

ON THE DERIVATIVES OF UNIMODULAR POLYNOMIALS

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ABSTRACT. Let D be the open unit disk of the complex plane. Its boundary, the unit circle of the complex plane, is denoted by ∂D . Let \mathcal{P}_n^c denote the set of all algebraic polynomials of degree at most n with complex coefficients. Associated with $\lambda \geq 0$ let

$$\mathcal{K}_n^\lambda := \left\{ p_n : p_n(z) = \sum_{k=0}^n a_k k^\lambda z^k, \quad a_k \in \mathbb{C}, \quad |a_k| = 1 \right\} \subset \mathcal{P}_n^c.$$

The class \mathcal{K}_n^0 is often called the collection of all (complex) unimodular polynomials of degree n . Given a sequence (ε_n) of positive numbers tending to 0, we say that a sequence (P_n) of polynomials $P_n \in \mathcal{K}_n^\lambda$ is (ε_n) -ultraflat if

$$(1 - \varepsilon_n) \frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}} \leq |P_n(z)| \leq (1 + \varepsilon_n) \frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}}, \quad z \in \partial D, \quad n \in \mathbb{N}.$$

Although we do not know in general, whether or not ultraflat sequences (P_n) of polynomials $P_n \in \mathcal{K}_n^\lambda$ exists, we make an effort to prove various interesting properties of them. These allow us to conclude that there are no sequences (P_n) of conjugate, plain, or skew reciprocal unimodular polynomials $P_n \in \mathcal{K}_n^0$ such that (Q_n) with $Q_n(z) = zP_n'(z) + 1$ is an ultraflat sequence of polynomials $Q_n \in \mathcal{K}_n^1$.

1. INTRODUCTION

Let D be the open unit disk of the complex plane. Its boundary, the unit circle of the complex plane, is denoted by ∂D . Let \mathcal{P}_n^c denote the set of all algebraic polynomials of degree at most n with complex coefficients. Associated with $\lambda \geq 0$ let

$$\mathcal{K}_n^\lambda := \left\{ p_n : p_n(z) = \sum_{k=0}^n a_k k^\lambda z^k, \quad a_k \in \mathbb{C}, \quad |a_k| = 1 \right\} \subset \mathcal{P}_n^c.$$

The class \mathcal{K}_n^0 is often called the collection of all (complex) unimodular polynomials of degree n . Given a sequence (ε_n) of positive numbers tending to 0, we say that a sequence (P_n) of polynomials $P_n \in \mathcal{K}_n^\lambda$ is (ε_n) -ultraflat if

$$(1 - \varepsilon_n) \frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}} \leq |P_n(z)| \leq (1 + \varepsilon_n) \frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}}, \quad z \in \partial D, \quad n \in \mathbb{N}.$$

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In 1957 the existence of an ultraflat sequence (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n^0$ seemed very unlikely in view of a conjecture of P. Erdős (Problem 22 in [Er1]) asserting that, for all $P_n \in \mathcal{K}_n^0$ with $n \geq 1$,

$$\max_{z \in \partial D} |P_n(z)| \geq (1 + \varepsilon)n^{1/2},$$

where $\varepsilon > 0$ is an absolute constant (independent of n). Yet, combining some probabilistic lemmas from Körner's paper [Kö] with some constructive methods (Gauss polynomials, etc.), which were completely unrelated to the deterministic part of Körner's paper, Kahane [Ka] proved that there exists a sequence (P_n) with $P_n \in \mathcal{K}_n^0$ which is (ε_n) -ultraflat, where $\varepsilon_n = O(n^{-1/17} \sqrt{\log n})$. Thus the Erdős conjecture was disproved for the classes \mathcal{K}_n^0 .

In this paper we study ultraflat sequences (P_n) of polynomials $P_n \in \mathcal{K}_n^\lambda$ in general, not necessarily those produced by Kahane in his paper [Ka]. We prove an extension of a conjecture of Saffari [Sa] (see also [QS2]).

2. THE PHASE PROBLEM: RESULTS AND CONJECTURES OF SAFFARI

Let (ε_n) be a sequence of positive numbers tending to 0. Suppose (P_n) is an ε_n -ultraflat sequence of polynomials $P_n \in \mathcal{K}_n^\lambda$. We write

$$P_n(e^{it}) = R_n(t)e^{i\alpha_n(t)}, \quad R_n(t) = |P_n(e^{it})|.$$

It is a simple exercise to show that α_n can be chosen to be in $C^\infty(\mathbb{R})$. This is going to be our understanding throughout the paper. The Lebesgue measure of a measurable set $A \subset \mathbb{R}$ or $\{\cdot\}$ will be denoted by $m(A)$ or $m\{\cdot\}$, respectively.

Theorem 2.1 (Distribution Theorem for the Angular Speed). *Let (P_n) be an ε_n -ultraflat sequence of polynomials $P_n \in \mathcal{K}_n^\lambda$. Then*

$$(2.2) \quad m \{t \in [0, 2\pi] : 0 \leq \alpha'_n(t) \leq nx\} = 2\pi x^{2\lambda+1} + o_n(x)$$

for every $x \in [0, 1]$, where $\lim_{n \rightarrow \infty} o_n(x) = 0$. As a consequence

$$(2.3) \quad m \left\{ t \in [0, 2\pi] : |P'_n(e^{it})| \leq \frac{n^{\lambda+3/2}x}{(2\lambda+1)^{1/2}} \right\} = 2\pi x^{2\lambda+1} + o_n(x)$$

for every $x \in [0, 1]$, where $\lim_{n \rightarrow \infty} o_n(x) = 0$ for every $x \in [0, 1]$.

