FLATNESS OF CONJUGATE RECIPROCAL UNIMODULAR POLYNOMIALS

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Abstract. A polynomial is called unimodular if each of its coefficients is a complex number of modulus 1. A polynomial $P$ of the form $P(z) = \sum_{j=0}^{n} a_j z^j$ is called conjugate reciprocal if $a_{n-j} = \overline{a}_j$, $a_j \in \mathbb{C}$ for each $j = 0, 1, \ldots, n$. Let $\partial D$ be the unit circle of the complex plane. We prove that there is an absolute constant $\varepsilon > 0$ such that

$$\max_{z \in \partial D} |f(z)| \geq (1 + \varepsilon) \sqrt{4/3} m^{1/2},$$

for every conjugate reciprocal unimodular polynomial of degree $m$. We also prove that there is an absolute constant $\varepsilon > 0$ such that there is an absolute constant $\varepsilon > 0$ such that

$$M_q(f') \leq \exp\left(\varepsilon(q-2)/q\right) \sqrt{1/3} m^{3/2}, \quad 1 \leq q < 2,$$

and

$$M_q(f') \geq \exp\left(\varepsilon(q-2)/q\right) \sqrt{1/3} m^{3/2}, \quad 2 < q,$$

for every conjugate reciprocal unimodular polynomial of degree $m$, where

$$M_q(g) = \left(\frac{1}{2\pi} \int_0^{2\pi} |g(e^{it})|^q dt\right)^{1/q}, \quad q > 0.$$

1. Introduction

Let $\mathcal{T}_n$ be the set of all real trigonometric polynomials of degree at most $n$. Let $\mathcal{P}^c_n$ be the set of all algebraic polynomials of degree at most $n$ with complex coefficients. Throughout this paper it will be comfortable for us to denote an appropriate period $[a, a + 2\pi)$ by $K$. Let

$$\mathcal{A}_n := \left\{ Q : Q(t) = \sum_{j=1}^{n} \cos(jt + \gamma_j), \quad \gamma_j \in \mathbb{R} \right\}.$$
and
\[
B_{n+1/2} := \left\{ Q : Q(t) = \sum_{j=1}^{n} \cos \left( \frac{2j + 1}{2} t + \gamma_j \right), \; \gamma_j \in \mathbb{R} \right\}.
\]

We use the notation
\[
\|Q\|_p := \left( \frac{1}{2\pi} \int_{K} |Q(t)|^p \right)^{1/p}, \quad p > 0,
\]
and
\[
\|Q\|_\infty := \max_{t \in K} |Q(t)|.
\]
The Bernstein-Szegő inequality (see page 232 in [5], for instance) gives that
\[
|Q'(t)|^2 + n^2 |Q(t)|^2 \leq n^2 \|Q\|_\infty^2, \quad Q \in T_n, \quad t \in \mathbb{R}.
\]

Integrating the left hand side on the period and using Parseval’s formula we obtain
\[
\frac{n(n+1)(2n+1)}{12} + \frac{n^3}{2} \leq n^2 \|Q\|_\infty^2, \quad Q \in A_n,
\]
and hence
\[
(1.1) \quad \|Q\|_\infty \geq \sqrt{4/3} \sqrt{n/2}, \quad Q \in A_n.
\]

One of the highlights of this paper to improve (1.1) by showing that there is an absolute constant \(\varepsilon > 0\) such that
\[
\|Q\|_\infty \geq (1 + \varepsilon) \sqrt{4/3} \sqrt{n/2}, \quad Q \in A_n.
\]

