THE ASYMPTOTIC DISTANCE BETWEEN AN ULTRAFLAT UNIMODULAR POLYNOMIAL AND ITS CONJUGATE RECIPROCAL

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Dedicated to the memory of Jean-Pierre Kahane

Abstract. Let

$$\mathcal{K}_n := \left\{ Q_n : Q_n(z) = \sum_{k=0}^{n} a_k z^k, \ \ a_k \in \mathbb{C}\,, \ \ |a_k| = 1 \right\}.$$

A sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ is called ultraflat if $(n+1)^{-1/2}|P_n(e^{it})|$ converge to 1 uniformly in $t \in \mathbb{R}$. In this paper we prove that

$$\frac{1}{2\pi}\int_0^{2\pi} \left|(P_n - P_n^*)(e^{it})\right|^q \, dt \sim \frac{2^q \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}} \, n^{q/2}$$

for every ultraflat sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ and for every $q \in (0,\infty)$, where P_n^* is the conjugate reciprocal polynomial associated with P_n, Γ is the usual gamma function, and the $\sim$ symbol means that the ratio of the left and right hand sides converges to 1 as $n \to \infty$. Another highlight of the paper states that

$$\frac{1}{2\pi}\int_0^{2\pi} \left|(P_n' - P_n^{*\prime})(e^{it})\right|^2 \, dt \sim \frac{2n^3}{3}$$

for every ultraflat sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$. We prove a few other new results and reprove some interesting old results as well.

1. Introduction

Let

$$\mathcal{K}_n := \left\{ Q_n : Q_n(z) = \sum_{k=0}^{n} a_k z^k, \ \ a_k \in \mathbb{C}\,, \ \ |a_k| = 1 \right\}.$$

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The class $\mathcal{K}_n$ is often called the collection of all (complex) unimodular polynomials of degree n. Let

$$\mathcal{L}_n := \left\{ Q_n : Q_n(z) = \sum_{k=0}^{n} a_k z^k, \ \ a_k \in \{-1, 1\} \right\} .$$

The class $\mathcal{L}_n$ is often called the collection of all (real) unimodular polynomials of degree n. By Parseval's formula,

$$\int_0^{2\pi} \left| P_n(e^{it}) \right|^2 \, dt = 2\pi(n+1)$$

for all $P_n \in \mathcal{K}_n$. Therefore

$$\min_{t \in \mathbb{R}} |P_n(e^{it})| \le \sqrt{n+1} \le \max_{t \in \mathbb{R}} |P_n(e^{it})| .$$

An old problem (or rather an old theme) is the following.

Problem 1.1 (Littlewood's Flatness Problem). *How close can a polynomial $P_n \in \mathcal{K}_n$ or $P_n \in \mathcal{L}_n$ come to satisfying*

$$|P_n(e^{it})| = \sqrt{n+1} , \qquad t \in \mathbb{R}? \tag{1.1}$$

Obviously (1.1) is impossible if $n \ge 1$. So one must look for less than (1.1), but then there are various ways of seeking such an "approximate situation". One way is the following. In his paper [Li1] Littlewood had suggested that, conceivably, there might exist a sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ (possibly even $P_n \in \mathcal{L}_n$) such that $(n+1)^{-1/2}|P_n(e^{it})|$ converge to 1 uniformly in $t \in \mathbb{R}$. We shall call such sequences of unimodular polynomials "ultraflat". More precisely, we give the following definition.

Definition 1.2. *Given a positive number ε, we say that a polynomial $P_n \in \mathcal{K}_n$ is ε-flat if*

$$(1-\varepsilon)\sqrt{n+1} \le |P_n(e^{it})| \le (1+\varepsilon)\sqrt{n+1} , \qquad t \in \mathbb{R} .$$

Definition 1.3. *Let (n_k) be an increasing sequence of positive integers. Given a sequence (ε_{n_k}) of positive numbers tending to 0, we say that a sequence (P_{n_k}) of polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is (ε_{n_k})-ultraflat if each P_{n_k} is (ε_{n_k})-flat. We simply say that a sequence (P_{n_k}) of polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is ultraflat if it is (ε_{n_k})-ultraflat with a suitable sequence (ε_{n_k}) of positive numbers tending to 0.*

The existence of an ultraflat sequence of unimodular polynomials seemed very unlikely, in view of a 1957 conjecture of P. Erdős (Problem 22 in [Er]) asserting that, for all $P_n \in \mathcal{K}_n$ with $n \ge 1$,

$$\max_{t \in \mathbb{R}} |P_n(e^{it})| \ge (1+\varepsilon)\sqrt{n+1} , \tag{1.2}$$

where $\varepsilon > 0$ is an absolute constant (independent of n). Yet, refining a method of Körner [Kö], Kahane [Ka] proved that there exists a sequence (P_n) with $P_n \in \mathcal{K}_n$ which is (ε_n)-ultraflat, where $\varepsilon_n = O\left(n^{-1/17}\sqrt{\log n}\right)$. (Kahane's paper contained though a slight error which was corrected in [QS2].) Thus the Erdős conjecture (1.2) was disproved for the classes $\mathcal{K}_n$. For the more restricted class $\mathcal{L}_n$ the analogous Erdős conjecture is unsettled to this date. It is a common belief that the analogous Erdős conjecture for $\mathcal{L}_n$ is true, and consequently there is no ultraflat sequence of polynomials $P_n \in \mathcal{L}_n$. An interesting result related to Kahane's breakthrough is given in [Be]. For an account of some of the work done till the mid 1960's, see Littlewood's book [Li2] and [QS2].

