ON THE DERIVATIVES OF UNIMODULAR POLYNOMIALS

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Dedicated to the memory of Andrei A. Gonchar and Herbert Stahl

Abstract. Let D be the open unit disk of the complex plane. Its boundary, the unit circle of the complex plane, is denoted by ∂D. Let $\mathcal{P}_n^c$ denote the set of all algebraic polynomials of degree at most n with complex coefficients. Associated with $\lambda \geq 0$ let

$$\mathcal{K}_n^\lambda := \left\{ p_n : p_n(z) = \sum_{k=0}^n a_k k^\lambda z^k, \ \ a_k \in \mathbb{C}\,, \ \ |a_k| = 1 \right\} \subset \mathcal{P}_n^c\,.$$

The class $\mathcal{K}_n^0$ is often called the collection of all (complex) unimodular polynomials of degree n. Given a sequence (ε_n) of positive numbers tending to 0, we say that a sequence (P_n) of polynomials $P_n \in \mathcal{K}_n^\lambda$ is (ε_n)-ultraflat if

$$(1-\varepsilon_n)\frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}} \leq |P_n(z)| \leq (1+\varepsilon_n)\frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}}\,, \qquad z \in \partial D\,, \quad n \in \mathbb{N}\,.$$

Although we do not know in general, whether or not ultraflat sequences (P_n) of polynomials $P_n \in \mathcal{K}_n^\lambda$ exists, we make an effort to prove various interesting properties of them. These allow us to conclude that there are no sequences (P_n) of conjugate, plain, or skew reciprocal unimodular polynomials $P_n \in \mathcal{K}_n^0$ such that (Q_n) with $Q_n(z) = zP_n'(z) + 1$ is an ultraflat sequence of polynomials $Q_n \in \mathcal{K}_n^1$.

1. Introduction

Let D be the open unit disk of the complex plane. Its boundary, the unit circle of the complex plane, is denoted by ∂D. Let $\mathcal{P}_n^c$ denote the set of all algebraic polynomials of degree at most n with complex coefficients. Associated with $\lambda \geq 0$ let

$$\mathcal{K}_n^\lambda := \left\{ p_n : p_n(z) = \sum_{k=0}^n a_k k^\lambda z^k, \ \ a_k \in \mathbb{C}\,, \ \ |a_k| = 1 \right\} \subset \mathcal{P}_n^c\,.$$

The class $\mathcal{K}_n^0$ is often called the collection of all (complex) unimodular polynomials of degree n. Given a sequence (ε_n) of positive numbers tending to 0, we say that a sequence

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(P_n) of polynomials $P_n \in \mathcal{K}_n^\lambda$ is (ε_n)-ultraflat if

$$(1-\varepsilon_n)\frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}} \le |P_n(z)| \le (1+\varepsilon_n)\frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}}\,, \qquad z \in \partial D\,, \quad n \in \mathbb{N}\,.$$

In 1957 the existence of an ultraflat sequence (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n^0$ seemed very unlikely in view of a conjecture of P. Erdős (Problem 22 in [Er1]) asserting that, for all $P_n \in \mathcal{K}_n^0$ with $n \ge 1$,

$$\max_{z\in\partial D} |P_n(z)| \ge (1+\varepsilon)n^{1/2},,$$

where $\varepsilon > 0$ is an absolute constant (independent of n). Yet, combining some probabilistic lemmas from Körner's paper [Kö] with some constructive methods (Gauss polynomials, etc.), which were completely unrelated to the deterministic part of Körner's paper, Kahane [Ka] proved that there exists a sequence (P_n) with $P_n \in \mathcal{K}_n^0$ which is (ε_n)-ultraflat, where $\varepsilon_n = O\left(n^{-1/17}\sqrt{\log n}\right)$. Thus the Erdős conjecture was disproved for the classes $\mathcal{K}_n^0$.

In this paper we study ultraflat sequences (P_n) of polynomials $P_n \in \mathcal{K}_n^\lambda$ in general, not necessarily those produced by Kahane in his paper [Ka]. We prove an extension of a conjecture of Saffari [Sa] (see also [QS2]).

2. The Phase Problem: Results and Conjectures of Saffari

Let (ε_n) be a sequence of positive numbers tending to 0. Suppose (P_n) is an ε_n-ultraflat sequence of polynomials $P_n \in \mathcal{K}_n^\lambda$. We write

$$P_n(e^{it}) = R_n(t)e^{i\alpha_n(t)}\,, \qquad R_n(t) = |P_n(e^{it})|\,.$$

It is a simple exercise to show that α_n can be chosen to be in $C^\infty(\mathbb{R})$. This is going to be our understanding throughout the paper. The Lebesgue measure of a measurable set $A \subset \mathbb{R}$ or $\{\cdot\}$ will be denoted by $m(A)$ or $m\{\cdot\}$, respectively.

Theorem 2.1 (Distribution Theorem for the Angular Speed). *Let (P_n) be an ε_n-ultraflat sequence of polynomials $P_n \in \mathcal{K}_n^\lambda$. Then*

$$m\left\{t \in [0,2\pi] : 0 \le \alpha_n'(t) \le nx\right\} = 2\pi x^{2\lambda+1} + o_n(x) \tag{2.2}$$

for every $x \in [0,1]$, where $\lim_{n\to\infty} o_n(x) = 0$. As a consequence

$$m\left\{t \in [0,2\pi] : |P_n'(e^{it})| \le \frac{n^{\lambda+3/2}x}{(2\lambda+1)^{1/2}}\right\} = 2\pi x^{2\lambda+1} + o_n(x) \tag{2.3}$$

for every $x \in [0,1]$, where $\lim_{n\to\infty} o_n(x) = 0$ for every $x \in [0,1]$.

