

REMEZ-TYPE INEQUALITY FOR NON-DENSE MÜNTZ SPACES WITH EXPLICIT BOUND

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ABSTRACT. Let $\Lambda := (\lambda_k)_{k=0}^\infty$ be a sequence of distinct nonnegative real numbers with $\lambda_0 := 0$ and $\sum_{k=1}^\infty 1/\lambda_k < \infty$. Let $\varrho \in (0, 1)$ and $\epsilon \in (0, 1 - \varrho)$ be fixed. An earlier work of the first two authors shows that

$$C(\Lambda, \epsilon, \varrho)$$

$$:= \sup \left\{ \|p\|_{[0, \varrho]} : p \in \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}, m\{x \in [\varrho, 1] : |p(x)| \leq 1\} \geq \epsilon \right\}$$

is finite. In this paper an explicit upper bound for $C(\Lambda, \epsilon, \varrho)$ is given. In the special case $\lambda_k := k^\alpha$, $\alpha > 1$, our bounds are essentially sharp.

1. INTRODUCTION

In this paper $\Lambda := (\lambda_k)_{k=0}^\infty$ always denotes a sequence of real numbers satisfying

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots$$

In [1] a Remez-type inequality for Müntz polynomials:

$$p(x) = \sum_{k=0}^n a_k x^{\lambda_k}$$

or equivalently for Dirichlet sums:

$$P(t) = \sum_{k=0}^n a_k e^{-\lambda_k t}$$

is established. The most common form of this inequality states that for every sequence $(\lambda_k)_{k=0}^\infty$ satisfying $\sum_{k=0}^\infty 1/\lambda_k < \infty$, there exists a constant $C(\Lambda, \epsilon)$ depending only on Λ and ϵ (and not on n , ϱ , or A) so that

$$\|p\|_{[0, \varrho]} \leq C(\Lambda, \epsilon) \|p\|_A$$

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for every Müntz polynomial p , as above, associated with the sequence $(\lambda_k)_{k=0}^{\infty}$, and for every set $A \subset [\varrho, 1]$ of Lebesgue measure at least $\epsilon > 0$. Throughout this paper $\|\cdot\|_A$ denotes the uniform norm on $A \subset \mathbb{R}$.

Using this Remez-type inequality, we resolved two reasonably long standing conjectures in [1]. In this paper we give an explicit upper bound for the best possible $C(\Lambda, \epsilon)$ in the above Remez-type inequality for non-dense Müntz spaces. Theorem 2.3 extends an inequality of Schwartz [4] in two directions. Theorem 2.1 offers a more explicit bound for the sequences $\Lambda := (k^\alpha)_{k=0}^{\infty}$, $\alpha > 1$. The sharpness of the Remez-type inequality of Theorem 2.1 is shown by Theorem 2.2.

2. RESULTS

Theorem 2.1. *Let $\lambda_k := k^\alpha$, $k = 0, 1, \dots$, $\alpha > 1$. Let $\varrho \in (0, 1)$, $\epsilon \in (0, 1 - \varrho)$, and $\epsilon \leq 1/2$. There exists a constant $c_\alpha > 0$ depending only on α so that*

$$\|p\|_{[0, \varrho]} \leq \exp\left(c_\alpha \epsilon^{1/(1-\alpha)}\right) \|p\|_A$$

for every $p \in \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ and for every set $A \subset [\varrho, 1]$ of Lebesgue measure at least $\epsilon > 0$.

The next theorem shows that the inequality of Theorem 2.1 is essentially the best possible.

Theorem 2.2. *Let $\lambda_k := k^\alpha$, $k = 0, 1, \dots$, $\alpha > 1$. For every $\alpha > 1$ and $\epsilon \in (0, 1/2]$, there exists a constant $c_\alpha > 0$ depending only on α and Müntz polynomials*

$$0 \neq p = p_{\alpha, \epsilon} \in \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

depending only on α and ϵ so that

$$|p(0)| \geq \exp\left(c_\alpha \epsilon^{1/(1-\alpha)}\right) \|p\|_{[1-\epsilon, 1]}.$$

Theorem 2.1 is a special case of the following more general, but less explicit result.

