DENSE MARKOV SPACES AND UNBOUNDED BERNSTEIN INEQUALITIES

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Abstract. An infinite Markov system $\{f_0, f_1, \dots\}$ of C^2 functions on $[a,b]$ has dense span in $C[a,b]$ if and only if there is an unbounded Bernstein inequality on every subinterval of $[a,b]$. That is if and only if, for each $[\alpha,\beta] \subset [a,b]$ and $\gamma > 0$, we can find $g \in \operatorname{span}\{f_0, f_1, \dots\}$ with $\|g'\|_{[\alpha,\beta]} > \gamma \|g\|_{[a,b]}$. This is proved under the assumption $(f_1/f_0)'$ does not vanish on (a,b).

Extension to higher derivatives are also considered. An interesting consequence of this is that functions in the closure of the span of a non-dense C^2 Markov system are always C^n on some subinterval.

The principal result of this paper will be a characterization of denseness of the span of a Markov system by whether or not it possesses an unbounded Bernstein Inequality. In order to make sense of this result we require the following definitions.

Definition 1 (Chebyshev System). *Let* $f_0, \dots, f_n$ *be elements of* $C[a,b]$ *the real valued continuous functions on* $[a,b]$. *Suppose that* $\operatorname{span}\{f_0, \dots, f_n\}$ *over* $\mathbb{R}$ *is an* $n+1$ *dimensional subspace of* $C[0,1]$. *Then* $\{f_0, \dots, f_n\}$ *is called a Chebyshev system of dimension* $n+1$ *if any element of* $\operatorname{span}\{f_0, \dots, f_n\}$ *that has* $n+1$ *distinct zeros in* $[0,1]$ *is identically zero. If* $\{f_0, \dots, f_n\}$ *is a Chebyshev system, then* $\operatorname{span}\{f_0, \dots, f_n\}$ *is called a Chebyshev space.*

Definition 2 (Markov System). *We say that* $\{f_0, \dots, f_n\}$ *is a Markov system on* $[a,b]$ *if each* $f_i \in C[a,b]$ *and* $\{f_0, \dots, f_m\}$ *is a Chebyshev system for every* $m \geq 0$. *(We allow* n *to tend* $+\infty$ *in which case we call the system an infinite Markov system.) If* $\{f_0, \cdots, f_n\}$ *is a Markov system then* $\operatorname{span}\{f_0, \dots, f_n\}$ *is called a Markov space.*

Definition 3 (Unbounded Bernstein Inequality). *Let* $\mathcal{A}$ *be a subset of* $C^1[a,b]$. *We say that* $\mathcal{A}$ *has an everywhere unbounded Bernstein inequality if for every* $[\alpha,\beta] \subset [a,b], \quad \alpha \neq \beta$

1991 *Mathematics Subject Classification.* 41A17, 41A540.

Key words and phrases. Denseness, Chebyshev system, Markov system, Bernstein Inequality.

Research of the first author supported, in part, by NSERC of Canada. Research of the second author supported, in part, by NSF under Grant No. DMS-9024901 and conducted while an NSERC International Fellow at Dalhousie University.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

$$\sup\left\{\frac{\|p'\|_{[\alpha,\beta]}}{\|p\|_{[a,b]}} : p \in \mathcal{A}, p \neq 0\right\} = \infty.$$

If for some $[\alpha, \beta]$ *the above sup is finite the Bernstein inequality is said to be bounded in* $[\alpha, \beta]$.

Note that the collection of all polynomials of the form

$$\{x^2 p(x) : p \text{ is a polynomial}\}$$

has an everywhere unbounded Bernstein inequality on $[-1, 1]$ despite the fact that every element has derivative vanishing at zero.

We now state the main result.

Theorem 1. *Suppose* $\mathcal{M} := \{f_0, f_1, f_2, \dots\}$ *is an infinite Markov system on* $[a, b]$ *with each* $f_i \in C^2[a, b]$, *and suppose that* $(f_1/f_0)'$ *does not vanish on* (a, b). *Then* span $\mathcal{M}$ *is dense in* $C[a, b]$ *if and only if* span $\mathcal{M}$ *has an everywhere unbounded Bernstein inequality.*

The additional assumption that $(f_1/f_0)'$ does not vanish on (a, b) is quite weak. It holds, for example, for any ECT system. Note that f_1/f_0 is strictly monotone if $\mathcal{M}$ is a Markov system.

The proof requires examining the Chebyshev polynomials associated with a Chebyshev system. These we now discuss.