In both statements the convergence of $o_n(x)$ is uniform on $[0,1]$ by Dini's Theorem.

When $\lambda = 0$, the basis of conjecturing Theorem 2.1 was that for the special ultraflat sequences of unimodular polynomials $P_n \in \mathcal{K}_n^0$ produced by Kahane [Ka], (2.2) is indeed true. In Section 4 we prove Theorem 2.1 for every $\lambda \ge 0$.

In the general case (2.2) can, by integration, be reformulated (equivalently) in terms of the moments of the angular speed $\alpha_n'(t)$. We will present the proof of this equivalence in Section 4 and will verify Theorem 2.1 by proving the following result first.

Theorem 2.2 (Reformulation of the Distribution Theorem). *Let (P_n) be a ε_n-ultraflat sequence of polynomials $P_n \in \mathcal{K}_n^\lambda$. Then, for any $q > 0$ we have*

$$\frac{1}{2\pi}\int_0^{2\pi} |\alpha_n'(t)|^q \, dt = \frac{(2\lambda+1)n^q}{q+2\lambda+1} + o_{n,q} n^q \,. \tag{2.4}$$

with suitable constants $o_{n,q}$ converging to 0 as $n \to \infty$ for every fixed $q > 0$.

An immediate consequence of (2.4) is the remarkable fact that for large values of $n \in \mathbb{N}$, the $L_q(\partial D)$ Bernstein factors

$$\frac{\int_0^{2\pi} |P_n'(e^{it})|^q \, dt}{\int_0^{2\pi} |P_n(e^{it})|^q \, dt}$$

of the elements of ultraflat sequences of polynomials $P_n \in \mathcal{K}_n^\lambda$ are essentially independent of the polynomials. More precisely (2.4) implies the following result.

Theorem 2.3 (The Bernstein Factors). *Let (P_n) be a ε_n-ultraflat sequence of polynomials $P_n \in \mathcal{K}_n^\lambda$. We have*

$$\frac{\int_0^{2\pi} |P_n'(e^{it})|^q \, dt}{\int_0^{2\pi} |P_n(e^{it})|^q \, dt} = \frac{(2\lambda+1)n^q}{q+2\lambda+1} + o_{n,q} n^q \,, \qquad q > 0 \,,$$

and as a limit case,

$$\frac{\max_{0\le t\le 2\pi} |P_n'(e^{it})|}{\max_{0\le t\le 2\pi} |P_n(e^{it})|} = n + o_n n \,.$$

with suitable constants $o_{n,q}$ and o_n converging to 0 as $n \to \infty$ for every fixed q.

In Section 3 we will show the following result which turns out to be stronger than Theorem 2.2.

Theorem 2.4 (Negligibility Theorem for Higher Derivatives). *Let (P_n) be a ε_n-ultraflat sequence of polynomials $P_n \in \mathcal{K}_n^\lambda$. For every integer $r \ge 2$, we have*

$$\max_{0\le t\le 2\pi} |\alpha_n^{(r)}(t)| \le o_{n,r} n^r$$

with suitable constants $o_{n,r} > 0$ converging to 0 as $n \to \infty$ for every fixed $r = 2, 3, \ldots$.

We will show in Section 4 how Theorem 2.1 follows from Theorem 2.4.

Finally we give an extension of Saffari's Uniform Distribution Conjecture to higher derivatives. This will be shown in Section 4 as well.

Theorem 2.5. *Suppose (P_n) be a ε_n-ultraflat sequence of polynomials $P_n \in \mathcal{K}_n^\lambda$. Then*

$$m\left\{ t \in [0, 2\pi] : |P_n^{(r)}(e^{it})| \le \frac{n^{r+\lambda+1/2} x^r}{(2\lambda+1)^{1/2}} \right\} = 2\pi x^{2\lambda+1} + o_{r,n}(x)$$

for every $x \in [0, 1]$, where $\lim_{n\to\infty} o_{r,n}(x) = 0$ for every fixed $r = 1, 2, \ldots$ and $x \in [0, 1]$.

For every fixed $r = 1, 2, \ldots$, the convergence of $o_{n,r}(x)$ is uniform on $[0, 1]$ by Dini's Theorem.

As a consequence of Theorems 2.1 and 2.4 we obtain

Theorem 2.6. *Let (P_n') be an ε_n-ultraflat sequence of polynomials $P_n' \in \mathcal{K}_n^{\lambda+1}$. Then*

$$\lim_{n\to\infty} n^{-(\lambda+1/2)} \max_{t\in\mathbb{R}} |P_n(e^{it})| = \infty\,.$$

Theorem 2.6 and Lemma 4.4 imply

Theorem 2.7. *Let (P_n) be a sequence of polynomials $P_n \in \mathcal{K}_n^0$ so that*

$$\max_{z\in\partial D} |P_n^{*\prime}(z)| = \max_{z\in\partial D} |P_n'(z)|\,, \qquad n = 1, 2, \ldots\,.$$

Then (Q_n) with $Q_n(z) := zP_n'(z) + 1$ is not an ultraflat sequence of polynomials $Q_n \in \mathcal{K}_n^1$.