Theorem 2.3. *Suppose $0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots$ and $\sum_{k=0}^{\infty} 1/\lambda_k < \infty$. Let $\varrho \in (0, 1)$ and $\epsilon \in (0, 1 - \varrho)$. Let $\delta := -\frac{1}{2} \log(1 - \epsilon)$. Let $N \in \mathbb{N}$ be chosen so that*

$$\sum_{k=N+1}^{\infty} \frac{1}{\lambda_k} \leq \frac{\delta}{3}.$$

Let

$$\sigma_k := A\lambda_k \quad \text{with} \quad A := \frac{\delta}{3N}.$$

Then, with $c := \|t^{-1} \sin t\|_{L_2(\mathbb{R})}$,

$$\|p\|_{[0, \varrho]} \leq \frac{3c}{\delta} \prod_{k=1}^N \left(2 + \frac{1}{\sigma_k}\right) \|p\|_A$$

for every $p \in \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ and for every set $A \subset [\varrho, 1]$ of Lebesgue measure at least $\epsilon > 0$.

3. LEMMAS

Our first lemma shows that $C(\Lambda, \epsilon)$ in the Remez-type inequality is related to a much simpler (Chebyshev-type) extremal problem. This is proved in both [1] and [2].

Lemma 3.1. *Suppose $0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots$, $\rho \in (0, 1)$, and $\epsilon \in (0, 1 - \rho)$. Then*

$$\begin{aligned} & \sup \{ \|p\|_{[0, \varrho]} : p \in \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}, m\{x \in [\varrho, 1] : |p(x)| \leq 1\} \geq \epsilon \} \\ &= \sup \left\{ \frac{|p(0)|}{\|p\|_{[1-\epsilon, 1]}} : p \in \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\} \right\}. \end{aligned}$$

Our key lemma is the following.

Lemma 3.2. *Suppose $0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots$ and $\sum_{k=0}^{\infty} 1/\lambda_k < \infty$. Given $\delta \in (0, 1)$, let $N \in \mathbb{N}$ be chosen so that*

$$\sum_{k=N+1}^{\infty} \frac{1}{\lambda_k} \leq \frac{\delta}{3}.$$

Let

$$\sigma_k := A\lambda_k \quad \text{with} \quad A := \frac{\delta}{3N}.$$

Then

$$|P(\infty)| \leq \frac{3c}{\delta} \prod_{k=1}^N \left(2 + \frac{1}{\sigma_k} \right) \|P\|_{[-\delta, \delta]}$$

for every $P \in \text{span}\{e^{-\lambda_0 t}, e^{-\lambda_1 t}, \dots\}$ with $c := \|t^{-1} \sin t\|_{L_2(\mathbb{R})}$.

In the proof of Lemma 3.2 we will need the following observation.

Lemma 3.3. *Let $0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots$. Suppose*

- (1) $F \in E^\delta \cap L_2(\mathbb{R})$;
- (2) $F(i\lambda_k) = 0$, $k = 1, 2, \dots$ (i is the imaginary unit);
- (3) $F(0) = 1$.

Then

$$|P(\infty)| \leq \|F\|_{L_2(\mathbb{R})} \|P\|_{L_2[-\delta, \delta]}$$

for every $P \in \text{span}\{e^{-\lambda_0 t}, e^{-\lambda_1 t}, \dots\}$.

An entire function f is called a function of *exponential type* δ if there exists a constant c depending only on f so that

$$|f(z)| \leq c \exp(\delta|z|), \quad z \in \mathbb{C}.$$

The collection of all such entire functions of exponential type δ is denoted by E^δ . The Paley-Wiener Theorem (see, for example, [3]) characterizes the functions F which can be written as the Fourier transform of some function $f \in L_2[-\delta, \delta]$. We will need it in the proof of Lemma 3.3.

Theorem (Paley-Wiener). *Let $\delta \in (0, \infty)$. Then $f \in E^\delta \cap L_2(\mathbb{R})$ if and only if there exists an $f \in L_2[-\delta, \delta]$ so that*

$$F(z) = \int_{-\delta}^{\delta} f(t)e^{itz} dt.$$

The following comparison theorem for Müntz polynomials is proved in [2]. We will need it in the proof of Theorem 2.3.