Suppose

$$H_n := \operatorname{span}\{f_0, \dots, f_n\}$$

is a Chebyshev space on $[a, b]$. We can define the Chebyshev polynomial

$$T_n(x) := T_n\{f_0, \dots, f_n; [a, b]\}(x)$$

associated with H_n

by

$$T_n(x) = c\left(f_n(x) - \sum_{k=0}^{n-1} a_k f_k(x)\right)$$

where the $\{a_k\}_{k=0}^{n-1}$ are chosen to minimize

$$\left\|f_n - \sum_{k=0}^{n-1} a_k f_k\right\|_{[a,b]}$$

and where c is a normalization constant chosen so that

$$\|T_n\|_{[a,b]} = 1 \quad \text{and} \quad T_n(b) > 0.$$

We will call T_n the associated Chebyshev polynomial for H_n. This is a unique "generalized" polynomial in span$\{f_0, \dots , f_n\}$ that alternates between ± 1 exactly

$n+1$ times and has exactly n zeros on $[a,b]$. With $f_i := x^i$, this generates the usual Chebyshev polynomials. These equioscillating polynomials encode much of the information of how the space H_n behaves with respect to the supremum norm. See [2], [3], [4] and [6].

Suppose

$$\mathcal{M} = \{f_0, f_1, \dots\}$$

is a fixed infinite Markov system on $[a,b]$. For each n

$$H_n := \{f_0, f_1, \dots, f_n\}$$

is then a Chebyshev system. So there is a sequence $\{T_n\}$ of associated Chebyshev polynomials where, for each n, T_n is associated with H_n. These we call the associated Chebyshev polynomials for the infinite Markov system $\mathcal{M}$.

Note that

$$\{T_0, T_1, \dots\}$$

is a Markov system again with the same span as $\mathcal{M}$.

In [2] we showed that the span of a C^1 Markov system $\mathcal{M}$ is dense in $C[a,b]$ in the uniform norm (i.e. the uniform closure of span $\mathcal{M}$ on $[a,b]$ equals $C[a,b]$) if and only if the zeros of the associated Chebyshev polynomials are dense. To state this result, which we will need, we require the following notation.

Suppose T_n has zeros $a \le x_1 < x_2 < \cdots < x_n \le b$, and let $x_0 := a$ and $x_{n+1} := b$. Then the mesh of T_n is defined by

$$M_n := M_n(T_n : [a,b]) := \max_{1 \le i \le n+1} |x_i - x_{i-1}|.$$

For a sequence of Chebyshev polynomials T_n from a fixed Markov system on $[a,b]$ we have

$$M_n \to 0 \quad \text{iff} \quad \underline{\lim} M_n = 0$$

as follows from the interlacing of the zeros of T_n and T_{n+1} (see [6]).

Our main result requires the following theorem from [2].

Theorem 2. *Suppose $\mathcal{M} := \{1, f_1, f_2, \dots\}$ is an infinite Markov system on $[a,b]$ with each $f_i \in C^1[a,b]$. Then span $\mathcal{M}$ is dense in $C[a,b]$ in the uniform norm if and only if*

$$M_n \to 0$$

(where M_n is the mesh of the associated Chebyshev polynomials).

The next result we need shows that in most instances the Chebyshev polynomial is close to extremal for Bernstein-type inequalities.

Theorem 3. *Let $H_n := \{1, f_1, \dots, f_n\}$ be a Chebyshev system of C^1 functions on $[a,b]$. Let T_n be the associated Chebyshev polynomial. Then*

$$\frac{|p_n'(x_0)|}{\|p_n\|_{[a,b]}} \le \frac{2}{1 - |T_n(x_0)|} |T_n'(x_0)|$$

for every $0 \neq p_n \in \operatorname{span}\{1, f_1, \dots, f_n\}$ *and every* $x_0 \in [a, b]$ *with* $|T_n(x_0)| \neq 1$.