Corollary 2.8. *There are no sequences (P_n) of conjugate, plain, or skew reciprocal unimodular polynomials $P_n \in \mathcal{K}_n^0$ such that (Q_n) with $Q_n(z) = zP_n'(z) + 1$ is an ultraflat sequence of polynomials $Q_n \in \mathcal{K}_n^1$.*

3. Proof of Theorem 2.4

To prove Theorem 2.4 we need a few lemmas. The first one is a standard polynomial inequality ascribed to Bernstein. Its proof is a simple exercise in complex analysis (an application of the Maximum Principle), and it may be found in a number of books. See [BE, p. 390], for example. We will use notation

$$D(z_0, R) := \{z \in \mathbb{C} : |z - z_0| < R\}\,, \qquad \partial D(z_0, R) := \{z \in \mathbb{C} : |z - z_0| = R\}\,,$$

and $\|f\|_A := \sup\{|f(z)| : z \in A\}$ for a complex-valued function f defined on a set A. As before, let $D := D(0, 1)$ and $\partial D := \partial D(0, 1)$.

Lemma 3.1. *We have*

$$|P(z)| \le |z|^n \|P\|_{\partial D}$$

for every polynomial P of degree at most n with complex coefficients, and for every $z \in \mathbb{C}$ with $|z| > 1$. As a corollary (consider $P(e^{it}) := e^{int}T(t)$), we have

$$|T(z)| \le e^{n\mathrm{Im}(z)} \|T\|_{\mathbb{R}}$$

for every trigonometric polynomial T of the form

$$T(t) = \sum_{k=-n}^{n} a_k e^{ikt}\,, a_k \in \mathbb{C}\,,$$

and for every $z \in \mathbb{C}$.

The main tool to prove Theorem 2.4 is the following well-known Jensen's Formula. For its proof, see, for example, E.10 c] of Section 4.2 in [BE].

Lemma 3.2 (Jensen's Formula). *Suppose h is a nonnegative integer and*

$$f(z) = \sum_{k=h}^{\infty} c_k z^k \,, \qquad c_h \neq 0 \,,$$

is analytic on a disk $D(0, R')$ with some $R' > R$. Suppose that the zeros of f in $D(0, R)\backslash\{0\}$ are $a_1, a_2, \ldots, a_m$, where each zero is listed as many times as its multiplicity. Then

$$\log|c_h| + h\log R + \sum_{k=1}^{m} \log\frac{R}{|a_k|} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta \,.$$

Lemma 3.3. *Suppose (ε_n) is a sequence of numbers from $(0, 1/3)$ tending to 0. Let (P_n) be an ε_n-ultraflat sequence of polynomials $P_n \in \mathcal{K}_n^\lambda$. Then P_n does not have a zero in the open annulus*

$$\left\{ z \in \mathbb{C} : 1 - \frac{1}{2n\delta_n} < |z| < 1 + \frac{1}{2n\delta_n} \right\} ,$$

where the positive numbers

$$\delta_n := \max\left\{ \frac{2}{-\log(3\varepsilon_n)}, \frac{1}{n} \right\}$$

tend to 0.

Proof of Lemma 3.3. Associated with a polynomial

$$Q_n(z) = \sum_{j=0}^{n} a_j z^j \,, \qquad a_j \in \mathbb{C} \,,$$

of degree n, we define the polynomial

$$Q_n^*(z) = z^n \overline{Q_n(1/\overline{z})} = \sum_{j=0}^{n} \overline{a}_{n-j} z^j \,. \tag{3.1}$$

of degree at most n. Let (P_n) be a ε_n-ultraflat sequence of polynomials $P_n \in \mathcal{K}_n^\lambda$, that is, $P_n \in \mathcal{K}_n^\lambda$ satisfies

$$(1 - \varepsilon_n)\frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}} < |P_n(z)| < (1 + \varepsilon_n)\frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}}$$

for every $z \in \mathbb{C}$ with $|z| = 1$. Then

$$(1 - \varepsilon_n)^2 \frac{n^{2\lambda+1}}{2\lambda+1} < z^{-n} P_n(z) P_n^*(z) = |P_n(z)|^2 < (1 + \varepsilon_n)^2 \frac{n^{2\lambda+1}}{2\lambda+1}$$

for every $z \in \partial D$. We define

$$Q_n(z) = P_n(z)P_n^*(z) - \frac{n^{2\lambda+1}}{2\lambda+1}\, z^n\,. \tag{3.2}$$

Then Q_n is a polynomial of degree $2n$ and

$$-3\varepsilon_n \frac{n^{2\lambda+1}}{2\lambda+1} < z^{-n}Q_n(z) = |P_n(z)|^2 - \frac{n^{2\lambda+1}}{2\lambda+1} < 3\varepsilon_n \frac{n^{2\lambda+1}}{2\lambda+1}$$

for every $z \in \partial D$. From this we conclude that

$$|Q_n(z)| < 3\varepsilon_n \frac{n^{2\lambda+1}}{2\lambda+1} \tag{3.3}$$

for every $z \in \partial D$. Using Lemma 3.1 and (3.3), we obtain that

$$|Q_n(z)| \le |z|^{2n} 3\varepsilon_n \frac{n^{2\lambda+1}}{2\lambda+1} < \frac{n^{2\lambda+1}}{2\lambda+1} \tag{3.4}$$

for every $z \in \mathbb{C}$ for which

$$1 \le |z| < 1 + \frac{1}{n\delta_n}\,,$$

where δ_n is defined in the lemma. Suppose that P_n has a zero in the annulus

$$\left\{ z \in \mathbb{C} : 1 - \frac{1}{2n\delta_n} < |z| < 1 + \frac{1}{2n\delta_n} \right\}.$$