In both statements the convergence of $o_n(x)$ is uniform on $[0, 1]$ by Dini's Theorem.

When $\lambda = 0$, the basis of conjecturing Theorem 2.1 was that for the special ultraflat sequences of unimodular polynomials $P_n \in \mathcal{K}_n^0$ produced by Kahane [Ka], (2.2) is indeed true. In Section 4 we prove Theorem 2.1 for every $\lambda \geq 0$.

In the general case (2.2) can, by integration, be reformulated (equivalently) in terms of the moments of the angular speed $\alpha'_n(t)$. We will present the proof of this equivalence in Section 4 and will verify Theorem 2.1 by proving the following result first.

Theorem 2.2 (Reformulation of the Distribution Theorem). *Let (P_n) be a ε_n -ultraflat sequence of polynomials $P_n \in \mathcal{K}_n^\lambda$. Then, for any $q > 0$ we have*

$$(2.4) \quad \frac{1}{2\pi} \int_0^{2\pi} |\alpha'_n(t)|^q dt = \frac{(2\lambda + 1)n^q}{q + 2\lambda + 1} + o_{n,q}n^q.$$

with suitable constants $o_{n,q}$ converging to 0 as $n \rightarrow \infty$ for every fixed $q > 0$.

An immediate consequence of (2.4) is the remarkable fact that for large values of $n \in \mathbb{N}$, the $L_q(\partial D)$ Bernstein factors

$$\frac{\int_0^{2\pi} |P'_n(e^{it})|^q dt}{\int_0^{2\pi} |P_n(e^{it})|^q dt}$$

of the elements of ultraflat sequences of polynomials $P_n \in \mathcal{K}_n^\lambda$ are essentially independent of the polynomials. More precisely (2.4) implies the following result.

Theorem 2.3 (The Bernstein Factors). *Let (P_n) be a ε_n -ultraflat sequence of polynomials $P_n \in \mathcal{K}_n^\lambda$. We have*

$$\frac{\int_0^{2\pi} |P'_n(e^{it})|^q dt}{\int_0^{2\pi} |P_n(e^{it})|^q dt} = \frac{(2\lambda + 1)n^q}{q + 2\lambda + 1} + o_{n,q}n^q, \quad q > 0,$$

and as a limit case,

$$\frac{\max_{0 \leq t \leq 2\pi} |P'_n(e^{it})|}{\max_{0 \leq t \leq 2\pi} |P_n(e^{it})|} = n + o_n n.$$

with suitable constants $o_{n,q}$ and o_n converging to 0 as $n \rightarrow \infty$ for every fixed q .

In Section 3 we will show the following result which turns out to be stronger than Theorem 2.2.

Theorem 2.4 (Negligibility Theorem for Higher Derivatives). *Let (P_n) be a ε_n -ultraflat sequence of polynomials $P_n \in \mathcal{K}_n^\lambda$. For every integer $r \geq 2$, we have*

$$\max_{0 \leq t \leq 2\pi} |\alpha_n^{(r)}(t)| \leq o_{n,r}n^r$$

with suitable constants $o_{n,r} > 0$ converging to 0 as $n \rightarrow \infty$ for every fixed $r = 2, 3, \dots$

We will show in Section 4 how Theorem 2.1 follows from Theorem 2.4.

Finally we give an extension of Saffari's Uniform Distribution Conjecture to higher derivatives. This will be shown in Section 4 as well.

Theorem 2.5. *Suppose (P_n) be a ε_n -ultraflat sequence of polynomials $P_n \in \mathcal{K}_n^\lambda$. Then*

$$m \left\{ t \in [0, 2\pi] : |P_n^{(r)}(e^{it})| \leq \frac{n^{r+\lambda+1/2}x^r}{(2\lambda+1)^{1/2}} \right\} = 2\pi x^{2\lambda+1} + o_{r,n}(x)$$

for every $x \in [0, 1]$, where $\lim_{n \rightarrow \infty} o_{r,n}(x) = 0$ for every fixed $r = 1, 2, \dots$ and $x \in [0, 1]$.

For every fixed $r = 1, 2, \dots$, the convergence of $o_{r,n}(x)$ is uniform on $[0, 1]$ by Dini's Theorem.

As a consequence of Theorems 2.1 and 2.4 we obtain

Theorem 2.6. *Let (P'_n) be an ε_n -ultraflat sequence of polynomials $P'_n \in \mathcal{K}_n^{\lambda+1}$. Then*

$$\lim_{n \rightarrow \infty} n^{-(\lambda+1/2)} \max_{t \in \mathbb{R}} |P'_n(e^{it})| = \infty.$$

Theorem 2.6 and Lemma 4.4 imply

Theorem 2.7. *Let (P_n) be a sequence of polynomials $P_n \in \mathcal{K}_n^0$ so that*

$$\max_{z \in \partial D} |P_n^{*'}(z)| = \max_{z \in \partial D} |P'_n(z)|, \quad n = 1, 2, \dots$$

Then (Q_n) with $Q_n(z) := zP'_n(z) + 1$ is not an ultraflat sequence of polynomials $Q_n \in \mathcal{K}_n^1$.