Proof. Let $a = y_0 < y_1 < \dots < y_n = b$ denote the extreme points of T_n, so

$$T_n(y_i) = (-1)^{n-i}, \qquad i = 0, 1, \dots, n.$$

Let $y_k \leq x_0 \leq y_{k+1}$ and $0 \neq p_n \in H_n$. If $p_n'(x_0) = 0$, then there is nothing to prove. So assume that $p_n'(x_0) \neq 0$. Then we may normalize p_n so that

$$\|p_n\|_{[a,b]} = 1$$

and

$$\operatorname{sign}(p_n'(x_0) = \operatorname{sign}(p(y_{k+1}) - p(y_k)).$$

Let $\delta := |T_n(x_0)|$. Let $\epsilon \in (0, 1)$ be fixed. Then there exists a constant η with $|\eta| \leq \delta + (1 - \delta)/2$ so that

$$\eta + \frac{(1-\delta)}{2}(1-\epsilon)p_n(x_0) = T_n(x_0).$$

Now let

$$q_n(x) := \eta + \frac{(1-\delta)}{2}(1-\epsilon)p_n(x).$$

Then

$$\|q_n\|_{[a,b]} \leq 1,$$
$$q_n(x_0) = T_n(x_0)$$

and

$$\operatorname{sign}(q_n'(x_0)) = \operatorname{sign}(T_n'(x_0)).$$

If the desired inequality does not hold for p_n then for a sufficiently small $\epsilon > 0$

$$|q_n'(x_0)| > |T_n'(x_0)|,$$

so

$$h_n(x) := q_n(x) - T_n(x)$$

will have at least 3 zeros in (y_k, y_{k+1}). But h_n has at least one zero in each of (x_i, x_{i+1}). Hence $h_n \in H_n$ has at least $n + 2$ zeros in $[a, b]$, which is a contradiction.

□

We need the following technical result concerning Chebyshev polynomials.

Lemma 1. *Suppose* $\mathcal{M} := \{1, f_1, f_2, \dots\}$ *is an infinite Markov system of* C^2 *functions on* $[a, b]$ *and* f_1' *does not vanish on* (a, b). *Suppose that the associated Chebyshev polynomials* $\{T_n\}$ *has a subsequence* $\{T_{n_i}\}$ *with no zeros on some subinterval of* $[a, b]$. *Then there exists another subinterval* $[c, d]$ *and another infinite subsequence* $\{T_{n_i}\}$ *so that for some* $\delta > 0$, $\gamma > 0$

$$\|T_{n_i}\|_{[c,d]} < 1 - \delta$$

and

$$\|T'_{n_i}\|_{[c,d]} < \gamma$$

for all n_i.

Proof. For both inequalities we first choose a subinterval $[c_1, d_1] \subset [a, b]$ and a subsequence $\{n_{i,1}\}$ of $\{n_i\}$ so that all oscillations of each $T_{n_{i,1}}$ take place away from $[c_1, d_1]$. We now choose a subsequence $\{n_{i,2}\}$ of $\{n_{i,1}\}$ so that either each $T_{n_{i,2}}$ is increasing or each $T_{n_{i,2}}$ is decreasing on $[c_1, d_1]$. We treat the first case, the second one is analogous. Let $[c_2, d_2]$ be the middle third of $[c_1, d_1]$. If the first inequality fails to hold with $[c_2, d_2]$ and $\{n_{i,2}\}$ then there is a subsequence $\{n_{i,3}\}$ of $\{n_{i,2}\}$ so that $\|T_{n_{i,3}}\|_{[c_2,d_2]} \to 1$ as $n_{i,3} \to \infty$. Hence, there is a subsequence $\{n_{i,4}\}$ of $\{n_{i,3}\}$ so that either

$$\max_{c_2 \le x \le d_2} T_{n_{i,4}}(x) \to 1 \quad \text{or} \quad \min_{c_2 \le x \le d_2} T_{n_{i,4}}(x) \to -1.$$

Once again we treat the first case, the second one is analogous. Since each $T_{n_{i,3}}$ is increasing on $[c_1, d_1]$,

$$\lim_{n_{i,4} \to \infty} \|1 - T_{n_{i,4}}\|_{[d_2,d_1]} = 0.$$

Now take $g := a_0 + a_1 f_1 + a_2 f_2$ so that g has two distinct zeros α_1 and α_2 in $[d_2, d_1]$, $\|g\|_{[\alpha_1,\alpha_2]} < 1$ and g is positive on (α_1, α_2). Let $\beta := \max\limits_{\alpha_1 \le x \le \alpha_2} g(x)$ and $\tilde{g} := g + 1 - \beta$. One can now deduce that $T_{n_{i,4}} - \tilde{g}$ has at least $n + 1$ distinct zeros in $[a, b]$ if $n_{i,4}$ is large enough, which is a contradiction.

