(J)} \leq c^{1+1/p}(N+M)\log(N+1)||fg||_{L_p(J)} \qquad (0 < p \leq \infty).$$

Exchanging the roles of f and g, and using the Bernstein inequality (1.8) with fg instead of f, we obtain (3.1). The sharpness follows from the sharpness of the trigonometric analogue (2.1).

Theorem 3. *Let χ be a nonnegative, nondecreasing, convex function defined on $[0,\infty)$. Then*

$$\int_{-1}^{1} \chi\left(\frac{|f'(x)|g(x)}{(N+M)^{2q}}\right) dx \le 2\int_{-1}^{1} \chi(c(f(x)g(x))^q)\,dx \tag{3.2}$$

for every $f \in \mathrm{GAP}_N$ of the form (1.6) with each $r_j \ge 1$, for every $g \in \mathrm{GAP}_M$, and for every $0 < q \le 1$. In particular,

$$||f'g||_{L_p(J)} \le c^{1+1/p}(N+M)^2||fg||_{L_p(J)} \qquad (0 < p \le \infty) \tag{3.3}$$

holds, where $c > 0$ is a suitable absolute constant. The latter inequality is sharp up to the factor $c^{1+1/p}$ for all $N, M \ge 1$.

Proof. (3.3) readily follows from (3.2) by putting $q = \min(1,p)$ and $\chi(x) = x^{\max(1,p)}$. In order to prove (3.2) we mention that (3.3) for $p = \infty$, i.e.

$$||f'g||_{L_\infty(J)} \le c(N+M)^2||fg||_{L_\infty(J)} \tag{3.4}$$

is nothing else but the corresponding inequality in (1.10) proved in [2].

Now consider an ordinary algebraic polynomial $h \ge 0$ on $[-1,1]$ of degree at most $N+M$ such that

$$\int_{-1}^{1} h(t)\,dt \le \frac{c}{(N+M)^2} \quad \text{and} \quad h(1) = 1 \tag{3.5}$$

with a suitable absolute constant $c > 0$. The existence of such a polynomial is guaranteed by known estimates for the Christoffel functions of the orthogonal Legendre polynomials (see e.g. Freud [5], Problem 10 on p. 132). Let $0 \le x \le 1$, y a fixed parameter to be specified later, and apply (3.4) on the interval $[x-1,x]$ instead of $[-1,1]$ for the generalized polynomials $f(x)$ and $g(x)h(x-y)^{1/q}$ (instead of $g(x)$):

$$|f'(x)|g(x)h(x-y)^{1/q} \le c_2 q^{-2}(N+M)^2 \max_{x-1\le t\le x} f(t)g(t)h(t-y)^{1/q} \qquad (0 \le x, y \le 1)$$

with a suitable absolute constant $c_1 > 0$. Using Nikolskii's inequality

$$||\chi(f)||_{L_p(J)} \le (c(2+qN))^{2/q-2/p}||\chi(f)||_{L_q(J)} \qquad (f \in \mathrm{GAP}_N,\ 0 < q < p \le \infty)$$

(see [1], Theorem A.4.4) with $\chi(x) = x$, $p = \infty$ and with $f(t)g(t)h(t-y)$ instead of f in the interval $[x-1,x]$ instead of J we get

$$(|f'(x)|g(x))^q h(x-y) \le c_1(N+M)^{2q+2}\int_{x-1}^{x} (f(t)g(t))^q h(t-y)\,dt \qquad (0 \le x, y \le 1).$$

Putting $y = x - 1$ and recalling (3.5), we deduce that

$$(|f'(x)|g(x))^q \le c(N+M)^{2q+2} \int_{x-1}^{x} (f(t)g(t))^q h(t-x+1)\,dt$$

$$\le c(N+M)^{2q} \int_{x-1}^{x} \frac{(f(t)g(t))^q h(t-x+1)}{\int_{x-1}^{x} h(u-x+1)\,du}\,dt, \qquad 0 \le x \le 1.$$

Rearranging this and using (2.13) with $[a,b] = [x-1,x]$, $S = c(fg)^q$, and

$$w(t) := \frac{h(t-x+1)}{\int_{x-1}^{x} h(u-x+1)\,du} = \frac{h(t-x+1)}{\int_0^1 h(v)\,dv}$$

(note that $\int_{x-1}^{x} w(t)\,dt = 1$), we obtain

$$\chi\left(\frac{|f'(x)|g(x)}{(N+M)^{2q}}\right) \le \frac{\int_{x-1}^{x} \chi(c(f(t)g(t))^q) h(t-x+1)\,dt}{\int_0^1 h(v)\,dv}$$

$$= \frac{\int_{-1}^{1} \chi(c(f(t)g(t))^q) \varphi_{[x-1,x]}(t) h(t-x+1)\,dt}{\int_0^1 h(v)\,dv},$$

where $\varphi_{[a,b]}(t)$ is the characteristic function of the interval $[a,b]$. Integrating both sides with respect to x on $[0,1]$ and using Fubini's theorem, we get

$$\int_0^1 \chi\left(\frac{|f'(x)|g(x)}{(N+M)^{2q}}\right) dx \le \frac{\int_{-1}^{1} \chi(c(f(t)g(t))^q) \int_0^1 \varphi_{[x-1,x]}(t) h(t-x+1)\,dx\,dt}{\int_0^1 h(v)\,dv}.$$