Let
\[
\mathcal{K}_m := \left\{ P : P(z) = \sum_{j=0}^{n} a_j z^j, \quad a_j \in \mathbb{C}, \; |a_j| = 1, \; j = 0, 1, \ldots, m \right\}
\]
be the set of all unimodular polynomials of degree \(m\). Associated with an algebraic polynomial \(P\) of the form
\[
P(z) = \sum_{j=0}^{m} a_j z^j, \quad a_j \in \mathbb{C}, \; a_m \neq 0,
\]
let
\[
\overline{P}(z) := \sum_{j=0}^{m} \overline{a_j} z^j \quad \text{and} \quad P^*(z) := z^m \overline{P}(1/z).
\]
The polynomial $P$ of degree $n$ is called conjugate reciprocal if $P^* = P$. The classes $A_n$, $B_{n+1/2}$, and $K_m$ and flatness properties of their elements were studied by many authors, see [1–40], for instance. Let

$$M_q(f) := \| f(e^{it}) \|_q = \left( \frac{1}{2\pi} \int_{K} |f(e^{it})|^q dt \right)^{1/q}, \quad q \in (0, \infty)$$

and

$$M_\infty(f) := \sup_{t \in K} |f(e^{it})|.$$

There is a beautiful short argument to see that

$$(1.2) \quad M_\infty(f) \geq \sqrt{4/3} m^{1/2}$$

for every conjugate reciprocal unimodular polynomial $f \in K_m$. Namely, Parseval’s formula gives

$$M_\infty(f') \geq M_2(f') \leq \left( \frac{m(m+1)(2m+1)}{3} \right)^{1/2}, \quad f \in K_m.$$

Combining this with Lax’s Bernstein type-inequality

$$M_\infty(f') \leq \frac{m}{2} M_\infty(f)$$

valid for all conjugate reciprocal algebraic polynomials $f \in P_n^c$ (see p. 438 in [5], for instance), we obtain

$$M_\infty(f) \geq \frac{2}{m} \left( \frac{m(m+1)(2m+1)}{3} \right)^{1/2} \geq \sqrt{4/3} m^{1/2}$$

for all $f \in K_m$. One of the highlights of this paper to improve (1.2) by showing that there is an absolute constant $\varepsilon > 0$ such that

$$M_\infty(f) \geq (1 + \varepsilon) \sqrt{4/3} m^{1/2},$$

for every conjugate reciprocal unimodular polynomial $f \in K_m$. We also prove that there is an absolute constant $\varepsilon > 0$ such that

$$M_q(f') \leq (1 - \varepsilon(2 - q)) \sqrt{1/3} m^{3/2}, \quad 1 < q < 2,$$

and

$$M_q(f') \geq (1 + \varepsilon(q - 2)) \sqrt{1/3} m^{3/2}, \quad 2 < q,$$

for every conjugate reciprocal unimodular polynomial of degree $m$. See Theorem 2.7.
2. New Results

**Theorem 2.1.** Let $Q \in A_n$ and $P = (Q')^2 + n^2 Q^2$. There is an absolute constant $\delta > 0$ such that
\[ \|P\|_{1/2} \leq (1 - \delta)\|P\|_1. \]

**Theorem 2.1*.** Let $Q \in B_{n+1/2}$ and $P = (Q')^2 + (n + 1/2)^2 Q^2$. There is an absolute constant $\delta > 0$ such that
\[ \|P\|_{1/2} \leq (1 - \delta)\|P\|_1. \]

**Theorem 2.2.** Let $Q \in A_n$ and $P = (Q')^2 + n^2 Q^2$. There is an absolute constant $\delta > 0$ such that
\[ \|P\|_{\infty} \geq (1 + \delta)\|P\|_1. \]

**Theorem 2.2*.** Let $Q \in B_{n+1/2}$ and $P = (Q')^2 + (n + 1/2)^2 Q^2$. There is an absolute constant $\delta > 0$ such that
\[ \|P\|_{\infty} \geq (1 + \delta)\|P\|_1. \]

**Theorem 2.3.** There is an absolute constant $\delta > 0$ such that
\[ \|Q\|_{\infty} \geq (1 + \delta)^{\sqrt{4/3} \sqrt{n/2}} \]
for every $Q \in A_n$.

**Theorem 2.3*.** There is an absolute constant $\delta > 0$ such that
\[ \|Q\|_{\infty} \geq (1 + \delta)^{\sqrt{4/3} \sqrt{n/2}} \]
for every $Q \in B_{n+1/2}$.