Corollary 2.8. *There are no sequences (P_n) of conjugate, plain, or skew reciprocal unimodular polynomials $P_n \in \mathcal{K}_n^0$ such that (Q_n) with $Q_n(z) = zP'_n(z) + 1$ is an ultraflat sequence of polynomials $Q_n \in \mathcal{K}_n^1$.*

3. PROOF OF THEOREM 2.4

To prove Theorem 2.4 we need a few lemmas. The first one is a standard polynomial inequality ascribed to Bernstein. Its proof is a simple exercise in complex analysis (an application of the Maximum Principle), and it may be found in a number of books. See [BE, p. 390], for example. We will use notation

$$D(z_0, R) := \{z \in \mathbb{C} : |z - z_0| < R\}, \quad \partial D(z_0, R) := \{z \in \mathbb{C} : |z - z_0| = R\},$$

and $\|f\|_A := \sup\{|f(z)| : z \in A\}$ for a complex-valued function f defined on a set A . As before, let $D := D(0, 1)$ and $\partial D := \partial D(0, 1)$.

Lemma 3.1. *We have*

$$|P(z)| \leq |z|^n \|P\|_{\partial D}$$

for every polynomial P of degree at most n with complex coefficients, and for every $z \in \mathbb{C}$ with $|z| > 1$. As a corollary (consider $P(e^{it}) := e^{int}T(t)$), we have

$$|T(z)| \leq e^{n\operatorname{Im}(z)} \|T\|_{\mathbb{R}}$$

for every trigonometric polynomial T of the form

$$T(t) = \sum_{k=-n}^n a_k e^{ikt}, \quad a_k \in \mathbb{C},$$

and for every $z \in \mathbb{C}$.

The main tool to prove Theorem 2.4 is the following well-known Jensen's Formula. For its proof, see, for example, E.10 c] of Section 4.2 in [BE].

Lemma 3.2 (Jensen's Formula). *Suppose h is a nonnegative integer and*

$$f(z) = \sum_{k=h}^{\infty} c_k z^k, \quad c_h \neq 0,$$

is analytic on a disk $D(0, R')$ with some $R' > R$. Suppose that the zeros of f in $D(0, R) \setminus \{0\}$ are a_1, a_2, \dots, a_m , where each zero is listed as many times as its multiplicity. Then

$$\log |c_h| + h \log R + \sum_{k=1}^m \log \frac{R}{|a_k|} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta.$$

Lemma 3.3. *Suppose (ε_n) is a sequence of numbers from $(0, 1/3)$ tending to 0. Let (P_n) be an ε_n -ultraflat sequence of polynomials $P_n \in \mathcal{K}_n^\lambda$. Then P_n does not have a zero in the open annulus*

$$\left\{ z \in \mathbb{C} : 1 - \frac{1}{2n\delta_n} < |z| < 1 + \frac{1}{2n\delta_n} \right\},$$

where the positive numbers

$$\delta_n := \max \left\{ \frac{2}{-\log(3\varepsilon_n)}, \frac{1}{n} \right\}$$

tend to 0.

Proof of Lemma 3.3. Associated with a polynomial

$$Q_n(z) = \sum_{j=0}^n a_j z^j, \quad a_j \in \mathbb{C},$$

of degree n , we define the polynomial

$$(3.1) \quad Q_n^*(z) = z^n \overline{Q_n(1/\bar{z})} = \sum_{j=0}^n \bar{a}_{n-j} z^j.$$

of degree at most n . Let (P_n) be a ε_n -ultraflat sequence of polynomials $P_n \in \mathcal{K}_n^\lambda$, that is, $P_n \in \mathcal{K}_n^\lambda$ satisfies

$$(1 - \varepsilon_n) \frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}} < |P_n(z)| < (1 + \varepsilon_n) \frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}}$$

for every $z \in \mathbb{C}$ with $|z| = 1$. Then

$$(1 - \varepsilon_n)^2 \frac{n^{2\lambda+1}}{2\lambda+1} < z^{-n} P_n(z) P_n^*(z) = |P_n(z)|^2 < (1 + \varepsilon_n)^2 \frac{n^{2\lambda+1}}{2\lambda+1}$$

for every $z \in \partial D$. We define

$$(3.2) \quad Q_n(z) = P_n(z)P_n^*(z) - \frac{n^{2\lambda+1}}{2\lambda+1} z^n.$$

Then Q_n is a polynomial of degree $2n$ and

$$-3\varepsilon_n \frac{n^{2\lambda+1}}{2\lambda+1} < z^{-n} Q_n(z) = |P_n(z)|^2 - \frac{n^{2\lambda+1}}{2\lambda+1} < 3\varepsilon_n \frac{n^{2\lambda+1}}{2\lambda+1}$$

for every $z \in \partial D$. From this we conclude that

$$(3.3) \quad |Q_n(z)| < 3\varepsilon_n \frac{n^{2\lambda+1}}{2\lambda+1}$$

for every $z \in \partial D$. Using Lemma 3.1 and (3.3), we obtain that

$$(3.4) \quad |Q_n(z)| \leq |z|^{2n} 3\varepsilon_n \frac{n^{2\lambda+1}}{2\lambda+1} < \frac{n^{2\lambda+1}}{2\lambda+1}$$

for every $z \in \mathbb{C}$ for which

$$1 \leq |z| < 1 + \frac{1}{n\delta_n},$$

where δ_n is defined in the lemma. Suppose that P_n has a zero in the annulus

$$\left\{ z \in \mathbb{C} : 1 - \frac{1}{2n\delta_n} < |z| < 1 + \frac{1}{2n\delta_n} \right\}.$$