Then $P_nP_n^*$ has a zero z_0 in the annulus

$$\left\{ z \in \mathbb{C} : 1 \le |z| < 1 + \frac{1}{n\delta_n} \right\}.$$

Hence by (3.2) we have

$$|Q_n(z_0)| = \left| P_n(z_0)P_n^*(z_0) - \frac{n^{2\lambda+1}}{2\lambda+1}\, z_0^n \right| = \frac{n^{2\lambda+1}}{2\lambda+1}\, |z_0|^n \ge \frac{n^{2\lambda+1}}{2\lambda+1}\,,$$

which is impossible by (3.4). □

Lemma 3.4. *Suppose (ε_n) is a sequence of numbers from $(0, 1/3)$ tending to 0. Let (P_n) be a ε_n-ultraflat sequence of polynomials $P_n \in \mathcal{P}_n^c$. Let $1/n \le R \le 2$. Let $z_0 \in \partial D$. Then P_n has at most $5nR$ zeros in the disk $D(z_0, R)$.*

Proof. We use Jensen's formula on the disk $D(z_0, 2R)$. Note that since (P_n) is a ε_n-ultraflat sequence of polynomials $P_n \in \mathcal{K}_n^\lambda$, we have

$$\begin{aligned} \log|P_n(z_0)| &\ge \log(1-\varepsilon_n) + (\lambda + 1/2)\log n - \frac{1}{2}\log(2\lambda+1) \\ &\ge -\frac{1}{2} + (\lambda+1/2)\log n - \frac{1}{2}\log(2\lambda+1)\,, \end{aligned}$$

while Bernstein inequality given by Lemma 3.1 yields

$$|P_n(z)| \le (1+\varepsilon_n)\frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}}(1+2R)^n\,, \qquad z \in \partial D(z_0, 2R)\,,$$

that is,

$$\log|P_n(z)| \le \frac{1}{3} + (\lambda+1/2)\log n - \frac{1}{2}\log(2\lambda+1) + n(2R)\,, \qquad z \in \partial D(z_0, 2R)\,.$$

Now if m denotes the number of zeros of P_n in $D(z_0, R)$, then by Jensen's formula

$$\begin{aligned} &-\frac{1}{2} + (\lambda+1/2)\log n - \frac{1}{2}\log(2\lambda+1) + m\log 2 \\ &\le \frac{1}{3} + (\lambda+1/2)\log n - \frac{1}{2}\log(2\lambda+1) + 2nR\,, \end{aligned}$$

whence

$$m \le \frac{3nR}{\log 2} \le 5nR\,,$$

and the lemma is proved. □

Our next lemma is a well-known inequality in approximation theory.

Lemma 3.5 (Bernstein's Inequality). *We have*

$$\|P'\|_{\partial D} \le n\|P\|_{\partial D}$$

for every $P \in \mathcal{P}_n^c$.

Now we are ready for the proof of Theorem 2.4.

Proof of Theorem 2.4. Observe that if $z_1, z_2, \dots z_n$ denote the zeros of P_n in the complex plane, then

$$\frac{zP_n'(z)}{P_n(z)} = \sum_{j=1}^n \frac{z}{z-z_j} = \sum_{j=1}^n \left(1 + \frac{z_j}{z-z_j}\right).$$

Since $P_n \in \mathcal{K}_n^\lambda$, we have

$$|z_1|, |z_2|, \dots, |z_n| < 2\,. \tag{3.7}$$

To see this, let

$$P_n(z) = \sum_{j=0}^n j^m a_j z^j, \qquad a_j \in \mathbb{C}\,, \quad |a_j| = 1\,.$$

Now if $z_0 \in \mathbb{C}$ and $|z_0| \ge 2$, then

$$\begin{aligned} \left|\sum_{j=0}^n j^\lambda a_j z_0^j\right| &\ge n^\lambda |z_0|^n - n^\lambda(|z_0|^{n-1} + |z_0|^{n-2} + \dots + |z_0|^1 + |z_0|^0) \\ &= n^\lambda\left(|z_0|^n - \frac{|z_0|^n - 1}{|z_0| - 1}\right) > 0\,. \end{aligned}$$

Using (3.5) and (3.7) and substituting $z_0 = e^{it_0}$, we can give the following upper bound for $|\alpha_n^{(r)}(t_0)|$ (the constants A_m below depend only on m).

(3.8)

$$\begin{aligned}
|\alpha_n^{(r)}(t_0)| &= \left| \frac{d^{r-1}}{dt^{r-1}} \left(\mathrm{Re} \left(\frac{e^{it} P_n'(e^{it})}{P_n(e^{it})} \right) \right) (t_0) \right| \le \left| \frac{d^{r-1}}{dt^{r-1}} \left(\frac{e^{it} P_n'(e^{it})}{P_n(e^{it})} \right) (t_0) \right| \\
&= \left| \sum_{m=0}^{r-1} A_m \frac{d^m}{dz^m} \left(\frac{z P_n'(z)}{P_n(z)} \right) (z_0)\, e^{imt_0} \right| \\
&= \left| \sum_{m=0}^{r} A_m \frac{d^m}{dz^m} \left(\sum_{k=1}^{n} \left(1 + \frac{z_k}{z - z_k} \right) \right) (z_0)\, e^{imt_0} \right| \\
&\le \left| A_0 \frac{z_0 P_n'(z_0)}{P_n(z_0)} \right| + \sum_{m=1}^{r-1} |A_m| m! \sum_{k=1}^{n} |z_k| |z_0 - z_k|^{-(m+1)} \\
&\le \left| A_0 \frac{z_0 P_n'(z_0)}{P_n(z_0)} \right| + 2 \sum_{m=1}^{r-1} |A_m| m! \sum_{k=1}^{n} |z_0 - z_k|^{-(m+1)} .
\end{aligned}$$