Let $\partial D$ be the unit circle of the complex plane. Let $f$ be a continuous function on $\partial D$ and let
\[ I_q(f) := M_q(f)^q = \frac{1}{2\pi} \int_K |f(e^{it})|^q \, dt \]
and
\[ M_\infty(f) := \max_{z \in \partial D} |f(z)|. \]

**Theorem 2.4.** There is an absolute constant $\varepsilon > 0$ such that
\[ M_1(f') \leq (1 - \varepsilon)^{\sqrt{1/3} m^{3/2}} \]
for every conjugate reciprocal unimodular polynomial $f \in K_m$.

**Theorem 2.5.** There is an absolute constant $\varepsilon > 0$ such that
\[ M_\infty(f') \geq (1 + \varepsilon)^{\sqrt{1/3} m^{3/2}} \]
for every conjugate reciprocal unimodular polynomial $f \in K_m$. 

Theorem 2.6. There is an absolute constant $\varepsilon > 0$ such that
\[ M_\infty(f) \geq (1 + \varepsilon)\sqrt{4/3}m^{1/2} \]
for every conjugate reciprocal unimodular polynomial $f \in \mathcal{K}_m$.

Theorem 2.7. There is an absolute constant $\varepsilon > 0$ such that
\[ M_q(f') \leq \exp(\varepsilon(q - 2)/q)\sqrt{1/3}m^{3/2}, \quad 1 \leq q < 2, \]
and
\[ M_q(f') \geq \exp(\varepsilon(q - 2)/q)\sqrt{1/3}m^{3/2}, \quad 2 < q, \]
for every conjugate reciprocal unimodular polynomial $f \in \mathcal{K}_m$.

The above results were well known before without the absolute constants $\delta > 0$ and $\varepsilon > 0$, respectively.

Remark 2.1. The factor $(1 + \delta)$ in Theorem 2.2 cannot be replaced by $(1 + \delta)\sqrt{3/2}$.

Remark 2.2. The factor $(1 + \varepsilon)\sqrt{4/3}$ in Theorem 2.3 cannot be replaced by $(1 + \varepsilon)\sqrt{2}$.

Remark 2.3. The factor $(1 + \varepsilon)\sqrt{1/3}$ in Theorem 2.5 cannot be replaced by $(1 + \varepsilon)\sqrt{1/2}$.

Remark 2.4. The factor $(1 + \varepsilon)\sqrt{4/3}$ in Theorem 2.6 cannot be replaced by $(1 + \varepsilon)\sqrt{2}$.

A polynomial $f \in \mathcal{P}_m^c$ is called skew-reciprocal if $f^*(z) = f(-z)$. A polynomial $f \in \mathcal{P}_m^c$ is called plain-reciprocal if $f^* = \overline{f}$, that is, $f(z) = z^m f(1/z)$ for all $z \in \mathbb{C} \setminus \{0\}$. Observe that Corollary 2.8 in [5] may be formulated as follows.

Remark 2.5. There is an absolute constant $\varepsilon > 0$ such that
\[ \max_{z \in \partial D} |f'(z)| - \min_{z \in \partial D} |f'(z)| \geq \varepsilon m^{3/2} \]
for all conjugate reciprocal, plain-reciprocal, and skew-reciprocal unimodular polynomials $f \in \mathcal{K}_m$.

Observe that for conjugate reciprocal unimodular polynomials Theorem 2.5 is stronger than Remark 2.5.

Problem 2.1. Is there an absolute constant $\varepsilon > 0$ such that
\[ M_\infty(f') \geq (1 + \varepsilon)\sqrt{1/3}m^{3/2} \]
holds for all plain-reciprocal and skew-reciprocal unimodular polynomials $f \in \mathcal{K}_m$?
Problem 2.2. Is there an absolute constant $\varepsilon > 0$ such that

$$M_{\infty}(f') \geq (1 + \varepsilon)\sqrt{1/3} m^{3/2}$$

or at least

$$\max_{z \in \partial D} |f'(z)| - \min_{z \in \partial D} |f'(z)| \geq \varepsilon m^{3/2}$$

holds for all unimodular polynomials $f \in \mathcal{K}_m$?