Then $P_n P_n^*$ has a zero z_0 in the annulus

$$\left\{ z \in \mathbb{C} : 1 \leq |z| < 1 + \frac{1}{n\delta_n} \right\}.$$

Hence by (3.2) we have

$$|Q_n(z_0)| = \left| P_n(z_0)P_n^*(z_0) - \frac{n^{2\lambda+1}}{2\lambda+1} z_0^n \right| = \frac{n^{2\lambda+1}}{2\lambda+1} |z_0|^n \geq \frac{n^{2\lambda+1}}{2\lambda+1},$$

which is impossible by (3.4). \square

Lemma 3.4. *Suppose (ε_n) is a sequence of numbers from $(0, 1/3)$ tending to 0. Let (P_n) be a ε_n -ultraflat sequence of polynomials $P_n \in \mathcal{P}_n^c$. Let $1/n \leq R \leq 2$. Let $z_0 \in \partial D$. Then P_n has at most $5nR$ zeros in the disk $D(z_0, R)$.*

Proof. We use Jensen's formula on the disk $D(z_0, 2R)$. Note that since (P_n) is a ε_n -ultraflat sequence of polynomials $P_n \in \mathcal{K}_n^\lambda$, we have

$$\begin{aligned} \log |P_n(z_0)| &\geq \log(1 - \varepsilon_n) + (\lambda + 1/2) \log n - \frac{1}{2} \log(2\lambda + 1) \\ &\geq -\frac{1}{2} + (\lambda + 1/2) \log n - \frac{1}{2} \log(2\lambda + 1), \end{aligned}$$

while Bernstein inequality given by Lemma 3.1 yields

$$|P_n(z)| \leq (1 + \varepsilon_n) \frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}} (1 + 2R)^n, \quad z \in \partial D(z_0, 2R),$$

that is,

$$\log |P_n(z)| \leq \frac{1}{3} + (\lambda + 1/2) \log n - \frac{1}{2} \log (2\lambda + 1) + n(2R), \quad z \in \partial D(z_0, 2R).$$

Now if m denotes the number of zeros of P_n in $D(z_0, R)$, then by Jensen's formula

$$\begin{aligned} & -\frac{1}{2} + (\lambda + 1/2) \log n - \frac{1}{2} \log (2\lambda + 1) + m \log 2 \\ & \leq \frac{1}{3} + (\lambda + 1/2) \log n - \frac{1}{2} \log (2\lambda + 1) + 2nR, \end{aligned}$$

whence

$$m \leq \frac{3nR}{\log 2} \leq 5nR,$$

and the lemma is proved. \square

Our next lemma is a well-known inequality in approximation theory.

Lemma 3.5 (Bernstein's Inequality). *We have*

$$\|P'\|_{\partial D} \leq n \|P\|_{\partial D}$$

for every $P \in \mathcal{P}_n^c$.

Now we are ready for the proof of Theorem 2.4.

Proof of Theorem 2.4. Observe that if z_1, z_2, \dots, z_n denote the zeros of P_n in the complex plane, then

$$\frac{zP'_n(z)}{P_n(z)} = \sum_{j=1}^n \frac{z}{z - z_j} = \sum_{j=1}^n \left(1 + \frac{z_j}{z - z_j}\right).$$

Since $P_n \in \mathcal{K}_n^\lambda$, we have

$$(3.7) \quad |z_1|, |z_2|, \dots, |z_n| < 2.$$

To see this, let

$$P_n(z) = \sum_{j=0}^n j^m a_j z^j, \quad a_j \in \mathbb{C}, \quad |a_j| = 1.$$

Now if $z_0 \in \mathbb{C}$ and $|z_0| \geq 2$, then

$$\begin{aligned} \left| \sum_{j=0}^n j^\lambda a_j z_0^j \right| & \geq n^\lambda |z_0|^n - n^\lambda (|z_0|^{n-1} + |z_0|^{n-2} + \dots + |z_0|^1 + |z_0|^0) \\ & = n^\lambda \left(|z_0|^n - \frac{|z_0|^n - 1}{|z_0| - 1} \right) > 0. \end{aligned}$$

Using (3.5) and (3.7) and substituting $z_0 = e^{it_0}$, we can give the following upper bound for $|\alpha_n^{(r)}(t_0)|$ (the constants A_m below depend only on m).

(3.8)

$$\begin{aligned}
|\alpha_n^{(r)}(t_0)| &= \left| \frac{d^{r-1}}{dt^{r-1}} \left(\operatorname{Re} \left(\frac{e^{it} P_n'(e^{it})}{P_n(e^{it})} \right) \right) (t_0) \right| \leq \left| \frac{d^{r-1}}{dt^{r-1}} \left(\frac{e^{it} P_n'(e^{it})}{P_n(e^{it})} \right) (t_0) \right| \\
&= \left| \sum_{m=0}^{r-1} A_m \frac{d^m}{dz^m} \left(\frac{z P_n'(z)}{P_n(z)} \right) (z_0) e^{imt_0} \right| \\
&= \left| \sum_{m=0}^r A_m \frac{d^m}{dz^m} \left(\sum_{k=1}^n \left(1 + \frac{z_k}{z - z_k} \right) \right) (z_0) e^{imt_0} \right| \\
&\leq \left| A_0 \frac{z_0 P_n'(z_0)}{P_n(z_0)} \right| + \sum_{m=1}^{r-1} |A_m| m! \sum_{k=1}^n |z_k| |z_0 - z_k|^{-(m+1)} \\
&\leq \left| A_0 \frac{z_0 P_n'(z_0)}{P_n(z_0)} \right| + 2 \sum_{m=1}^{r-1} |A_m| m! \sum_{k=1}^n |z_0 - z_k|^{-(m+1)}.
\end{aligned}$$