If Q_n is a polynomial of degree n of the form

$$Q_n(z) = \sum_{k=0}^{n} a_k z^k \,, \qquad a_k \in \mathbb{C} \,,$$

then its conjugate polynomial is defined by

$$Q_n^*(z) := z^n \overline{Q}_n(1/z) := \sum_{k=0}^{n} \overline{a}_{n-k} z^k \,.$$

Let (ε_n) be a sequence of positive numbers tending to 0. Let the sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ be (ε_n)-ultraflat. We write

$$P_n(e^{it}) = R_n(t) e^{i\alpha_n(t)} \,, \qquad R_n(t) = |P_n(e^{it})| \,, \qquad t \in \mathbb{R} \,. \tag{1.3}$$

It is simple to show that α_n can be chosen to be in $C^\infty(\mathbb{R})$. This is going to be our understanding throughout the paper. It is easy to find a formula for $\alpha_n(t)$ in terms of P_n. We have

$$\alpha_n'(t) = \mathrm{Re}\left(\frac{e^{it} P_n'(e^{it})}{P_n(e^{it})}\right) \,, \tag{1.4}$$

see formulas (7.1) and (7.2) on p. 564 and (8.2) on p. 565 in [Sa1]. The angular function α_n^* and modulus function $R_n^* = R_n$ associated with the polynomial P_n^* are defined by

$$P_n^*(e^{it}) = R_n^*(t) e^{i\alpha_n^*(t)} \,, \qquad R_n^*(t) = |P_n^*(e^{it})| \,.$$

Similarly to α_n, the angular function α_n^* can also be chosen to be in $C^\infty(\mathbb{R})$ on $\mathbb{R}$. By applying formula (1.4) to P_n^*, it is easy to see that

$$\alpha_n'(t) + \alpha_n^{*\prime}(t) = n \,, \qquad t \in \mathbb{R} \,. \tag{1.5}$$

The structure of ultraflat sequences of unimodular polynomials is studied in [Er1], [Er2], [Er3], and [Er4], where several conjectures of Saffari are proved. In [Er6], based on the results in [Er1], we proved yet another conjecture of Queffelec and Saffari, see (1.30) in [QS2]. Namely we proved asymptotic formulas for the L_q norms of the real part and the

derivative of the real part of ultraflat unimodular polynomials on the unit circle. A recent paper of Bombieri and Bourgain [BB] is devoted to the construction of ultraflat sequences of unimodular polynomials. In particular, they obtained a much improved estimate for the error term. A major part of their paper deals also with the long-standing problem of the effective construction of ultraflat sequences of unimodular polynomials.

For $\lambda \geq 0$, let

$$\mathcal{K}_n^\lambda := \left\{ P_n : P_n(z) = \sum_{k=0}^{n} a_k k^\lambda z^k, \;\; a_k \in \mathbb{C}\,, \;\; |a_k| = 1 \right\}.$$

Ultraflat sequences (P_n) of polynomials $P_n \in \mathcal{K}_n^\lambda$ are defined and studied thoroughly in [EN] where various extensions of Saffari's conjectures have been proved.

In [Er2] we examined how far an ultraflat unimodular polynomial is from being conjugate reciprocal, and we proved the following three theorems.

Theorem 1.4. *Let (P_n) be an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. We have*

$$\frac{1}{2\pi}\int_0^{2\pi} \left(|P_n'(e^{it})| - |P_n^{*\prime}(e^{it})|\right)^2 dt = \left(\frac{1}{3} + \gamma_n\right) n^3\,,$$

where (γ_n) is a sequence of real numbers converging to 0.

Theorem 1.5. *Let (P_n) be an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. If the coefficients of P_n are denoted by $a_{k,n}$, that is,*

$$P_n(z) = \sum_{k=0}^{n} a_{k,n} z^k\,, \qquad k = 0, 1, \ldots, n, \quad n = 1, 2, \ldots,$$

then

$$\sum_{k=0}^{n} k^2 \left|a_{k,n} - \overline{a}_{n-k,n}\right|^2 = \frac{1}{2\pi}\int_0^{2\pi} \left|(P_n' - P_n^{*\prime})(e^{it})\right|^2 dt \geq \left(\frac{1}{3} + h_n\right) n^3\,,$$

where (h_n) is a sequence of real numbers converging to 0.

Theorem 1.6. *Let (P_n) be an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Using the notation of Theorem 1.5 we have*

$$\sum_{k=0}^{n} \left|a_{k,n} - \overline{a}_{n-k,n}\right|^2 = \frac{1}{2\pi}\int_0^{2\pi} \left|(P_n - P_n^{*})(e^{it})\right|^2 dt \geq \left(\frac{1}{3} + h_n\right) n\,,$$

where (h_n) is the same sequence of real numbers converging to 0 as in Theorem 1.5.

There are quite a few recent publications on or related to ultraflat sequences of unimodular polynomials. Some of them (not mentioned before) are are [Bo], [Sa2], [QS1], [Od], and [Mo].

2. Results

Theorems 2.1–2.4 and 2.6 are new, Theorems 2.5 and 2.7 recapture old results.

In our results below Γ denotes the usual gamma function, and the $\sim$ symbol means that the ratio of the left and right hand sides converges to 1 as $n \to \infty$.