Our method to prove Theorem 2.5 does not seem to work for all unimodular polynomials $f \in \mathcal{K}_m$. In an e-mail communication several years ago B. Saffari speculated that the answer to Problem 2.2 is no. However we do not know the answer even to Problem 2.1.

Let $\mathcal{L}_m$ is the collection of all polynomials of degree $m$ with each of their coefficients in $\{-1, 1\}$.

Problem 2.3. Is there is an absolute constant $\varepsilon > 0$ such that

$$M_{\infty}(f') \geq (1 + \varepsilon)\sqrt{1/3} m^{3/2}$$

or at least

$$\max_{z \in \partial D} |f'(z)| - \min_{z \in \partial D} |f'(z)| \geq \varepsilon m^{3/2}$$

holds for all Littlewood polynomials $f \in \mathcal{L}_m$?

The following problem due to Erdős [6] is open for a long time.

Problem 2.4. Is there an absolute constant $\varepsilon > 0$ such that

$$M_{\infty}(f) \geq (1 + \varepsilon)m^{1/2}$$

or at least

$$\max_{z \in \partial D} |f(z)| - \min_{z \in \partial D} |f(z)| \geq \varepsilon m^{1/2}$$

holds for all Littlewood polynomials $f \in \mathcal{L}_m$?

The same problem may be raised only for all skew reciprocal Littlewood polynomials $f \in \mathcal{L}_m$ and as far as we know, it is also open.

3. Lemmas

Let $m(A)$ denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. The following lemma is due to Littlewood, see Theorem 1 in [8].

Lemma 3.1. Let $R \in \mathcal{T}_n$ be of the form

$$R(t) = R_n(t) = \sum_{j=1}^{n} a_m \cos(jt + \gamma_j).$$
Let \( s_m := \sum_{j=1}^{m} a_j^2 \), \( m = 1, 2, \ldots, n \), and let \( \mu := \|R\|_2 \), that is, \( \mu^2 = s_n \). Suppose

\[
\|R\|_1 \geq c \mu,
\]

where \( c > 0 \) is a constant (necessarily not greater than 1). Suppose also that the coefficients of \( R \) satisfy

\[
s_{\lfloor n/h \rfloor} / s_n = \mu^{-2} \sum_{1 \leq j \leq n/h} a_j^2 \leq 2^{-9} c^6
\]

for some constant \( h > 0 \). Let \( V = 2^{-5} c^3 \). Then there exists a constant \( B > 0 \) depending only on \( c \) and \( h \) such that

\[
m(\{ t \in K : v_1 \mu \leq |R(t)| \leq v_2 \mu \}) \geq B (v_2 - v_1)^2
\]

for every \( v_1 \) and \( v_2 \) such that \(-V \leq v_1 < v_2 \leq V\).

**Lemma 3.2.** Associated with \( Q \in T_n \) we define the sets

\[
E_\delta := \{ t \in K : |Q'(t)| < \delta n^{3/2} \}
\]

and

\[
F_\delta := \{ t \in K : |Q''(t)| \leq \delta^{1/2} n^{5/2} \}.
\]

We have

\[
m(E_\delta \setminus F_\delta) \leq 8 \delta^{1/2}.
\]

**Proof of Lemma 3.2.** Observe that \( E_\delta \setminus F_\delta \) is the union of at most \( 4n \) pairwise disjoint open subintervals of the period. Let these intervals be \((x_j, y_j), j = 1, 2, \ldots, \mu\), where \( \mu \leq 4n \). By the Mean Value Theorem we can deduce that there are \( \xi_j \in (x_j, y_j) \) such that

\[
2\delta n^{3/2} \geq |Q'(y_j) - Q'(x_j)| = |Q''(\xi_j)| (y_j - x_j) \geq \delta^{1/2} n^{5/2} (y_j - x_j),
\]

and hence

\[
y_j - x_j \leq 2\delta^{1/2} n^{-1}, \quad j = 1, 2, \ldots, \mu.
\]

Hence

\[
m(E_\delta \setminus F_\delta) = \sum_{j=1}^{\mu} (y_j - x_j) \leq \mu (2\delta^{1/2} n^{-1}) \leq 8 \delta^{1/2}.
\]