Now we define the annulus

$$E_\mu = D(z_0, 2^\mu (2n\delta_n)^{-1}) \setminus D(z_0, 2^{\mu-1}(2n\delta_n)^{-1}) , \qquad \mu = 1, 2, \dots ,$$

where $\delta_n := \max\{2/(-\log(3\varepsilon_n)), 1/n\}$ as in Lemma 3.3. We denote the number of zeros of P_n in E_μ by m_μ. By Lemma 3.4 we have $m_\mu \le 5n2^\mu/(2n\delta_n)$. Combining this with (3.8) and Lemmas 3.5 and 3.3, we obtain

$$\begin{aligned}
|\alpha_n^{(r)}(t)| &\le C_0 \frac{n(1+\varepsilon_n)(2\lambda+1)^{-1/2} n^{\lambda+1/2}}{(1-\varepsilon_n)(2\lambda+1)^{-1/2} n^{\lambda+1/2}} + C_r \sum_{m=1}^{r-1} \sum_{k=1}^{n} |z_0 - z_k|^{-(m+1)} \\
&\le 2C_0 n + C_r \sum_{m=1}^{r-1} \sum_{\mu=1}^{\infty} m_\mu \left(\frac{2^{\mu-1}}{2n\delta_n} \right)^{-(m+1)} \\
&\le 2C_0 n + C_r \sum_{m=1}^{r-1} \sum_{\mu=1}^{\infty} \frac{5n2^\mu}{2n\delta_n} \left(\frac{2^{\mu-1}}{2n\delta_n} \right)^{-(m+1)} \\
&\le 2C_0 n + C_r \sum_{m=1}^{r-1} \sum_{\mu=1}^{\infty} 2 \cdot 2^{-(\mu-1)m} 5n (2n\delta_n)^m \\
&\le 2C_0 n + C_r' n^r \delta_n \le C_r'' n^r \delta_n ,
\end{aligned}$$

where C_r, C_r', and C_r'' are positive constants depending only on r. Since

$$\delta_n := \max\{2/(-\log(3\varepsilon_n)), 1/n\}$$

tends to 0 together with $\varepsilon_n > 0$, the theorem is proved. □

4. Proof of Theorems 2.1, 2.2, and Theorem 2.3

First we prove Theorem 2.2 for positive integers q. To this end we need the lemmas below. The first one is a classical polynomial inequality of Bernstein available in many books. See [BE, Corollary 5.1.5], for example.

Lemma 4.1 (Bernstein's Inequality for Trigonometric Polynomials). *We have*

$$\max_{0\le t\le 2\pi} |T^{(m)}(t)| \le n^m \max_{0\le t\le 2\pi} |T(t)|\,, \qquad m=1,2,\dots\,,$$

for every trigonometric polynomial T of degree at most n with complex coefficients.

Lemma 4.2. *Suppose (P_n) is an ε_n-ultraflat sequence of polynomials $P_n\in\mathcal{K}_n^\lambda$. Using the notation (2.1) we have*

$$o_n \le \alpha_n'(t) \le n+o_n\,, \qquad t\in\mathbb{R}\,,$$

with some real numbers o_n tending to 0.

Lemma 4.3. *Suppose (ε_n) is a sequence of numbers from $(0,1/3)$ tending to 0. Suppose (P_n) is a (ε_n)-ultraflat sequence of polynomials $P_n\in\mathcal{K}_n^\lambda$. Using notation (2.1), we have*

$$\max_{0\le t\le 2\pi} |R_n^{(m)}(t)| = o_{n,m} n^{m+\lambda+1/2}\,, \qquad m=1,2,\dots\,, \tag{4.1}$$

with suitable constants $o_{n,m}$ converging to 0 as $n\to\infty$ for every $m=1,2,\dots$.

Proof of Lemma 4.2. It is easy to find a formula for $\alpha_n(t)$ in terms of P_n. One can easily verify formula (8.2) from Saffari's paper [Sa], which asserts that

$$\alpha_n'(t) = \mathrm{Re}\left(\frac{e^{it}P_n'(e^{it})}{P_n(e^{it})}\right)\,. \tag{3.5}$$

Combining this with Lemma 3.5 (Bernstein's inequality) and the ultraflatness of the sequence (P_n), we obtain the upper bound of the lemma. Associated with the (ε_n)-ultraflat sequence (P_n) of polynomials $P_n\in\mathcal{K}_n^\lambda$, we study the (ε_n)-ultraflat sequence (P_n^*) of the corresponding conjugate polynomials P_n^* (see (3.1) for the definition). The associated angular function α_n^* and modulus function R_n^* are defined by

$$P_n^*(e^{it}) = R_n^*(t)e^{i\alpha_n^*(t)}\,, \qquad R_n^*(t)=|P_n^*(e^{it})|\,. \tag{3.6}$$