Theorem 2.1. *If (P_n) is an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$ and $q \in (0, \infty)$, then*

$$\frac{1}{2\pi} \int_0^{2\pi} \left|(P_n - P_n^*)(e^{it})\right|^q \, dt \sim \frac{2^q \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}} \, n^{q/2} \, .$$

Our next theorem is a special case ($q = 2$) of Theorem 2.1. Compare it with Theorem 1.6.

Theorem 2.2. *Let (P_n) be an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. If the coefficients of P_n are denoted by $a_{k,n}$, that is,*

$$P_n(z) = \sum_{k=0}^{n} a_{k,n} z^k \, , \qquad k = 0, 1, \dots, n, \quad n = 1, 2, \dots ,$$

then

$$\sum_{k=0}^{n} \left|a_{k,n} - \overline{a}_{n-k,n}\right|^2 = \frac{1}{2\pi} \int_0^{2\pi} \left|(P_n - P_n^*)(e^{it})\right|^2 \, dt \sim 2n \, .$$

Our next theorem should be compared with Theorem 1.5.

Theorem 2.3. *Let (P_n) be an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Using the notation in Theorem 2.2 we have*

$$\sum_{k=0}^{n} k^2 \left|a_{k,n} - \overline{a}_{n-k,n}\right|^2 = \frac{1}{2\pi} \int_0^{2\pi} \left|(P_n' - P_n^{*\prime})(e^{it})\right|^2 \, dt \sim \frac{2n^3}{3} \, .$$

We also prove the following result.

Theorem 2.4. *If (P_n) is an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$ and $q \in (0, \infty)$, then*

$$\frac{1}{2\pi} \int_0^{2\pi} \left|\frac{d}{dt}|(P_n - P_n^*)(e^{it})|\right|^q \, dt \sim \frac{\Gamma\left(\frac{q+1}{2}\right)}{(q+1)\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}} \, n^{3q/2} \, .$$

As a Corollary of Theorem 2.2 we can recapture Saffari's "near orthogonality conjecture" raised in [Sa] and proved first in [Er4].

Theorem 2.5. *Let (P_n) be an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Using the notation in Theorem 2.2 we have*

$$\sum_{k=0}^{n} a_{k,n} a_{n-k,n} = o(n) \, .$$

As a Corollary of Theorem 2.3 we can easily prove a new "near orthogonality" formula.

Theorem 2.6. *Let (P_n) be an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Using the notation in Theorem 2.2 we have*

$$\sum_{k=0}^{n} k^2 a_{k,n} a_{n-k,n} = o(n^3)\,.$$

Finally we recapture the asymptotic formulas for the real part and the derivative of the real part of ultraflat unimodular polynomials proved in [Er5] first.

Theorem 2.7. *If (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$, and $q \in (0,\infty)$, then for $f_n(t) := \mathrm{Re}(P_n(e^{it}))$ we have*

$$\frac{1}{2\pi}\int_0^{2\pi} |f_n(t)|^q \, dt \sim \frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}}\, n^{q/2}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} |f_n'(t)|^q \, dt \sim \frac{\Gamma\left(\frac{q+1}{2}\right)}{(q+1)\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}}\, n^{3q/2}\,.$$

We remark that trivial modifications of the proof of Theorem 2.1–2.7 yield that the statement of the above theorem remains true if the ultraflat sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ is replaced by an ultraflat sequence (P_{n_k}) of polynomials $P_{n_k} \in \mathcal{K}_{n_k}$, where (n_k) is an increasing sequence of positive integers.

3. Lemmas

To prove Theorems 2.1 and 2.2 we need a few lemmas. The first two are from [Er1].

Lemma 3.1 (Uniform Distribution Theorem for the Angular Speed). *Suppose (P_n) is an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Then, with the notation (1.3), in the interval $[0,2\pi]$, the distribution of the normalized angular speed $\alpha_n'(t)/n$ converges to the uniform distribution as $n \to \infty$. More precisely, we have*

$$\mathrm{meas}(\{t \in [0,2\pi] : 0 \le \alpha_n'(t) \le nx\}) = 2\pi x + \gamma_n(x)$$

for every $x \in [0,1]$, where $\lim_{n\to\infty} \max_{x\in[0,1]} |\gamma_n(x)| = 0$.

Our next lemma is a simple observation of Saffari [Sa1], which follows from (1.4), Bernstein's inequality, and the ultraflatness property given by Definition 1.3.

Lemma 3.2. *Suppose (P_n) is an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Then, with the notation (1.3), we have*

$$o_n n \le \alpha_n'(t) \le n - o_n n\,, \qquad t \in \mathbb{R}\,, \tag{3.1}$$

with real numbers o_n converging to 0.

Lemma 3.3 (Negligibility Theorem for Higher Derivatives). *Suppose (P_n) is an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Then, with the notation (1.3), for every integer $r \geq 2$, we have*

$$\max_{0 \leq t \leq 2\pi} |\alpha_n^{(r)}(t)| \leq \gamma_{n,r} n^r$$

with real numbers $\gamma_{n,r} > 0$ converging to 0 for every fixed $r = 2, 3, \ldots$.

Our next lemma is a special case of Lemma 4.2 from [Er1].

Lemma 3.4. *Suppose (P_n) is an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Using notation (1.3), we have*

$$\max_{0 \leq t \leq 2\pi} |R_n'(t)| = \varphi_n n^{3/2}, \qquad m = 1, 2, \ldots,$$

with real numbers φ_n converging to 0.