Now we define the annulus

$$E_\mu = D(z_0, 2^\mu (2n\delta_n)^{-1}) \setminus D(z_0, 2^{\mu-1} (2n\delta_n)^{-1}), \quad \mu = 1, 2, \dots,$$

where $\delta_n := \max\{2/(-\log(3\varepsilon_n)), 1/n\}$ as in Lemma 3.3. We denote the number of zeros of P_n in E_μ by m_μ . By Lemma 3.4 we have $m_\mu \leq 5n2^\mu/(2n\delta_n)$. Combining this with (3.8) and Lemmas 3.5 and 3.3, we obtain

$$\begin{aligned}
|\alpha_n^{(r)}(t)| &\leq C_0 \frac{n(1 + \varepsilon_n)(2\lambda + 1)^{-1/2} n^{\lambda+1/2}}{(1 - \varepsilon_n)(2\lambda + 1)^{-1/2} n^{\lambda+1/2}} + C_r \sum_{m=1}^{r-1} \sum_{k=1}^n |z_0 - z_k|^{-(m+1)} \\
&\leq 2C_0 n + C_r \sum_{m=1}^{r-1} \sum_{\mu=1}^{\infty} m_\mu \left(\frac{2^{\mu-1}}{2n\delta_n} \right)^{-(m+1)} \\
&\leq 2C_0 n + C_r \sum_{m=1}^{r-1} \sum_{\mu=1}^{\infty} \frac{5n2^\mu}{2n\delta_n} \left(\frac{2^{\mu-1}}{2n\delta_n} \right)^{-(m+1)} \\
&\leq 2C_0 n + C_r \sum_{m=1}^{r-1} \sum_{\mu=1}^{\infty} 2 \cdot 2^{-(\mu-1)m} 5n(2n\delta_n)^m \\
&\leq 2C_0 n + C_r' n^r \delta_n \leq C_r'' n^r \delta_n,
\end{aligned}$$

where C_r , C_r' , and C_r'' are positive constants depending only on r . Since

$$\delta_n := \max\{2/(-\log(3\varepsilon_n)), 1/n\}$$

tends to 0 together with $\varepsilon_n > 0$, the theorem is proved. \square

4. PROOF OF THEOREMS 2.1, 2.2, AND THEOREM 2.3

First we prove Theorem 2.2 for positive integers q . To this end we need the lemmas below. The first one is a classical polynomial inequality of Bernstein available in many books. See [BE, Corollary 5.1.5], for example.

Lemma 4.1 (Bernstein's Inequality for Trigonometric Polynomials). *We have*

$$\max_{0 \leq t \leq 2\pi} |T^{(m)}(t)| \leq n^m \max_{0 \leq t \leq 2\pi} |T(t)|, \quad m = 1, 2, \dots,$$

for every trigonometric polynomial T of degree at most n with complex coefficients.

Lemma 4.2. *Suppose (P_n) is an ε_n -ultraflat sequence of polynomials $P_n \in \mathcal{K}_n^\lambda$. Using the notation (2.1) we have*

$$o_n \leq \alpha'_n(t) \leq n + o_n, \quad t \in \mathbb{R},$$

with some real numbers o_n tending to 0.

Lemma 4.3. *Suppose (ε_n) is a sequence of numbers from $(0, 1/3)$ tending to 0. Suppose (P_n) is a (ε_n) -ultraflat sequence of polynomials $P_n \in \mathcal{K}_n^\lambda$. Using notation (2.1), we have*

$$(4.1) \quad \max_{0 \leq t \leq 2\pi} |R_n^{(m)}(t)| = o_{n,m} n^{m+\lambda+1/2}, \quad m = 1, 2, \dots,$$

with suitable constants $o_{n,m}$ converging to 0 as $n \rightarrow \infty$ for every $m = 1, 2, \dots$.

Proof of Lemma 4.2. It is easy to find a formula for $\alpha_n(t)$ in terms of P_n . One can easily verify formula (8.2) from Saffari's paper [Sa], which asserts that

$$(3.5) \quad \alpha'_n(t) = \operatorname{Re} \left(\frac{e^{it} P'_n(e^{it})}{P_n(e^{it})} \right).$$

Combining this with Lemma 3.5 (Bernstein's inequality) and the ultraflatness of the sequence (P_n) , we obtain the upper bound of the lemma. Associated with the (ε_n) -ultraflat sequence (P_n) of polynomials $P_n \in \mathcal{K}_n^\lambda$, we study the (ε_n) -ultraflat sequence (P_n^*) of the corresponding conjugate polynomials P_n^* (see (3.1) for the definition). The associated angular function α_n^* and modulus function R_n^* are defined by

$$(3.6) \quad P_n^*(e^{it}) = R_n^*(t) e^{i\alpha_n^*(t)}, \quad R_n^*(t) = |P_n^*(e^{it})|.$$