Lemma 3.4. *Let $\Lambda := (\lambda_k)_{k=0}^\infty$ and $\Gamma := (\gamma_k)_{k=0}^\infty$ be increasing sequences of non-negative real numbers with $\lambda_0 = 0$, $\gamma_0 = 0$, and $\lambda_k \leq \gamma_k$ for each k . Let $0 < a < b$. Then*

$$\begin{aligned} & \max \left\{ \frac{|p(0)|}{\|p\|_{[a,b]}} : p \in \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\} \right\} \\ & \geq \max \left\{ \frac{|p(0)|}{\|p\|_{[a,b]}} : p \in \text{span}\{x^{\gamma_0}, x^{\gamma_1}, \dots, x^{\gamma_n}\} \right\}. \end{aligned}$$

4. PROOFS

Proof of Lemma 3.3. By the Paley-Wiener Theorem

$$F(z) = \int_{-\delta}^{\delta} f(t)e^{itz} dt$$

for some $f \in L_2[-\delta, \delta]$. Now if

$$P(t) = a_0 + \sum_{k=1}^n a_k e^{-\lambda_k t},$$

then

$$\begin{aligned} \int_{-\delta}^{\delta} f(t)P(t) dt &= a_0 \int_{-\delta}^{\delta} f(t) dt + \sum_{k=1}^n a_k \int_{-\delta}^{\delta} f(t)e^{-\lambda_k t} dt \\ &= a_0 F(0) + \sum_{k=1}^n a_k F(i\lambda_k) = a_0 = P(\infty). \end{aligned}$$

Hence by the Cauchy-Schwartz Inequality and the L_2 inversion theorem of Fourier transforms, we obtain

$$|P(\infty)| \leq \|f\|_{L_2[-\delta, \delta]} \|P\|_{L_2[-\delta, \delta]} \leq \|F\|_{L_2(\mathbb{R})} \|P\|_{L_2[-\delta, \delta]}$$

and the lemma is proved. \square

Proof of Lemma 3.2. We define

$$F(z) := \frac{\sin(\delta z/3)}{\delta z/3} \prod_{k=1}^N \left(\left(1 - \frac{z}{i\lambda_k} \right) \frac{\sin(\sigma_k z/\lambda_k)}{\sigma_k z/\lambda_k} \right) \prod_{k=N+1}^{\infty} \left(1 - \left(\frac{\sin(z/\lambda_k)}{\sin i} \right)^4 \right),$$

where i is the imaginary unit. It follows easily that

$$F \in E^\delta, \quad F(0) = 1, \quad F(i\lambda_k) = 0, \quad k = 1, 2, \dots$$

and

$$|F(t)| \leq \frac{\sin(\delta t/3)}{(\delta t/3)} \prod_{k=1}^N \left(2 + \frac{1}{\sigma_k}\right), \quad t \in \mathbb{R}.$$

Hence Lemma 3.3 implies that

$$|P(\infty)| \leq \frac{3c}{\delta} \prod_{k=1}^N \left(2 + \frac{1}{\sigma_k}\right) \|P\|_{[-\delta, \delta]}$$

for every $P \in \text{span}\{e^{-\lambda_0 t}, e^{-\lambda_1 t}, \dots\}$ with $c := \|t^{-1} \sin t\|_{L_2(\mathbb{R})}$. \square

Proof of Theorem 2.3. When $A = [1 - \epsilon, 1]$, the theorem follows from Lemma 3.2 by the substitution $x = e^{-\delta} e^{-t}$. The general case follows from Lemma 3.1. \square

Proof of Theorem 2.1. Let

$$(4.1) \quad \delta := -\frac{1}{2} \log(1 - \epsilon).$$

Observe that N in Theorem 2.1 can be chosen so that

$$(4.2) \quad N := \left\lceil \left(\frac{\delta(\alpha - 1)}{3} \right)^{1/(1-\alpha)} \right\rceil + 1.$$

Also, σ_k in Lemma 3.2 is of the form

$$\sigma_k = \frac{\delta k^\alpha}{3N}.$$

Let $M + 1$ be the smallest value of $k \in \mathbb{N}$ for which

$$\frac{1}{\sigma_k} < 1, \quad \text{that is,} \quad \frac{3N}{k^\alpha \delta} \leq 1.$$