Here an easy calculation shows that

$$\int_0^1 \varphi_{[x-1,x]}(t) h(t-x+1)\,dx = \begin{cases} \int_0^{t+1} h(v)\,dv & \text{if } -1 \le t \le 0, \\ \int_t^1 h(v)\,dv & \text{if } 0 \le t \le 1, \end{cases}$$

which can be estimated by $\int_0^1 h(v)\,dv$ in both cases. Hence (3.2) is proved without the factor 2 when the integral is taken over $[0,1]$ rather than $[-1,1]$ on the left hand side. Similar arguments yield (3.2) without the factor 2 when the integral is takeb over $[-1,0]$ rather than $[-1,1]$ on the left hand side. In conclusion (3.2) holds with the factor 2.

To prove the sharpness of (3.3), let $u_n^{(\lambda)}(x)$ be the ultraspherical Jacobi polynomial of degree n with parameter $\lambda \ge 0$ normalized such that $u_n^{(\lambda)}(1) = 1$. Then the absolute maximum of $u_n^{(\lambda)}(x)$ is attained at ± 1. Without loss of generality we may assume that N and M are positive integers. Let f_N be the monic polynomial of degree N which has N roots of the polynomial $u_{N+M}^{(\lambda)}(x)$ closest to 1, and let g_M be defined by $f_N g_M = u_{N+M}^{(\lambda)}$. Then for $p = \infty$ we get

$$f_N'(1) g_M(1) = \frac{f_N'(1)}{f_N(1)} u_{N+M}^{(\lambda)}(1) \ge \frac{||f_N g_M||_{L_p(J)}}{1-x_1} \ge c(N+M)^2 ||f_N g_M||_{L_p(J)},$$

where x_1 is the largest root of $u_n^{(\lambda)}$ (cf. [6], (6.6.6)).

Now let $0 < p < \infty$ and $\lambda > 2/p$. Using the estimates

$$|u_n^{(\lambda)}(\cos t)| \leq \begin{cases} c_2 & \text{if } 0 \leq t \leq c_1/n, \\ c_3(nt)^{-\lambda} & \text{if } c_1/n \leq t \leq \pi/2 \end{cases}$$

(cf. [6], (7.33.6)), we get

$$||f_N g_M||_{L_p(J)} = ||u_n^{(\lambda)}||_{L_p(J)} = 2^{1/p}||u_n^{(\lambda)}||_{L_p([0,1])} = 2^{1/p}\left(\int_0^{\pi/2} |u_n^{(\lambda)}(\cos t)|^p \sin t\, dt\right)^{1/p}$$

$$\leq c_4^{1/p}\left(\int_0^{c_1/n} t\, dt + n^{-p\lambda}\int_{c_1/n}^{\pi/2} t^{1-p\lambda}\, dt\right)^{1/p} \leq c_5^{1/p} n^{-2/p}.$$

On the other hand, by Markov's inequality we have

$$u_n^{(\lambda)}(x) \geq \frac{1}{2} \quad \text{if} \quad 1 - \frac{\mu}{n^2} \leq x \leq 1$$

and $x_1 \leq 1 - \frac{2\mu}{n^2}$ with a suitable constant $\mu > 0$ depending only on λ. Thus

$$||f_N' g_M||_{L_p(J)} \geq \left(\int_{1-\mu/(N+M)^2}^1 \left|\frac{f_N'(x)}{f_N(x)} u_{N+M}^{(\lambda)}(x)\right|^p dx\right)^{1/p} \geq$$

$$\geq \left(\int_{1-\mu/(N+M)^2}^1 \left|\frac{u_{N+M}^{(\lambda)}(x)}{x - x_1}\right|^p dx\right)^{1/p} \geq \left(\frac{\mu}{(N+M)^2}\left(\frac{(N+M)^2}{\mu}\right)^p \left(\frac{1}{2}\right)^p\right)^{1/p} =$$

$$= \frac{1}{2}\mu^{1/p-1}(N+M)^{2-2/p} = \frac{1}{2}\mu^{1/p-1} c_5^{-1/p} ||f_N g_M||_{L_p(J)},$$

which proves the sharpness of (3.3).

References

[1] P. Borwein and T. Erdélyi, *Polynomials and Polynomial Inequalities*, Graduate Texts in Mathematics, Springer-Verlag (New York, 1995).

[2] T. Erdélyi, Weighted Markov and Bernstein type inequalities for generalized polynomials, *J. Approx. Theory,* **68** (1992), 283-305.

[3] T. Erdélyi, Remez-type inequalities on the size of generalized polynomials, *J. London Math. Soc.,* **45** (1992), 255-264.

[4] T. Erdélyi, Remez-type inequalities and their applications, *J. Computational and Applied Math.,* **47** (1993),167-209.

[5] G. Freud, *Orthogonal Polynomials,* Akadémiai Kiadó (Budapest, 1971).

[6] G. Szegö, *Orthogonal Polynomials,* AMS Coll. Publ., Vol. 23 (Providence, 1978).

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