It is a simple exercise to show that α_n^* can be chosen to be in $C^\infty(\mathbb{R})$ on \mathbb{R} . By applying formula (3.5) to P_n^* , it is easy to see that

$$\alpha_n(t) + \alpha_n^*(t) = n, \quad t \in \mathbb{R}.$$

Since the upper bound of the lemma is valid for α_n^* as well, the lower bound of the lemma follows from this. \square

Proof of Lemma 4.3. The proof is very similar to that of Lemma 3.2. Let

$$\delta_n := \max\{2/(-\log(3\varepsilon_n)), 1/n\},$$

as in the proof Lemma 3.3. Let (P_n) be a (ε_n) -ultraflat sequence of polynomials $P_n \in \mathcal{K}_n^\lambda$, that is, $P_n \in \mathcal{K}_n^\lambda$ satisfies

$$(4.2) \quad (1 - \varepsilon_n) \frac{n^{\lambda+1/2}}{(2\lambda+1)^{1/2}} < |P_n(z)| < (1 + \varepsilon_n) \frac{n^{\lambda+1/2}}{(2\lambda+1)^{1/2}}, \quad z \in \partial D.$$

(In fact, in this proof we will not use that $P_n \in \mathcal{K}_n^\lambda$, we will use only that P_n is a polynomial of degree n with complex coefficients that satisfies (4.2).) To denote the conjugate polynomial of a polynomial $Q_n \in \mathcal{P}_n^c \setminus \mathcal{P}_{n-1}^c$, we use the notation Q_n^* introduced by (3.1).

Step 1. By Lemma 3.3,

$$(4.3) \quad T_n(t) := e^{-int} P_n(e^{it}) P_n^*(e^{it})$$

has no zeros in the strip

$$(4.4) \quad \mathcal{E}_n := \left\{ z \in \mathbb{C} : |\operatorname{Im}(z)| \leq \frac{1}{4n\delta_n} \right\}.$$

Therefore

$$\tilde{T}_n(t) := \sqrt{T_n(t)} = \sqrt{e^{-int} P_n(e^{it}) P_n^*(e^{it})}$$

is a well-defined analytic function in the strip \mathcal{E}_n .

Step 2. We show that

$$|\tilde{T}_n'(t)| \leq o_n n^{\lambda+3/2}, \quad t \in \mathbb{R},$$

with suitable constants o_n converging to 0. Indeed, T_n is a trigonometric polynomial of degree n (with complex coefficients). Note that (4.2) implies that

$$(4.5) \quad -3\varepsilon_n \frac{n^{2\lambda+1}}{2\lambda+1} < T_n(t) - \frac{n^{2\lambda+1}}{2\lambda+1} < 3\varepsilon_n \frac{n^{2\lambda+1}}{2\lambda+1}.$$

Combining this with Lemma 4.1 (Bernstein's inequality for trigonometric polynomials), we obtain

$$(4.6) \quad \begin{aligned} \max_{0 \leq t \leq 2\pi} |T_n'(t)| &= \max_{0 \leq t \leq 2\pi} \left| \frac{d}{dt} \left(T_n(t) - \frac{n^{2\lambda+1}}{2\lambda+1} \right) \right| \\ &\leq n \max_{0 \leq t \leq 2\pi} \left| T_n(t) - \frac{n^{2\lambda+1}}{2\lambda+1} \right| \leq n 3\varepsilon_n \frac{n^{2\lambda+1}}{2\lambda+1} \\ &\leq \frac{3\varepsilon_n}{2\lambda+1} n^{2\lambda+2}. \end{aligned}$$

Now

$$(4.7) \quad \begin{aligned} |\tilde{T}'_n(t)| &= \left| \frac{T'_n(t)}{2\sqrt{T_n(t)}} \right| \leq \frac{3\varepsilon_n}{2\lambda+1} \frac{n^{2\lambda+2}}{2(1-\varepsilon_n)(2\lambda+1)^{-1/2}n^{\lambda+1/2}} \leq \frac{3\varepsilon_n}{(1-\varepsilon_n)} n^{3/2} \\ &\leq \frac{3}{2} \varepsilon_n n^{\lambda+3/2} = o_n n^{\lambda+3/2}, \quad t \in \mathbb{R}, \end{aligned}$$

with suitable constants o_n converging to 0.

Step 3. Let

$$\mathcal{F}_{c,n} := \left\{ z \in \mathbb{C} : |\operatorname{Im}(z)| \leq \frac{c}{n} \right\}.$$

We show that there is a sufficiently small absolute constant $c > 0$ such that

$$(4.8) \quad |\tilde{T}'_n(t)| \leq o_n n^{\lambda+3/2}, \quad t \in \mathcal{F}_{c,n},$$

with suitable constants o_n converging to 0 as $n \rightarrow \infty$. To see this, first note that

$$(4.9) \quad |\tilde{T}'_n(t)| = \left| \frac{T'_n(t)}{2\sqrt{T_n(t)}} \right|,$$

where T_n is defined by (4.3). Using (4.6) and Lemma 3.1 we obtain that

$$(4.10) \quad |T'_n(t)| \leq o'_n n^{2\lambda+2} e^{n(c/n)} = o_n n^{2\lambda+2}, \quad t \in \mathcal{F}_{c,n}$$

with suitable constants o'_n and o_n converging to 0 as $n \rightarrow \infty$ and with a sufficiently small absolute constant $c > 0$. Similarly, (4.5), $\varepsilon_n \in (0, 1/6)$, and Lemma 3.1 give

$$(4.11) \quad |T_n(t)| \geq \frac{n^{2\lambda+1}}{4(2\lambda+1)}, \quad t \in \mathcal{F}_{c,n},$$

for a sufficiently small absolute constant $c > 0$. Now (4.9) – (4.11) imply that

$$(4.12) \quad |\tilde{T}'_n(t)| \leq o_n n^{\lambda+3/2}, \quad t \in \mathcal{F}_{c,n},$$

with suitable constants o_n converging to 0 as $n \rightarrow \infty$ and with a sufficiently small absolute constant $c > 0$.