For the second inequality, by [8], $\{f'_1, f'_2, \dots\}$ is a weak Markov system on $[a, b]$, and so is

$$\left\{ (T'_2 / T'_1)', (T'_3 / T'_1)', \dots \right\}$$

on every closed subinterval of (a, b). (In the definitions of weak Markov systems and weak Chebyshev systems we only count zeros where the sign changes.) The assumption that f'_1 does not vanish on (a, b) implies that T'_1 does not vanish on (a, b).

¿From this we deduce that each $(T'_{n_{i,2}}/T'_1)'$ has at most one sign change in $[c_2, d_2]$. Choose a subinterval $[c_3, d_3] \subset [c_2, d_2]$ and a subsequence $\{n_{i,5}\}$ of $\{n_{i,2}\}$ so that none of $(T'_{n_{i,5}}/T'_1)'$ changes sign in $[c_3, d_3]$. Choose a subsequence $\{n_{i,6}\}$ of $\{n_{i,5}\}$ so that either each $T'_{n_{i,6}}/T'_1$ is increasing or each $T'_{n_{i,6}}/T'_1$ is decreasing on $[c_3, d_3]$. We only study the first case, the second one is similar. Let $[c_4, d_4]$ be the middle third of $[c_3, d_3]$. If the second inequality fails to hold with $[c_4, d_4]$ and $\{n_{i,6}\}$ then there is a subsequence $\{n_{i,7}\}$ so that either

$$\max_{c_4 \le x \le d_4} T'_{n_{i,7}}(x) \,/\, T'_1(x) \to \infty$$

or

$$\min_{c_4 \le x \le d_4} T'_{n_{i,7}}(x) \,/\, T'_1(x) \to -\infty.$$

Again we treat only the first case, the second one is analogous. Then for every $K > 0$ there is N so that for every $n_{i,7} \ge N$ we have

$$T'_{n_i,7}(x) > K, \qquad x \in [d_4, d_3],$$

hence

$$K(d_3 - d_4) \le \int_{d_4}^{d_3} T'_{n_i,7}(x)dx = T_{n_i,7}(d_3) - T_{n_i,7}(d_4) \le 2,$$

which is a contradiction. □

Lemma 2. *Suppose $\mathcal{M} := \{f_0, f_1, \dots\}$ is a $C^1[a,b]$ infinite Markov system and suppose $g \in C^1[a,b]$ and g is strictly positive on $[a,b]$. Then $\mathcal{N} = \{gf_0, gf_1, \dots\}$ is also a $C^1[a,b]$ infinite Markov system. Furthermore* span $\mathcal{M}$ *has a bounded Bernstein inequality on $[\alpha,\beta] \subset [a,b]$ if and only if* span $\mathcal{N}$ *also has bounded Bernstein inequality on $[\alpha,\beta]$.*

Proof. Consider differentiating gf with $f \in$ span $\mathcal{M}$ by the product rule. If span $\mathcal{M}$ has a bounded Bernstein inequality on $[\alpha,\beta]$ then

$$\begin{aligned}\|(gf)'\|_{[\alpha,\beta]} &\le \|g'f\|_{[\alpha,\beta]} + \|gf'\|_{[\alpha,\beta]} \\ &\le c_1\|gf\|_{[\alpha,\beta]} + c_2\|gf\|_{[a,b]}\end{aligned}$$

where the first constant arises since

$$g'(x)/g(x)$$

is uniformly bounded on $[a,b]$ and the second constant comes from the bounded Bernstein inequality for f. □

Proof of Theorem 1. The only if part of this theorem is obvious. A good uniform approximation to a function with uniformly large derivative on a subinterval $[\alpha,\beta] \subset [a,b]$ must have large derivative at some points in $[\alpha,\beta]$.

In the other direction we first note that by Lemma 2 we may assume $f_0 \equiv 1$. We use Theorem 2 and Lemma 1 in the following way. If span $\mathcal{M}$ is not dense then there exists a subinterval $[\alpha,\beta] \subset [a,b]$ by Theorem 2, where a subsequence of the associated Chebyshev polynomials have no zeros. By Lemma 1 from this subsequence we can pick another subsequence T_{n_i} and a subinterval $[c,d] \subset [\alpha,\beta]$ with

$$\|T_{n_i}\|_{[c,d]} < 1 - \delta$$

and

$$\|T'_{n_i}\|_{[c,d]} < \gamma$$

for some positive constants δ and γ. The result now follows from Theorem 3. □

Corollary 1. *Suppose $\mathcal{M} = \{f_0, f_1, \dots\}$ is an infinite Markov system of C^2 functions on $[a,b]$ so that* span $\mathcal{M}$ *fails to be dense in $C[a,b]$ in the uniform norm. Then there exists a subinterval $[\alpha,\beta]$ of $[a,b]$ so that if g is in the uniform closure of* span $\mathcal{M}$ *then g is differentiable on $[\alpha,\beta]$.*