The next lemma follows simply from the ultraflatness property given by Definition 1.3.

Lemma 3.5. *Let $q \in (0, \infty)$. Using the notation (1.3) and*

$$\beta_n(t) := \frac{1}{2}(\alpha_n(t) - \alpha_n^*(t)) = \alpha_n(t) - \frac{nt}{2} - t_0$$

we have

$$\frac{1}{2\pi} \int_0^{2\pi} \left|(P_n - P_n^*)(e^{it})\right|^q \, dt = \int_0^{2\pi} \left| n^{1/2}(1 + \delta_n(t)) 2 \sin(\beta_n(t)) \right|^q \, dt$$

with real numbers $\delta_n(t)$ converging to 0 uniformly on $[0, 2\pi]$.

Lemma 3.5*. *Let (P_n) be an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$, $q \in (0, \infty)$, and $f_n(t) := \mathrm{Re}(P_n(e^{it}))$. Using the notation (1.3) we have*

$$\frac{1}{2\pi} \int_0^{2\pi} |f_n(t)|^q \, dt = \int_0^{2\pi} \left| n^{1/2}(1 + \delta_n(t)) \cos(\alpha_n(t)) \right|^q \, dt$$

with real numbers $\delta_n(t)$ converging to 0 uniformly on $[0, 2\pi]$.

The next lemma follows simply from the ultraflatness property given by Definition 1.3 and Lemma 3.4.

Lemma 3.6. *Let $q \in (0, \infty)$. Using the notation (1.3) and*

$$\beta_n(t) := \frac{1}{2}(\alpha_n(t) - \alpha_n^*(t)) = \alpha_n(t) - \frac{nt}{2} - t_0$$

we have

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{d}{dt} |(P_n - P_n^*)(e^{it})| \right|^q \, dt$$

$$= \int_0^{2\pi} \left| n^{1/2}(1 + \delta_n(t)) 2 \cos(\beta_n(t)) \beta_n'(t) + \eta_n(t) n^{3/2} \right|^q \, dt$$

with real numbers $\delta_n(t)$ and $\eta_n(t)$ converging to 0 uniformly on $[0, 2\pi]$.

Lemma 3.6*. *Let (P_n) be an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$, $q \in (0,\infty)$, and $f_n(t) := \mathrm{Re}(P_n(e^{it}))$. Using the notation (1.3) we have*

$$\frac{1}{2\pi}\int_0^{2\pi} |f_n'(t)|^q \, dt$$

$$= \int_0^{2\pi} \left| n^{1/2}(1+\delta_n(t))\sin(\alpha_n(t))\alpha_n'(t) + \eta_n(t) n^{3/2} \right|^q \, dt$$

with real numbers $\delta_n(t)$ converging to 0 uniformly on $[0,2\pi]$.

To prove Lemmas 3.8 and 3.9 we need the technical lemma below that follows by a simple calculation using formulas (6.2.1), (6.2.2), and (6.1.8) on pages 258 and 255 in [AS].

Lemma 3.7. *Assume that $A, B \in \mathbb{R}$, $B \neq 0$, $q > 0$, and $I \subset [0,2\pi]$ is an interval. Then*

$$\int_I |\cos(Bt+A)|^q \, dt = K(q)\mathrm{meas}(I) + \delta_1(A,B,q)$$

and

$$\int_I |\sin(Bt+A)|^q \, dt = K(q)\mathrm{meas}(I) + \delta_2(A,B,q)\,,$$

where

$$K(q) := \frac{1}{2\pi}\int_0^{2\pi} |\sin t|^q \, dt = \frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}}$$

and

$$|\delta_1(A,B,q)| \le \pi\,|B|^{-1} \quad \text{and} \quad |\delta_2(A,B,q)| \le \pi\,|B|^{-1}\,.$$

Our next lemma takes care of the most difficult part of the proof of Theorem 2.1.

Lemma 3.8. *Suppose that β_n, $n = 1,2,\ldots,$ are real-valued functions defined on $[0,2\pi]$ such that their second derivatives β_n'' are continuous on $[0,2\pi]$. Suppose also that*

$$\mathrm{meas}(\{t \in [0,2\pi] : |2\beta_n'(t)| \le nx\}) = \gamma(x) + \gamma_n(x)\,, \qquad x \in [0,1]\,, \tag{3.2}$$

where

$$\lim_{x\to 0+} \gamma(x) = \lim_{n\to\infty} \max_{x\in[0,1]} |\gamma_n(x)| = 0\,, \tag{3.3}$$

and

$$\max_{0\le t\le 2\pi} |\beta_n''(t)| \le \gamma_{n,2} n^2 \tag{3.4}$$

with real numbers $\gamma_{n,2} > 0$ converging to 0. Then

$$\frac{1}{2\pi}\int_0^{2\pi} |\sin(\beta_n(t))|^q \, dt \sim K(q) := \frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}}\,. \tag{3.5}$$