Step 4. From Step 3 we conclude by the Cauchy Integral Formula that

$$\begin{aligned} |\tilde{T}_n^{(m)}(t)| &= (m-1)! \left| \int_{\partial D(t, c/n)} \frac{T'_n(\xi) d\xi}{(\xi-t)^m} \right| \\ &\leq \frac{2\pi c}{n} (m-1)! o_{n,1} n^{\lambda+3/2} \left(\frac{c}{n}\right)^{-m} = o_{n,m} n^{m+\lambda+1/2}, \end{aligned}$$

with suitable constants $o_{n,m}$ converging to 0 as $n \rightarrow \infty$ for every fixed $m = 1, 2, \dots$

Step 5. Note that for $t \in \mathbb{R}$ we have

$$(4.13) \quad R_n(t) = |P_n(e^{it})| = \sqrt{e^{-int} P_n(e^{it}) P_n^*(e^{it})} = \tilde{T}_n(t),$$

hence by Step 4

$$\max_{0 \leq t \leq 1} |R_n^{(m)}(t)| = o_{n,m} n^{m+\lambda+1/2}$$

with suitable constants $o_{n,m}$ converging to 0 as $n \rightarrow \infty$ for every fixed $m = 1, 2, \dots$. This proves the lemma. \square

Now we are ready to prove Theorem 2.2 for positive integers q .

Proof of Theorem 2.2 for integers $q \geq 0$. Let (P_n) be a (ε_n) -ultraflat sequence of polynomials $P_n \in \mathcal{K}_n^\lambda$. We define

$$(4.14) \quad S_n(t) := P_n(e^{it}) = \sum_{k=0}^n k^\lambda a_{k,n} e^{ikt}, \quad |a_{k,n}| = 1.$$

We calculate

$$\frac{1}{2\pi} \int_0^{2\pi} S_n^{(q)}(t) \overline{S_n(t)} dt$$

in two different ways. On one hand, using orthogonality, we have

$$(4.15) \quad \frac{1}{2\pi} \int_0^{2\pi} S_n^{(q)}(t) \overline{S_n(t)} dt = i^q \sum_{k=0}^n k^{q+2\lambda} |a_{k,n}|^2 = i^q \frac{n^{q+2\lambda+1}}{q+2\lambda+1} + o_{n,q} n^{q+\lambda+1},$$

with suitable constants $o_{n,q}$ converging to 0 as $n \rightarrow \infty$ for every fixed $q = 0, 1, \dots$.

On the other hand, with our standard notation introduced by (2.1), Theorem 2.4 and Lemmas 4.2 and 4.3 give

$$(4.16) \quad \begin{aligned} S_n^{(q)}(t) &= \sum_{k=0}^q \binom{q}{k} \frac{d^k}{dt^k} \left(e^{i\alpha_n(t)} \right) R_n^{(q-k)}(t) \\ &= \frac{d^q}{dt^q} \left(e^{i\alpha_n(t)} \right) R_n(t) + \sum_{k=0}^{q-1} \binom{q}{k} \frac{d^k}{dt^k} \left(e^{i\alpha_n(t)} \right) R_n^{(q-k)}(t) \\ &= \frac{d^q}{dt^q} \left(e^{i\alpha_n(t)} \right) R_n(t) + \sum_{k=0}^{q-1} \binom{q}{k} c_{n,k}(t) n^k o_{n,q-k}(t) n^{q-k+\lambda+1/2} \\ &= \left(e^{i\alpha_n(t)} \alpha_n'(t) \right)^q i^q + o'_{n,q}(t) n^q R_n(t) + o''_{n,q}(t) n^{q+\lambda+1/2} \end{aligned}$$

with suitable numbers $o_{n,q-k}(t)$, $c_{n,k}(t)$, $o'_{n,q}(t)$, and $o''_{n,q}(t)$, where

$$\max_{0 \leq t \leq 2\pi} |o_{n,q-k}(t)|$$

converge to 0 for every fixed q and $k = 0, 1, \dots, q-1$,

$$\max_{0 \leq t \leq 2\pi} |c_{n,k}(t)|$$

is bounded by a constant independent of n for every fixed $k = 0, 1, \dots, q-1$, and

$$\max_{0 \leq t \leq 2\pi} |o'_{n,q}(t)| \quad \text{and} \quad \max_{0 \leq t \leq 2\pi} |o''_{n,q}(t)|$$

converge to 0 as $n \rightarrow \infty$ for every fixed q . Now (4.13), (4.14), and (4.16) yield

(4.17)