It is a simple exercise to show that α_n^* can be chosen to be in $C^\infty(\mathbb{R})$ on $\mathbb{R}$. By applying formula (3.5) to P_n^*, it is easy to see that

$$\alpha_n(t)+\alpha_n^*(t)=n\,, \qquad t\in\mathbb{R}\,.$$

Since the upper bound of the lemma is valid for α_n^* as well, the lower bound of the lemma follows from this. $\square$

Proof of Lemma 4.3. The proof is very similar to that of Lemma 3.2. Let

$$\delta_n := \max\{2/(-\log(3\varepsilon_n)), 1/n\}\,,$$

as in the proof Lemma 3.3. Let (P_n) be a (ε_n)-ultraflat sequence of polynomials $P_n \in \mathcal{K}_n^\lambda$, that is, $P_n \in \mathcal{K}_n^\lambda$ satisfies

$$(1-\varepsilon_n)\frac{n^{\lambda+1/2}}{(2\lambda+1)^{1/2}} < |P_n(z)| < (1+\varepsilon_n)\frac{n^{\lambda+1/2}}{(2\lambda+1)^{1/2}}\,, \qquad z \in \partial D\,. \tag{4.2}$$

(In fact, in this proof we will not use that $P_n \in \mathcal{K}_n^\lambda$, we will use only that P_n is a polynomial of degree n with complex coefficients that satisfies (4.2).) To denote the conjugate polynomial of a polynomial $Q_n \in \mathcal{P}_n^c \setminus \mathcal{P}_{n-1}^c$, we use the notation Q_n^* introduced by (3.1).
Step 1. By Lemma 3.3,

$$T_n(t) := e^{-int}P_n(e^{it})P_n^*(e^{it}) \tag{4.3}$$

has no zeros in the strip

$$\mathcal{E}_n := \left\{ z \in \mathbb{C} : |\mathrm{Im}(z)| \le \frac{1}{4n\delta_n} \right\}\,. \tag{4.4}$$

Therefore

$$\widetilde{T}_n(t) := \sqrt{T_n(t)} = \sqrt{e^{-int}P_n(e^{it})P_n^*(e^{it})}$$

is a well-defined analytic function in in the strip $\mathcal{E}_n$.
Step 2. We show that

$$|\widetilde{T}_n'(t)| \le o_n n^{\lambda+3/2}\,, \qquad t \in \mathbb{R}\,,$$

with suitable constants o_n converging to 0. Indeed, T_n is a trigonometric polynomial of degree n (with complex coefficients). Note that (4.2) implies that

$$-3\varepsilon_n\frac{n^{2\lambda+1}}{2\lambda+1} < T_n(t) - \frac{n^{2\lambda+1}}{2\lambda+1} < 3\varepsilon_n\frac{n^{2\lambda+1}}{2\lambda+1}\,. \tag{4.5}$$

Combining this with Lemma 4.1 (Bernstein's inequality for trigonometric polynomials), we obtain

$$\begin{aligned} \max_{0\le t\le 2\pi}|T_n'(t)| &= \max_{0\le t\le 2\pi}\left|\frac{d}{dt}\left(T_n(t) - \frac{n^{2\lambda+1}}{2\lambda+1}\right)\right| \\ &\le n\max_{0\le t\le 2\pi}\left|T_n(t) - \frac{n^{2\lambda+1}}{2\lambda+1}\right| \le n3\varepsilon_n\,\frac{n^{2\lambda+1}}{2\lambda+1} \\ &\le \frac{3\varepsilon_n}{2\lambda+1}\,n^{2\lambda+2}\,. \end{aligned} \tag{4.6}$$

Now

(4.7)
$$|\widetilde{T}_n'(t)| = \left|\frac{T_n'(t)}{2\sqrt{T_n(t)}}\right| \le \frac{3\varepsilon_n}{2\lambda+1}\,\frac{n^{2\lambda+2}}{2(1-\varepsilon_n)(2\lambda+1)^{-1/2}n^{\lambda+1/2}} \le \frac{3\varepsilon_n}{(1-\varepsilon_n)}n^{3/2}$$
$$\le \frac{3}{2}\,\varepsilon_n n^{\lambda+3/2} = o_n n^{\lambda+3/2}\,, \qquad t\in\mathbb{R}\,,$$

with suitable constants o_n converging to 0.
Step 3. Let
$$\mathcal{F}_{c,n} := \left\{z\in\mathbb{C} : |\mathrm{Im}(z)| \le \frac{c}{n}\right\}\,.$$
We show that there is a sufficiently small absolute constant $c>0$ such that

$$|\widetilde{T}_n'(t)| \le o_n n^{\lambda+3/2}\,, \qquad t\in\mathcal{F}_{c,n}\,, \tag{4.8}$$

with suitable constants o_n converging to 0 as $n\to\infty$. To see this, first note that

$$|\widetilde{T}_n'(t)| = \left|\frac{T_n'(t)}{2\sqrt{T_n(t)}}\right|\,, \tag{4.9}$$

where T_n is defined by (4.3). Using (4.6) and Lemma 3.1 we obtain that

$$|T_n'(t)| \le o_n' n^{2\lambda+2}e^{n(c/n)} = o_n n^{2\lambda+2}\,, \qquad t\in\mathcal{F}_{c,n} \tag{4.10}$$

with suitable constants o_n' and o_n converging to 0 as $n\to\infty$ and with a sufficiently small absolute constant $c>0$. Similarly, (4.5), $\varepsilon_n\in(0,1/6)$, and Lemma 3.1 give

$$|T_n(t)| \ge \frac{n^{2\lambda+1}}{4(2\lambda+1)}\,, \qquad t\in\mathcal{F}_{c,n}\,, \tag{4.11}$$