Proof of Lemma 3.8. Let $\varepsilon > 0$ be fixed. Let $L_n := \gamma_{n,2}^{-1/4}$. We divide the interval $[0, 2\pi]$ into subintervals

$$I_m := [a_{m-1}, a_m) := \left[\frac{(m-1)L_n}{n}, \frac{mL_n}{n}\right), \qquad m = 1, 2, \dots, N-1 := \left\lfloor \frac{2\pi n}{L_n} \right\rfloor,$$

and

$$I_N := [a_{N-1}, a_N) := \left[\frac{(N-1)L_n}{n}, 2\pi\right).$$

For the sake of brevity let

$$A_{m-1} := \beta_n(a_{m-1}), \qquad m = 1, 2, \dots, N,$$

and

$$B_{m-1} := \beta_n'(a_{m-1}), \qquad m = 1, 2, \dots, N.$$

Using Taylor's Theorem and assumption (3.4) we obtain that

$$|\beta_n(t) - (A_{m-1} + B_{m-1}(t - a_{m-1})| \le \gamma_{n,2} n^2 (L_n/n)^2 \le \gamma_{n,2}\gamma_{n,2}^{-1/2} \le \gamma_{n,2}^{1/2}$$

for every $t \in I_m$, where $\lim_{n\to\infty} \gamma_{n,2}^{1/2} = 0$. Hence the functions

$$G_{n,q}(t) := \begin{cases} |\sin(A_0 + B_0(t - a_0))|^q, & t \in I_1, \\ |\sin(A_1 + B_1(t - a_0))|^q, & t \in I_2, \\ \quad\vdots & \quad\vdots \\ |\sin(A_{N-1} + B_{N-1}(t - a_{N-1}))|^q, & t \in I_N, \end{cases}$$

and

$$F_{n,q}(t) := |\sin(\beta_n(t))|^q$$

satisfy

$$\lim_{n\to\infty} \sup_{t\in[0,2\pi)} |G_{n,q}(t) - F_{n,q}(t)| = 0. \tag{3.10}$$

Therefore, in order to prove (3.5), it is sufficient to prove that

$$\int_0^{2\pi} G_{n,q}(t)\, dt \sim 2\pi K(q). \tag{3.11}$$

If $|B_{m-1}| \ge n\varepsilon$, then Lemma 3.7 gives

$$\left| \int_{I_m} G_{n,q}(t)\, dt - K(q)\mathrm{meas}(I_m) \right| \le \frac{\pi}{n\varepsilon}.$$

By assumption (3.4) we have $\lim_{n\to\infty} L_n = \lim_{n\to\infty} \gamma_{n,2}^{-1/4} = \infty$, and hence

(3.12)
$$\left|\sum_m \int_{I_m} G_{n,q}(t)\,dt - K(q)\sum_m \mathrm{meas}(I_m)\right| \le N\frac{\pi}{n\varepsilon} \le \left(\frac{2\pi n}{L_n}+1\right)\frac{\pi}{n\varepsilon} \le \eta_n(\varepsilon)\,,$$

where the summation is taken over all $m = 1, 2, \dots, N$ for which $|B_{m-1}| \ge n\varepsilon$, and where $(\eta_n(\varepsilon))$ is a sequence of real numbers tending to 0. Now let

$$E_{n,\varepsilon} := \bigcup_{m:\,|B_{m-1}|\le n\varepsilon} I_m\,.$$

If $|B_{m-1}| \le n\varepsilon$, then by assumption (3.4) we have

$$|\beta_n'(t)| \le |B_{m-1}| + \frac{L_n}{n}\max_{t\in I_m}|\beta_n''(t)| \le |B_{m-1}| + \frac{\gamma_{n,2}^{-1/4}}{n}\gamma_{n,2}n^2 \le 2n\varepsilon$$

for every $t \in I_m$ if n is sufficiently large. So

$$E_{n,\varepsilon} \subset \{t \in [0, 2\pi] : |\beta_n'(t)| \le 2n\varepsilon\}$$

for every sufficiently large n. Hence, by assumptions (3.2) we have

$$\mathrm{meas}(E_{n,\varepsilon}) \le \gamma(4\varepsilon) + \gamma_n(4\varepsilon)$$

for every sufficiently large n. Combining this with $0 \le G_{n,q}(t) \le 1$, $t \in [0, 2\pi)$, we obtain

$$\left|\sum_m \int_{I_m} G_{n,q}(t)\,dt - K(q)\sum_m \mathrm{meas}(I_m)\right| \le (\gamma(4\varepsilon) + \gamma_n(4\varepsilon)(1 + K(q))\,, \tag{3.13}$$

for every sufficiently large n, where the summation is taken over all $m = 1, 2, \dots, N$ for which $|B_{m-1}| < n\varepsilon$, and where $\lim_{\varepsilon\to 0+}\gamma(4\varepsilon) = 0$ and $\lim_{n\to\infty}\gamma_n(4\varepsilon) = 0$ by assumption (3.3). Since $\varepsilon > 0$ is arbitrary, (3.11) follows from (3.12) and (3.13). As we have already pointed out (3.5) follows from (3.11). □

Our final lemma takes care of the most difficult part of the proof of Theorem 2.4.