\( \square \)

**Lemma 3.3.** Let \( Q \in A_n \), \( P := (Q')^2 + n^2 Q^2 \), \( \delta \in (0, 1) \), and

\[
G_\delta := \{ t \in K : |P(t)|^{1/2} - \|P\|_1^{1/2} \leq \delta^{1/4} \|P\|_1^{1/2} \}.
\]

Suppose

\[
\|P\|_1^{1/2} \geq (1 - \delta) \|P\|_1.
\]
Then
\[ m(K \setminus G_\delta) \leq 4\pi\delta^{1/2}. \]

**Proof of Lemma 3.3.** Using (3.1) we have
\[ (3.2) \quad \|P\|_{1/2}^{1/2} \geq (1 - \delta)^{1/2}\|P\|_1^{1/2} \geq (1 - \delta)\|P\|_{1/2}^{1/2}. \]
Hence
\[
\int_K \left| |P(t)|^{1/2} - \|P\|_{1/2}^{1/2} \right|^2 dt = 4\pi \|P\|_1^{1/2} (\|P\|_{1/2}^{1/2} - \|P\|^{1/2}) \\
\leq 4\pi \|P\|_{1/2}^{1/2} \delta \|P\|_1 = 4\pi\delta \|P\|_1.
\]
Letting \( a := m(K \setminus G_\delta) \) we have
\[ a\delta^{1/2}\|P\|_1 \leq 4\pi\delta\|P\|_1, \]
and the lemma follows. \(\square\)

**Lemma 3.4.** Let \( Q \in \mathcal{A}_n \), \( P := (Q')^2 + n^2Q^2 \), and \( \|P\|_{1/2} \geq \frac{31}{32} \|P\|_1 \). Then \( R := n^2Q^2 + Q'' \) satisfies the assumptions of Lemma 3.1 with \( c := 1/32 \) and \( h = 2^932^6 \) for all sufficiently large \( n \).

**Proof of Lemma 3.4.** Let \( n \geq 2 \). Observe that
\[ R(t) = \sum_{j=1}^{n} a_j \cos(jt + \gamma_j), \quad a_j := n^2 - j^2, \quad j = 1, 2, \ldots, n. \]
Then
\[ s_n = \mu^2 = \|R\|_2^2 = \frac{1}{2} \sum_{j=1}^{n} (n^2 - j^2)^2, \]
hence
\[ \frac{n^5}{6} \leq s_n = \mu^2 \leq \frac{n^5}{2}. \]
Using Parseval’s formula, we have
\[ (3.3) \quad \|P\|_1 = \frac{1}{2} \left( \sum_{j=1}^{n} j^2 + n^2 \right) \geq \frac{2n^3}{3}. \]

(3.4) \[ \|Q''\|_1 \leq \|Q''\|_2 \leq \frac{1}{2} \left( \sum_{j=1}^{n} j^4 \right)^{1/2} \leq (1 + o(1)) \frac{n^{5/2}}{\sqrt{10}}. \]
and

\[ \|Q\|_2 \leq \frac{1}{2} \left( \sum_{j=1}^{n} j^2 \right)^{1/2} \leq (1 + o(1)) \frac{n^{3/2}}{\sqrt{6}} \]

and hence

\[ \|nQ\|_1 = (2\pi)^{-1} \int_K |P(t) - Q(t)|^{1/2} dt \]

\[ \geq (2\pi)^{-1} \int_K |P(t)|^{1/2} dt - (2\pi)^{-1} \int_K |Q(t)| dt \]

\[ = \|P\|^{1/2} - \|Q\| \geq \left( \frac{31}{32} \right)^2 \|P\|^{1/2} - \|Q\|_2 \]

\[ \geq \frac{31}{32} \sqrt{\frac{2}{3}} n^{3/2} - (1 + o(1)) \sqrt{\frac{1}{6}} n^{3/2}. \]