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} S_n^{(q)}(t) \overline{S_n(t)} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left((e^{i\alpha_n(t)} \alpha'_n(t)^q i^q + o'_{n,q}(t) n^q) R_n(t) + o''_{n,q}(t) n^{q+\lambda+1/2} \right) R_n(t) e^{-i\alpha_n(t)} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{n^{2\lambda+1}}{2\lambda+1} (1 - o_n(t)) (\alpha'_n(t)^q i^q + o'_{n,q}(t) n^q) + o'''_{n,q}(t) n^{q+\lambda+1/2+\lambda+1/2} \right) dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} i^q \left(\frac{n^{2\lambda+1}}{2\lambda+1} (1 - o_n(t)) \right) \alpha'_n(t)^q dt + o_{n,q}^* n^{q+2\lambda+1}, \end{aligned}$$

with suitable numbers $o_n(t)$, $o'_{n,q}(t)$, $o''_{n,q}(t)$, $o'''_{n,q}(t)$, and $o_{n,q}^*$, where

$$\max_{0 \leq t \leq 2\pi} |o_n(t)|, \quad \max_{0 \leq t \leq 2\pi} |o'_{n,q}(t)|, \quad \max_{0 \leq t \leq 2\pi} |o''_{n,q}(t)|, \quad \max_{0 \leq t \leq 2\pi} |o'''_{n,q}(t)|,$$

and $o_{n,q}^*$ converge to 0 as $n \rightarrow \infty$ for every fixed q .

Now (4.15) and (4.17) give the statement of the theorem for integers $q \geq 0$. \square

Proof of Theorem 2.1. We introduce the normalized distribution functions

$$(4.18) \quad F_n(x) := m\{t \in [0, 2\pi] : 0 \leq \alpha'_n(t) \leq nx\}, \quad x \in [0, 1].$$

Each F_n is continuous and nondecreasing on $[0, 1]$, and

$$0 \leq F_n(x) \leq 2\pi, \quad x \in [0, 1].$$

Suppose that the conjecture is not true. Then we can find a subsequence (F_{n_k}) of the sequence (F_n) , and numbers $y \in [0, 1]$ and $\varepsilon > 0$ such that

$$(4.19) \quad |F_{n_k}(y) - 2\pi y^{2\lambda+1}| \geq \varepsilon, \quad k = 1, 2, \dots$$

Then by Helly's Selection Theorem, there is a subsequence (m_k) of (n_k) such that

$$(4.20) \quad \lim_{k \rightarrow \infty} F_{m_k}(x) = F(x)$$

exists for every $x \in [0, 1]$. Then Theorem 2.2, (4.18), Lemma 4.2, and the Lebesgue Dominated Convergence Theorem imply that

$$\int_0^1 x^q dF(x) = \frac{2\pi(2\lambda + 1)}{q + 2\lambda + 1}, \quad q = 1, 2, \dots$$

That is, all the corresponding moments of the measures $dF(x)$ and $dG(x)$ with $G(x) := 2\pi x^{2\lambda+1}$ are the same on $[0, 1]$. Therefore, using the uniqueness part of the Riesz Representation Theorem describing all continuous linear functionals on $C[0, 1]$, we obtain that $F(x) = 2\pi x^{2\lambda+1}$ for all $x \in [0, 1]$. However, this contradicts (4.19) and (4.20). So

$$m\{t \in [0, 2\pi] : 0 \leq \alpha'_n(t) \leq nx\} = 2\pi x^{2\lambda+1} + o_n(x)$$

for every $x \in [0, 1]$, where $\lim_{n \rightarrow \infty} o_n(x) = 0$ for every $x \in [0, 1]$.

To see the second statement of the theorem, we argue as follows. Using notation (2.1) and Lemma 4.2 we have $R'_n(t) = o_n(t)n^{\lambda+3/2}$ with a constant $o_n(t)$ tending to 0 as $n \rightarrow \infty$ for every $t \in \mathbb{R}$. Therefore

$$(4.21) \quad \begin{aligned} |P'_n(e^{it})| &= |R'_n(t)e^{i\alpha_n(t)} + i\alpha'_n(t)e^{i\alpha_n(t)}R_n(t)| \\ &= o_n(t)n^{\lambda+3/2} + |\alpha'_n(t)|(1 + \varepsilon_n(t))\frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}}, \end{aligned}$$

where $o_n(t)$ and $\varepsilon_n(t)$ tend to 0 as $n \rightarrow \infty$ for every $t \in \mathbb{R}$. Now the result follows from the first part of the theorem. \square

Proof of Theorem 2.2 for all real $q > 0$. This follows from the already proved Theorem 2.1 in a routine fashion. \square

Proof of Theorem 2.3. This follows immediately from Theorem 2.2 and (4.21). \square

Proof of Theorem 2.5. To see the second part of the theorem, we write, as in (2.1),

$$P_n(e^{it}) = R_n(t)e^{i\alpha_n(t)},$$

where, as before, $R_n(t) = |P_n(e^{it})|$. Then

$$P_n^{(r)}(z) = \sum_{k=0}^r \binom{r}{k} R_n^{(k)}(t) \frac{d^{(r-k)}}{dt^{r-k}} \left(e^{i\alpha_n(t)} \right)$$

Now the theorem follows from (2.1), Theorem 2.4, Lemma 4.2, and Theorem 2.1. \square

To prove Theorem 2.7 we need the lemma below. This is stated as Theorem 3.1.27 and proved on page 689 of [MMR].

Lemma 4.4. *We have*

$$\max_{z \in \partial D} (|P'(z)| + |P^{*'}(z)|) = n \max_{z \in \partial D} |P(z)|$$

for every $P \in \mathcal{P}_n \setminus \mathcal{P}_{n-1}$.

The proof of Theorem 2.6 is a simple consequence of Theorems 2.1 and 2.4, while Theorem 2.7 follows easily from Lemma 4.4. These proofs are left to the reader.

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