Lemma 3.9. *Suppose that β_n, $n = 1, 2, \dots,$ are real-valued functions defined on $[0, 2\pi]$ such that their second derivatives β_n'' are continuous on $[0, 2\pi]$ and*

$$\mathrm{meas}(\{t \in [0, 2\pi] : |2\beta_n'(t)| \le nx\}) = 2\pi x + \gamma_n(x)\,, \qquad x \in [0, 1]\,, \tag{3.6}$$

where (3.3) holds (with $\gamma(x) := 2\pi x$). Suppose also that (3.4) holds and

$$\max_{t\in[0,2\pi]} |\beta_n'(t)| \le cn \tag{3.7}$$

with an absolute constant $c > 0$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\cos(\beta_n(t))n^{-1}\beta_n'(t)\right|^q \, dt \sim \frac{K(q)}{2^q(q+1)} := \frac{\Gamma\left(\frac{q+1}{2}\right)}{2^q(q+1)\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}} \,. \tag{3.8}$$

We note that conditions (3.6), (3.3), and (3.7) imply in a standard fashion that

$$\frac{1}{2\pi}\int_0^{2\pi} |2\beta_n'(t)|^q \, dt = \frac{n^q}{q+1} + \delta_{n,q} n^q \tag{3.9}$$

with real numbers $\delta_{n,q}$ converging to 0 for every fixed $q > 0$.

Proof of Lemma 3.9. Let $\varepsilon > 0$ be fixed. Let $L_n := \gamma_{n,2}^{-1/4}$ be the same as in Lemma 3.8. Let the intervals I_m and the numbers A_m and B_m, $m = 1, 2, \dots, N$, be the same as in the proof of Lemma 3.8. We define

$$F_{n,q}(t) := |\cos(\beta_n(t))|^q \,,$$

$$\widetilde{F}_{n,q}(t) := F_{n,q}(t)\left|n^{-1}\beta_n'(t)\right|^q \,,$$

$$G_{n,q}(t) := \begin{cases} |\cos(A_0 + B_0(t-a_0))|^q, & t \in I_1 \,, \\ |\cos(A_1 + B_1(t-a_0))|^q, & t \in I_2 \,, \\ \quad\vdots & \quad\vdots \\ |\cos(A_{N-1} + B_{N-1}(t-a_{N-1}))|^q, & t \in I_N \,, \end{cases}$$

$$H_{n,q}(t) := \begin{cases} |n^{-1}B_0|^q, & t \in I_1 \,, \\ |n^{-1}B_1|^q, & t \in I_2 \,, \\ \quad\vdots & \quad\vdots \\ |n^{-1}B_{N-1}|^q, & t \in I_N \,, \end{cases}$$

and

$$\widetilde{G}_{n,q}(t) := G_{n,q}(t)H_{n,q}(t) \,. \tag{3.14}$$

Similarly to the corresponding argument in the proof of Lemma 3.8, we obtain

$$\lim_{n\to\infty} \sup_{t\in[0,2\pi)} |G_{n,q}(t) - F_{n,q}(t)| = 0 \,. \tag{3.15}$$

It follows from assumption (3.4) that

$$\begin{aligned} \left|\left|n^{-1}\beta_n'(t)\right| - \left|n^{-1}B_{m-1}\right|\right| &= \left|\left|n^{-1}\beta_n'(t)\right| - \left|n^{-1}\beta_n'(a_{m-1})\right|\right| \\ &\le \left|n^{-1}\beta_n'(t) - n^{-1}\beta_n'(a_{m-1})\right| \\ &\le \frac{L_n}{n}\max_{t\in I_m}\left|n^{-1}\beta_n''(t)\right| \le \frac{\gamma_{n,2}^{-1/4}}{n} n^{-1}\gamma_{n,2}n^2 = \gamma_{n,2}^{3/4} \end{aligned}$$

for every $t \in I_m$, where $\lim_{n\to\infty} \gamma_{n,2}^{3/4} = 0$. Hence

$$\lim_{n\to\infty} \sup_{t\in[0,2\pi)} \left| H_{n,q}(t) - \left|n^{-1}\beta_n'(t)\right|^q \right| = 0\,. \tag{3.16}$$

Observe that

$$\sup_{t\in[0,2\pi)} |\cos(\beta_n(t)|^q \le 1\,, \tag{3.17}$$

and by the assumption (3.7) we have

$$\sup_{t\in[0,2\pi)} \left|n^{-1}\beta_n'(t)\right|^q \le c^q\,. \tag{3.18}$$

Now (3.14), (3.16), (3.17), (3.18), and (3.14) imply

$$\lim_{n\to\infty} \sup_{t\in[0,2\pi)} |\widetilde{G}_{n,q}(t) - \widetilde{F}_{n,q}(t)| = 0\,.$$

Therefore, in order to prove (3.8), it is sufficient to prove that

$$\int_0^{2\pi} \widetilde{G}_{n,q}(t)\,dt \sim \frac{2\pi K(q)}{q+1}\,. \tag{3.19}$$

As a special case of (3.18), we have

$$\left|n^{-1}B_{m-1}\right|^q \le c^q\,, \qquad m = 1, 2, \dots, N\,. \tag{3.20}$$

If $|B_{m-1}| \ge n\varepsilon$, then (3.14), (3.20), and Lemma 3.7 give that

$$\left| \int_{I_m} \widetilde{G}_{n,q}(t)\,dt - K(q)\mathrm{meas}(I_m)\left|n^{-1}B_{m-1}\right|^q \right| \le c^q \frac{\pi}{n\varepsilon}\,.$$

It follows from assumption (3.4) that $\lim_{n\to\infty} L_n = \lim_{n\to\infty} \gamma_{n,2}^{-1/4} = \infty$, and hence