Note that

$$M := \left\lceil \left(\frac{3N}{\delta} \right)^{1/\alpha} \right\rceil.$$

If $0 < M < N$, then

$$\begin{aligned} \prod_{k=1}^N \left(2 + \frac{1}{\sigma_k}\right) &= \prod_{k=1}^N \left(2 + \frac{3N}{\delta k^\alpha}\right) \\ &\leq \left(\prod_{k=1}^M \frac{9N}{\delta k^\alpha} \right) \left(\prod_{k=M+1}^N 3 \right) = \left(\frac{9N}{\delta} \right)^M \left(\frac{M}{e} \right)^{-\alpha M} 3^{N-M} \\ &= \left(\frac{9e^\alpha N}{\delta} \right)^M M^{-\alpha M} 3^{N-M} \\ &\leq \left(\frac{9e^\alpha N}{\delta} \right)^M \left(\frac{1}{2} \left(\frac{3N}{\delta} \right)^{1/\alpha} \right)^{-\alpha M} 3^{N-M} \\ &\leq (3(2e)^\alpha)^M 3^{N-M} \leq (3(2e)^\alpha)^N, \end{aligned}$$

and the theorem follows by (4.1), (4.2), and Theorem 2.1.

If $N \leq M$, then

$$\begin{aligned} \prod_{k=1}^N \left(2 + \frac{1}{\sigma_k}\right) &= \prod_{k=1}^N \left(2 + \frac{3N}{\delta k^\alpha}\right) \\ &\leq \left(\prod_{k=1}^N \frac{9N}{\delta k^\alpha}\right) = \left(\frac{9N}{\delta}\right)^N \left(\frac{N}{e}\right)^{-\alpha N} \\ &= \left(\frac{9e^\alpha N^{(1-\alpha)}}{\delta}\right)^N \leq \left(\frac{9e^\alpha}{\delta}\right)^N \left(\left(\frac{\delta(\alpha-1)}{3}\right)^{1/(1-\alpha)}\right)^{(1-\alpha)N} \\ &\leq \left(\frac{9e^\alpha}{\delta}\right)^N \left(\frac{\delta(\alpha-1)}{3}\right)^N \leq (3e^\alpha(\alpha-1))^N, \end{aligned}$$

and the theorem follows by (4.1), (4.2), and Theorem 2.1.

If $M = 0$, then

$$\prod_{k=1}^N \left(2 + \frac{1}{\sigma_k}\right) \leq \prod_{k=1}^N 3 = 3^N,$$

and the theorem follows by (4.1), (4.2), and Theorem 2.1. \square

Proof of Theorem 2.2. Let $n \in \mathbb{N}$ be a fixed. We define $\gamma_k := kn^{\alpha-1}$, $k = 0, 1, \dots$. Let $T_n(x) := \left(\frac{1}{2}(x-1)\right)^n$ and

$$Q_n(x) := T_n \left(\frac{2x^{n^{\alpha-1}}}{1 - (1-\epsilon)^{n^{\alpha-1}}} - \frac{1 + (1-\epsilon)^{n^{\alpha-1}}}{1 - (1-\epsilon)^{n^{\alpha-1}}} \right)^n \in \text{span}\{x^{\gamma_0}, x^{\gamma_1}, \dots, x^{\gamma_n}\}.$$

Then, by Lemma 3.4,

$$\begin{aligned} \sup \left\{ \frac{|p(0)|}{\|p\|_{[1-\epsilon, 1]}} : p \in \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\} \right\} &\geq \frac{|Q_n(0)|}{\|Q_n\|_{[1-\epsilon, 1]}} = |Q_n(0)| \\ &= \left(\frac{1}{1 - (1-\epsilon)^{n^{\alpha-1}}} \right)^n. \end{aligned}$$

Now let n be the smallest integer satisfying $n^{\alpha-1} \geq \epsilon^{-1}$. Since $(1-\epsilon)^{1/\epsilon}$ is bounded away from 0 on $(0, 1/2]$, the result follows. \square

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