for a sufficiently small absolute constant $c>0$, Now (4.9) – (4.11) imply that

$$|\widetilde{T}_n'(t)| \le o_n n^{\lambda+3/2}\,, \qquad t\in\mathcal{F}_{c,n}\,, \tag{4.12}$$

with suitable constants o_n converging to 0 as $n\to\infty$ and with a sufficiently small absolute constant $c>0$.
Step 4. From Step 3 we conclude by the Cauchy Integral Formula that

$$|\widetilde{T}_n^{(m)}(t)| = (m-1)!\left|\int_{\partial D(t,c/n)}\frac{T_n'(\xi)\,d\xi}{(\xi-t)^m}\right|$$
$$\le \frac{2\pi c}{n}\,(m-1)!o_{n,1}n^{\lambda+3/2}\left(\frac{c}{n}\right)^{-m} = o_{n,m}n^{m+\lambda+1/2}\,,$$

with suitable constants $o_{n,m}$ converging to 0 as $n\to\infty$ for every fixed $m=1,2,\ldots$.

Step 5. Note that for $t \in \mathbb{R}$ we have

$$R_n(t) = |P_n(e^{it})| = \sqrt{e^{-int} P_n(e^{it}) P_n^*(e^{it})} = \widetilde{T}_n(t)\,, \tag{4.13}$$

hence by Step 4

$$\max_{0 \le t \le 1} |R_n^{(m)}(t)| = o_{n,m} n^{m+\lambda+1/2}$$

with suitable constants $o_{n,m}$ converging to 0 as $n \to \infty$ for every fixed $m = 1, 2, \ldots$. This proves the lemma. $\square$

Now we are ready to prove Theorem 2.2 for positive integers q.

Proof of Theorem 2.2 for integers $q \ge 0$. Let (P_n) be a (ε_n)-ultraflat sequence of polynomials $P_n \in \mathcal{K}_n^\lambda$. We define

$$S_n(t) := P_n(e^{it}) = \sum_{k=0}^{n} k^\lambda a_{k,n} e^{ikt}\,, \qquad |a_{k,n}| = 1\,. \tag{4.14}$$

We calculate

$$\frac{1}{2\pi} \int_0^{2\pi} S_n^{(q)}(t) \overline{S_n(t)}\, dt$$

in two different ways. On one hand, using orthogonality, we have

$$\frac{1}{2\pi} \int_0^{2\pi} S_n^{(q)}(t) \overline{S_n(t)}\, dt = i^q \sum_{k=0}^{n} k^{q+2\lambda} |a_{k,n}|^2 = i^q \frac{n^{q+2\lambda+1}}{q+2\lambda+1} + o_{n,q} n^{q+\lambda+1}\,, \tag{4.15}$$

with suitable constants $o_{n,q}$ converging to 0 as $n \to \infty$ for every fixed $q = 0, 1, \ldots$.

On the other hand, with our standard notation introduced by (2.1), Theorem 2.4 and Lemmas 4.2 and 4.3 give

(4.16)

$$\begin{aligned}
S_n^{(q)}(t) &= \sum_{k=0}^{q} \binom{q}{k} \frac{d^k}{dt^k} \left(e^{i\alpha_n(t)}\right) R_n^{(q-k)}(t) \\
&= \frac{d^q}{dt^q} \left(e^{i\alpha_n(t)}\right) R_n(t) + \sum_{k=0}^{q-1} \binom{q}{k} \frac{d^k}{dt^k} \left(e^{i\alpha_n(t)}\right) R_n^{(q-k)}(t) \\
&= \frac{d^q}{dt^q} \left(e^{i\alpha_n(t)}\right) R_n(t) + \sum_{k=0}^{q-1} \binom{q}{k} c_{n,k}(t) n^k o_{n,q-k}(t) n^{q-k+\lambda+1/2} \\
&= \left(e^{i\alpha_n(t)} \alpha_n'(t)^q i^q + o_{n,q}'(t) n^q\right) R_n(t) + o_{n,q}''(t) n^{q+\lambda+1/2}
\end{aligned}$$

with suitable numbers $o_{n,q-k}(t)$, $c_{n,k}(t)$, $o_{n,q}'(t)$, and $o_{n,q}''(t)$, where

$$\max_{0 \le t \le 2\pi} |o_{n,q-k}(t)|$$

converge to 0 for every fixed q and $k = 0, 1, \dots, q-1$,

$$\max_{0 \le t \le 2\pi} |c_{n,k}(t)|$$

is bounded by a constant independent of n for every fixed $k = 0, 1, \dots, q-1$, and

$$\max_{0 \le t \le 2\pi} |o'_{n,q}(t)| \qquad \text{and} \qquad \max_{0 \le t \le 2\pi} |o''_{n,q}(t)|$$

converge to 0 as $n \to \infty$ for every fixed q. Now (4.13), (4.14), and (4.16) yield