(3.21)
$$\begin{aligned}\left| \sum_m \int_{I_m} G_{n,q}(t)\,dt - K(q)\sum_m \mathrm{meas}(I_m)\left|n^{-1}B_{m-1}\right|^q \right| &\le Nc^q \frac{\pi}{n\varepsilon} \\ &\le c^q \left(\frac{2\pi n}{L_n} + 1\right)\frac{\pi}{n\varepsilon} \\ &\le \eta_n(\varepsilon, q)\,,\end{aligned}$$

where the summation is taken over all $m = 1, 2, \dots, N$ for which $|B_{m-1}| \ge n\varepsilon$, and where $(\eta_n(\varepsilon, q))$ is a sequence of real numbers tending to 0 for every fixed $\varepsilon > 0$ and $q > 0$. Now let

$$E_{n,\varepsilon} := \bigcup_{m:\,|B_{m-1}|\le n\varepsilon} I_m\,.$$

As in the proof of Lemma 3.8 we have

$$\operatorname{meas}(E_{n,\varepsilon}) \le 8\pi\varepsilon + \gamma_n(4\varepsilon)\,,$$

for every sufficiently large n. Combining this with (3.17) and (3.20), and recalling the definition of $\widetilde{G}_{n,q}$, we obtain

$$\left|\sum_m \int_{I_m} \widetilde{G}_{n,q}(t)\,dt - K(q)\sum_m \operatorname{meas}(I_m)\left|n^{-1}B_{m-1}\right|^q\right| \le (8\pi\varepsilon+\gamma_n(4\varepsilon))c^q(1+K(q)) \tag{3.22}$$

for every sufficiently large n, where the summation is taken over all $m = 1, 2, \dots, N$ for which $|B_{m-1}| < n\varepsilon$, and where

$$\lim_{n\to\infty} \gamma_n(4\varepsilon)| = 0$$

by assumption (3.3). Since $\varepsilon > 0$ is arbitrary, from (3.21) and (3.22) we obtain that

$$\int_0^{2\pi} \widetilde{G}_{n,q}(t)\,dt \sim K(q)\int_0^{2\pi} H_{n,q}(t)\,dt\,. \tag{3.23}$$

However, (3.16) and (3.9) imply that

$$\int_0^{2\pi} H_{n,q}(t)\,dt \sim n^{-q}\int_0^{2\pi} |\beta_n'(t)|^q\,dt \sim \frac{2\pi}{2^q(q+1)}\,. \tag{3.24}$$

The statement under (3.19) now follows by combining (3.23), and (3.24). As we have remarked before, (3.19) implies (3.8). □

4. Proofs of Theorems 2.1–2.7

Proof of Theorem 2.1. Using the notation (1.3) observe that (1.5) implies that the functions

$$\beta_n(t) := \frac{1}{2}(\alpha_n(t) - \alpha_n^*(t)) = \alpha_n(t) - \frac{nt}{2} - t_0 \tag{4.1}$$

satisfy

$$\beta_n'(t) = \alpha_n'(t) - n/2\,, \qquad t \in \mathbb{R}\,,$$

and

$$\beta_n''(t) = \alpha_n''(t)\,, \qquad t \in \mathbb{R}\,,$$

and hence Lemmas 3.1, 3.2, and 3.3 imply that the functions β_n satisfy assumptions (3.2), (3.3), and (3.4) of Lemma 3.8. Hence the theorem follows from Lemmas 3.5 and 3.8. □

Proof of Theorem 2.3. Using the notation (1.3) let

$$P_n(e^{it}) = R_n(t)e^{i\alpha_n(t)} \qquad \text{and} \qquad R_n^*(t)e^{i\alpha_n^*(t)}\,,$$

where

$$R_n(t) = |P_n(e^{it})| = |P_n^*(e^{it})| = R_n^*(t)\,,$$

and both R_n and α_n are in $C^\infty(\mathbb{R})$. Let (P_n) be an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. We have

$$R_n(t)^2 = n(1+\delta_n(t))\,, \qquad t \in \mathbb{R}\,, \quad \lim_{n\to\infty} \max_{t\in[0,2\pi]} |\delta_n(t)| = 0\,. \tag{4.2}$$

Let $\beta_n(t)$ be defined by (4.1). Let $\varepsilon > 0$ be fixed. Using the cosine theorem for triangles, Lemmas 3.2, and 3.4, and (4.1), (4.2), and (1.5), we obtain

(4.3)

$$\begin{aligned}
&\left|(P_n' - P_n^{*\prime}(e^{it})\right|^2 - \left|P_n'(e^{it})\right|^2 - \left|P_n^{*\prime})(e^{it})\right|^2 \\
&= 2((R_n(t)\alpha_n'(t))^2 + (R_n'(t))^2)^{1/2}(R_n(t)(\alpha_n^{*\prime}(t))^2 + (R_n'(t))^2)^{1/2}\cos(2\beta_n(t)) \\
&= 2(R_n(t)\alpha_n'(t))(R_n(t)\alpha_n^{*\prime}(t))\cos(2\beta_n(t)) + \eta_n(t)n^3 \\
&= 2(R_n(t)^2(n\alpha_n'(t) - \alpha_n'(t)^2)\cos(2\beta_n(t)) + \eta_n(t)n^3 \\
&= 2(n+\delta_n(t))(n\alpha_n'(t) - \alpha_n'(t)^2)\cos(2\beta_n(t)) + \eta_n(t)n^3 \\
&= 2n(n\alpha_n'(t) - \alpha_n'(t)^2)\cos(2\beta_n(t)) + \varphi_n(t)n^3 + \eta_n(t)n^3 \\
&= 2n\left(\frac{n^2}{4} - \beta_n'(t)^2\right)\cos(2\beta_n(t)) + \varphi_n(t)n^3 + \eta_n(t)n^3
\end{aligned}$$

with some real numbers $\eta_n(t)$ and $\varphi_n(t)$ satisfying

$$\max_{t\in[0,2\pi]} |\varphi_n(t) + \eta_n(t)| < \varepsilon$$

for every sufficiently large n. Observe that

$$\int_0^{2\pi} \left(\left|P_n'(e^{it})\right|^2 + \left|P_n^{*\prime}(e^{it})\right|^2\right) dt = 4\pi\,\frac{n(n+1)(2n+1)}{6} \tag{4.4}$$

by the Parseval formula. The integration by parts formula and Lemma 3.3 give that