Proof. By Theorem 1, there exists an interval $[\alpha,\beta]$ where $\|h'\|_{[\alpha,\beta]}/\|h\|_{[a,b]}$ is uniformly bounded for every $h \in$ span $\mathcal{M}$. Suppose $h_n \to g, h_n \in$ span $\mathcal{M}$. Then we can choose n_i so that

$$\|g - h_{n_i}\|_{[a,b]} \le \frac{1}{2^i} \qquad i = 0, 1, 2, \dots$$

and hence

$$g = \sum_{i=1}^{\infty}(h_{n_i} - h_{n_{i-1}}) + h_{n_0}.$$

Since

$$\|(h_{n_i} - h_{n_{i-1}})'\|_{[\alpha,\beta]} \le \frac{c}{2^i}$$

for some constant c independent of i, if follows that g is differentiable on $[\alpha, \beta]$. □

Suppose $\mathcal{M} = \{f_0, f_1, \dots\}$ is an extended complete Markov system of C^∞ functions on $[a, b]$ (the extra requirement being that the multiplicity of the zeros matters in the definition: so if $f := \sum_{i=0}^{n} a_i f_i$ has $n + 1$ zeros by counting multiplicities then $f = 0$ identically). In this case the differential operator D defined by

$$D(f) := \left(\frac{f}{f_0}\right)'$$

maps $\mathcal{M}$ to $\mathcal{M}_D$ where

$$\mathcal{M}_D = \left\{\left(\frac{f_1}{f_0}\right)', \left(\frac{f_2}{f_0}\right)', \dots\right\}$$

and $\mathcal{M}_D$ is once again an extended complete Markov system of C^∞ functions (see Nürnberger [5]). We define the differential operators $D^{(n)}(f)$ for n times differentiable functions f by

$$F_{i,0} := f_i, \qquad F_{i,n} := \left(\frac{F_{i+1,n-1}}{F_{0,n-1}}\right)', \qquad i = 0, 1, \dots, \quad n = 1, 2, \dots,$$

$$D^{(0)}(f) := f, \qquad D^{(n)}(f) := \left(\frac{D^{(n-1)}(f)}{F_{0,n-1}}\right)', \qquad n = 1, 2, \dots.$$

Note that if span $\mathcal{M}_D$ is dense in $C[a, b]$ in the uniform norm then so is span $\mathcal{M}$. The "if" part of the next theorem can be proved from Theorem 1 by induction on n, while the "only if" part is obvious.

Theorem 4. *Suppose $\mathcal{M} = \{f_0, f_1, \dots\}$ is an extended complete Markov system of C^∞ functions on $[a, b]$. Let n be a fixed positive integer. Then* span $\mathcal{M}$ *is dense in $C[a, b]$ in the uniform norm if and only if*

$$\sup\left\{\frac{\|D^{(n)}(f)\|_{[\alpha,\beta]}}{\|f\|_{[a,b]}} : f \in \text{span}\,\mathcal{M}, f \ne 0\right\} = \infty$$

for every $[\alpha, \beta] \subset [a, b]$, $\alpha \ne \beta$.

Corollary 2. *Suppose $\mathcal{M}$ is an extended complete Markov system of C^∞ functions on $[a,b]$ so that* span $\mathcal{M}$ *fails to be dense in $C[a,b]$ in the uniform norm. Then for each n there exists an interval $[\alpha_n,\beta_n]\subset[a,b]$ of positive length where all elements of the uniform closure of* span $\mathcal{M}$ *are n times continuously differentiable.*

Proof. Use Theorem 4 as in Corollary 1. We omit the technical details. □

Suppose that $\mathcal{M}$, as in Corollary 2, has the property that span $\mathcal{M}$ fails to be dense in the uniform norm on any proper subinterval of $[a,b]$, as in the case of Müntz systems

$$\mathcal{M}:=\{x^{\lambda_0},x^{\lambda_1},\dots\},\qquad 0\le\lambda_0<\lambda_1<\cdots,\qquad \sum_{i=1}^{\infty}\frac{1}{\lambda_i}<\infty,\qquad 0\le a<b.$$

Then the uniform closure of span $\mathcal{M}$ on $[a,b]$ contains only functions that are C^∞ on a dense subset of $[a,b]$. In this non-dense Müntz case the closure actually contains only analytic functions on (a,b) (Achiezer [1], Schwartz [7]).