(4.17)

$$\begin{aligned}
&\frac{1}{2\pi} \int_0^{2\pi} S_n^{(q)}(t) \overline{S_n(t)}\, dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left(\left(e^{i\alpha_n(t)} \alpha_n'(t)^q i^q + o'_{n,q}(t) n^q \right) R_n(t) + o''_{n,q}(t) n^{q+\lambda+1/2} \right) R_n(t) e^{-i\alpha_n(t)}\, dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{n^{2\lambda+1}}{2\lambda+1} (1 - o_n(t)) \left(\alpha_n'(t)^q i^q + o'_{n,q}(t) n^q \right) + o'''_{n,q}(t) n^{q+\lambda+1/2+\lambda+1/2} \right) dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} i^q \left(\frac{n^{2\lambda+1}}{2\lambda+1} (1 - o_n(t)) \right) \alpha_n'(t)^q\, dt + o^*_{n,q} n^{q+2\lambda+1}\,,
\end{aligned}$$

with suitable numbers $o_n(t)$, $o'_{n,q}(t)$, $o''_{n,q}(t)$, $o'''_{n,q}(t)$, and $o^*_{n,q}$, where

$$\max_{0 \le t \le 2\pi} |o_n(t)|\,, \qquad \max_{0 \le t \le 2\pi} |o'_{n,q}(t)|\,, \qquad \max_{0 \le t \le 2\pi} |o''_{n,q}(t)|\,, \qquad \max_{0 \le t \le 2\pi} |o'''_{n,q}(t)|\,,$$

and $o^*_{n,q}$ converge to 0 as $n \to \infty$ for every fixed q.

Now (4.15) and (4.17) give the statement of the theorem for integers $q \ge 0$. □

Proof of Theorem 2.1. We introduce the normalized distribution functions

$$F_n(x) := m\{t \in [0, 2\pi] : 0 \le \alpha_n'(t) \le nx\}\,, \qquad x \in [0, 1]\,. \tag{4.18}$$

Each F_n is continuous and nondecreasing on $[0, 1]$, and

$$0 \le F_n(x) \le 2\pi\,, \qquad x \in [0, 1]\,.$$

Suppose that the conjecture is not true. Then we can find a subsequence (F_{n_k}) of the sequence (F_n), and numbers $y \in [0, 1]$ and $\varepsilon > 0$ such that

$$|F_{n_k}(y) - 2\pi y^{2\lambda+1}| \ge \varepsilon\,, \qquad k = 1, 2, \dots\,. \tag{4.19}$$

Then by Helly's Selection Theorem, there is a subsequence (m_k) of (n_k) such that

$$\lim_{k \to \infty} F_{m_k}(x) = F(x) \tag{4.20}$$

exists for every $x \in [0,1]$. Then Theorem 2.2, (4.18), Lemma 4.2, and the Lebesgue Dominated Convergence Theorem imply that

$$\int_0^1 x^q \, dF(x) = \frac{2\pi(2\lambda+1)}{q+2\lambda+1}\,, \qquad q = 1, 2, \dots .$$

That is, all the corresponding moments of the measures $dF(x)$ and $dG(x)$ with $G(x) := 2\pi x^{2\lambda+1}$ are the same on $[0,1]$. Therefore, using the uniqueness part of the Riesz Representation Theorem describing all continuous linear functionals on $C[0,1]$, we obtain that $F(x) = 2\pi x^{2\lambda+1}$ for all $x \in [0,1]$. However, this contradicts (4.19) and (4.20). So

$$m\{t \in [0,2\pi] : 0 \le \alpha_n'(t) \le nx\} = 2\pi x^{2\lambda+1} + o_n(x)$$

for every $x \in [0,1]$, where $\lim_{n\to\infty} o_n(x) = 0$ for every $x \in [0,1]$.

To see the second statement of the theorem, we argue as follows. Using notation (2.1) and Lemma 4.2 we have $R_n'(t) = o_n(t) n^{\lambda+3/2}$ with a constant $o_n(t)$ tending to 0 as $n \to \infty$ for every $t \in \mathbb{R}$. Therefore

$$\begin{aligned} |P_n'(e^{it})| &= |R_n'(t) e^{i\alpha_n(t)} + i\alpha_n'(t) e^{i\alpha_n(t)} R_n(t)| \\ &= o_n(t) n^{\lambda+3/2} + |\alpha_n'(t)|(1+\varepsilon_n(t)) \frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}}\,, \end{aligned} \tag{4.21}$$

where $o_n(t)$ and $\varepsilon_n(t)$ tend to 0 as $n \to \infty$ for every $t \in \mathbb{R}$. Now the result follows from the first part of the theorem. □

Proof of Theorem 2.2 for all real $q > 0$. This follows from the already proved Theorem 2.1 in a routine fashion. □

Proof of Theorem 2.3. This follows immediately from Theorem 2.2 and (4.21). □

Proof of Theorem 2.5. To see the second part of the theorem, we write, as in (2.1),

$$P_n(e^{it}) = R_n(t) e^{i\alpha_n(t)}\,,$$

where, as before, $R_n(t) = |P_n(e^{it})|$. Then

$$P_n^{(r)}(z) = \sum_{k=0}^{r} \binom{r}{k} R_n^{(k)}(t) \frac{d^{(r-k)}}{dt^{r-k}} \left(e^{i\alpha_n(t)}\right)$$

Now the theorem follows from (2.1), Theorem 2.4, Lemma 4.2, and Theorem 2.1. □

To prove Theorem 2.7 we need the lemma below. This is stated as Theorem 3.1.27 and proved on page 689 of [MMR].

Lemma 4.4. *We have*

$$\max_{z\in\partial D} (|P'(z)| + |P^{*\prime}(z)|) = n \max_{z\in\partial D} |P(z)|$$

for every $P \in \mathcal{P}_n \setminus \mathcal{P}_{n-1}$.

The proof of Theorem 2.6 is a simple consequence of Theorems 2.1 and 2.4, while Theorem 2.7 follows easily from Lemma 4.4. These proofs are left to the reader.

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