(4.5)

$$\begin{aligned}
&\left|\int_0^{2\pi} \beta_n'(t)^2\cos(2\beta_n(t))\,dt\right| \\
&= \left|\left[\frac{1}{2}(\sin(2\beta_n(t))\beta_n'(t))\right]_0^{2\pi} - \int_0^{2\pi} \beta_n''(t)\sin(2\beta_n(t))\,dt\right| \\
&= \left|\int_0^{2\pi} \beta_n''(t)\sin(2\beta_n(t))\,dt\right| \le \int_0^{2\pi} |\beta_n''(t)|\,dt = \int_0^{2\pi} |\alpha_n''(t)|\,dt \\
&\le 2\pi\gamma_{n,2}n^2 < \varepsilon n^2
\end{aligned}$$

for every sufficiently large n. Observe also that Lemma 3.8 gives that

$$\int_0^{2\pi} \cos(2\beta_n(t))\,dt = \int_0^{2\pi} \left(2\sin^2(\beta_n(t)) - 1\right)\,dt = 2\pi(2K(2) + h_n - 1) \tag{4.6}$$
$$= 2\pi\left(2\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)\sqrt{\pi}} + h_n - 1\right) = 2\pi h_n$$

with a sequence (h_n) converging to 0. Combining (4.3)–(4.6) we conclude

$$\left|\frac{1}{2\pi}\int_0^{2\pi} \left|(P_n' - P_n^{*\prime})(e^{it})\right|^2\,dt - \frac{2n^3}{3}\right| \le \varepsilon n^3$$

for every sufficiently large n. As $\varepsilon > 0$ is arbitrary, this finishes the proof. □

Proof of Theorem 2.4. Using the notation (1.3) observe that (1.5) implies that the functions β_n defined by (4.1) satisfy

$$\beta_n'(t) = \alpha_n'(t) - n/2\,, \qquad t \in \mathbb{R}\,,$$

and

$$\beta_n''(t) = \alpha_n''(t)\,, \qquad t \in \mathbb{R}\,,$$

and hence Lemmas 3.1, 3.2, and 3.3 imply that the functions β_n satisfy assumptions (3.6), (3.3), (3.4), and (3.7) of Lemma 3.9. Hence the theorem follows from Lemmas 3.6 and 3.9. □

Proof of Theorem 2.5. Let (P_n) be an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Theorem 2.2 gives that

$$\sum_{k=0}^{n} |a_{k,n} - \overline{a}_{n-k,n}|^2 \sim 2n\,,$$

which is equivalent to

$$2\mathrm{Re}\left(\sum_{k=0}^{n} a_{k,n}a_{n-k,n}\right) = o(n)\,.$$

Now let $c \in \mathbb{C}$, $|c| = 1$, and let Q_n be defined by $Q_n(z) = P_n(cz)$. Observe that (Q_n) is an ultraflat sequence of polynomials $Q_n \in \mathcal{K}_n$ and hence

$$\mathrm{Re}\left(\sum_{k=0}^{n} c^n a_{k,n}a_{n-k,n}\right) = o(n)\,,$$

and hence

$$\left|\sum_{k=0}^{n} a_{k,n}a_{n-k,n}\right| = o(n)\,.$$

□

Proof of Theorem 2.6. Let (P_n) be an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Theorem 2.3 gives that

$$\sum_{k=0}^{n} k^2 \left|a_{k,n} - \overline{a}_{n-k,n}\right|^2 \sim \frac{2n^3}{3}\,,$$

which is equivalent to

$$2\mathrm{Re}\left(\sum_{k=0}^{n} k^2 a_{k,n} a_{n-k,n}\right) = o(n^3)\,.$$

Now let $c \in \mathbb{C}$, $|c| = 1$, and let Q_n be defined by $Q_n(z) = P_n(cz)$. Observe that (Q_n) is an ultraflat sequence of polynomials $Q_n \in \mathcal{K}_n$ and hence

$$\mathrm{Re}\left(\sum_{k=0}^{n} c^n k^2 a_{k,n} a_{n-k,n}\right) = o(n^3)\,,$$

and hence

$$\left|\sum_{k=0}^{n} k^2 a_{k,n} a_{n-k,n}\right| = o(n^3)\,.$$

□

Proof of Theorem 2.7.. Using Lemma 3.5* and then applying Lemma 3.8 with β_{2n} defined by $\beta_{2n}(t) := \alpha_n(t) + \pi/2$ we obtain the first asymptotic formula of the theorem. Using Lemma 3.6* and then applying Lemma 3.9 with β_{2n} defined by $\beta_{2n}(t) := \alpha_n(t) + \pi/2$ we obtain the second asymptotic formula of the theorem. □

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