We record one final corollary.

Corollary 3. *Suppose $\{\alpha_k\}\subset\mathbb{R}\setminus[-1,1]$ is a sequence of distinct numbers. Then*

$$\operatorname{span}\left\{1,\frac{1}{x-\alpha_1},\frac{1}{x-\alpha_2},\dots\right\}$$

is dense in $C[-1,1]$ if and only if

$$\sum_{k=1}^{\infty}\sqrt{\alpha_k^2-1}=\infty.$$

Proof. The inequality

$$|p'(x)|\le\frac{1}{\sqrt{1-x^2}}\sum_{k=1}^{n}\frac{\sqrt{\alpha_k^2-1}}{|\alpha_k-x|}\|p\|_{[-1,1]}$$

holds for any

$$p\in\operatorname{span}\left\{1,\frac{1}{x-\alpha_1},\dots,\frac{1}{x-\alpha_n}\right\}.$$

See [3]. This together with Theorem 1 gives the “only if” part of the corollary.

In [3] the Chebyshev “polynomials” T_n (of the first kind) and U_n (of the second kind) for the Chebyshev space

$$\operatorname{span}\left\{1,\frac{1}{x-\alpha_1},\dots,\frac{1}{x-\alpha_n}\right\}$$

are introduced. Properties of

$$\widetilde{T}_n(t):=T_n(\cos t)$$

and

$$\widetilde{U}_n(t) := U_n(\cos t)\sin t$$

established in [3] include

$$\|\widetilde{T}_n\|_{\mathbb{R}} = 1 \qquad \textit{and} \qquad \|\widetilde{U}_n\|_{\mathbb{R}} = 1, \tag{1}$$

$$\widetilde{T}_n(t)^2 + \widetilde{U}_n(t)^2 = 1, \qquad t \in \mathbb{R}, \tag{2}$$

$$\widetilde{T}_n'(t)^2 + \widetilde{U}_n'(t)^2 = \widetilde{B}_n(t)^2, \qquad t \in \mathbb{R}, \tag{3}$$

$$\widetilde{T}_n'(t) = -\widetilde{B}_n(t)\widetilde{U}_n(t), \qquad t \in \mathbb{R}, \tag{4}$$

$$\widetilde{U}_n'(t) = \ \widetilde{B}_n(t)\widetilde{T}_n(t), \qquad t \in \mathbb{R} \tag{5}$$

where

$$\widetilde{B}_n(t) = \sum_{k=1}^{n} \frac{\sqrt{\alpha_k^2 - 1}}{|\alpha_k - \cos t|}, \qquad t \in \mathbb{R}.$$

Suppose

$$\sum_{k=1}^{\infty} \sqrt{\alpha_k^2 - 1} = \infty.$$

Then

$$\lim_{n\to\infty} \min_{t\in[\alpha,\beta]} \widetilde{B}_n(t) = \infty, \qquad 0 < \alpha < \beta < \pi. \tag{6}$$

Assume that there is a subinterval $[a,b]$ of $(-1,1)$ so that

$$\sup_{n\in\mathbb{N}} \|T_n'\|_{[a,b]} < \infty.$$

Let $\alpha := \arccos b$ and $\beta := \arccos a$. Then by properties (4) and (6)

$$\lim_{n\to\infty} \|\widetilde{U}_n\|_{[\alpha,\beta]} = 0$$

hence by property (2)

$$\lim_{n\to\infty} \|\widetilde{T}_n^2 - 1\|_{[\alpha,\beta]} = 0.$$

Thus by properties (5) and (6)

$$\lim_{n\to\infty} \min_{t\in[\alpha,\beta]} |\widetilde{U}_n'(t)| = \infty$$

that is

$$\lim_{n\to\infty} |\widetilde{U}_n(\beta) - \widetilde{U}_n(\alpha)| = \infty$$

which contradicts property (1). Hence

$$\sup_{n\in\mathbb{N}} \frac{\|T_n'\|_{[a,b]}}{\|T_n\|_{[-1,1]}} = \sup_{n\in\mathbb{N}} \|T_n'\|_{[a,b]} = \infty.$$

for every subinterval $[a,b]$ of $(-1,1)$ which together with Theorem 1 finishes the "if" part of the proof. □

Corollary 3 is to be found in Achieser [1, p. 255] proven by entirely different methods.

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