Combining (3.5) and (3.4) we conclude

\[ \|R\|_1 = \|n^2Q + Q\|_1 \geq \|n^2Q\|_1 - \|Q\|_1 \]

\[ \geq \frac{31}{32} \sqrt{\frac{2}{3}} n^{5/2} - (1 + o(1)) \sqrt{\frac{1}{6}} n^{5/2} - (1 + o(1)) \frac{n^{5/2}}{\sqrt{10}} \]

\[ \geq \frac{1}{32} n^{5/2} \]

for all sufficiently large \( n \). Also, \( s_{[n/h]} \leq (n/h)n^4 = n^5/h \), hence \( s_{[n/h]}/s_n \leq 1/h \). Therefore the assumptions of Lemma 3.1 are satisfied with \( c := 1/32 \) and \( h = 2^{9/32^6} \). □

4. PROOF OF THE THEOREMS

Proof of Theorem 2.1. Let \( Q \in A_n \), \( P = (Q')^2 + n^2Q^2 \), and \( R := n^2Q^2 + Q' \). Let \( \delta \in (0, 1) \). Suppose \( \|P\|_{1/2} \geq (1 - \delta)\|P\|_1 \). Lemma 3.4 states that if \( 0 < \delta 1/32 \) then \( R \) satisfies the assumptions of Lemma 3.1 with \( c = 1/32 \) and \( h = 2^{9/32^6} \). Now let

\[ E_\delta := \{ t \in K : |Q'(t)| < \delta n^{3/2} \}, \]

\[ F_\delta := \{ t \in K : |Q''(t)| \leq \delta n^{5/2} \}, \]

\[ G_\delta := \{ t \in K : |P(t)|^{1/2} - \|P\|_{1/2}^{1/2} \leq \delta^{1/4} \|P\|_{1/2}^{1/2} \}, \]

and

\[ H_\gamma := \{ t \in K : \gamma n^{5/2} \leq |R(t)| < 2\gamma n^{5/2} \}. \]

Recall that by the Parseval’s formula we have

\[ \|P\|_1 = \frac{n^3}{2} + \frac{n(n + 1)(2n + 1)}{12} \]

(4.1)
Hence, if \( t \in G_\delta \cap E_\delta \cap F_\delta \) and the absolute constant \( \delta > 0 \) is sufficiently small, then

\[
|R(t)| \geq n^2|Q(t)| - |Q''(t)| = n(P(t) - Q'(t)^2)^{1/2} - |Q''(t)| \\
\geq n \left((1 - \delta^{1/4})^2 \left(\frac{n^3}{2} + \frac{n(n+1)(2n+1)}{12}\right) - \delta^2 n^3\right)^{1/2} - \delta n^{5/2} \\
\geq \frac{1}{2} n^{5/2},
\]

that is,

\begin{equation}
|R(t)| \geq \frac{1}{2} n^{5/2}, \quad t \in G_\delta \cap E_\delta \cap F_\delta.
\end{equation}

By Lemma 3.2 we have

\begin{equation}
m(E_\delta \setminus F_\delta) \leq 8\delta^{1/2}.
\end{equation}

By Lemma 3.3 we have

\begin{equation}
m(K \setminus G_\delta) \leq 4\pi\delta^{1/2}
\end{equation}

Observe that if \( 0 < \gamma < 1/4 \) then (4.2) implies that \( H_\gamma \subset K \setminus (G_\delta \cap E_\delta \cap F_\delta) \), hence

\[
H_\gamma \cap E_\delta \subset (E_\delta \setminus G_\delta) \cup (E_\delta \setminus F_\delta).
\]

Therefore, by (4.3) and (4.4) we can deduce that

\begin{equation}
m(H_\gamma \cap E_\delta) \leq m(E_\delta \setminus G_\delta) + m(E_\delta \setminus F_\delta) \\
\leq 4\pi\delta^{1/2} + 8\delta^{1/2}.
\end{equation}

By Lemmas 3.1 and 3.4 there are absolute constants \( 0 < \gamma < 1/4 \) and \( B > 0 \) such that

\begin{equation}
m(H_\gamma) \geq B\gamma^2.
\end{equation}

It follows from (4.5) and (4.6) that

\begin{equation}
m(H_\gamma \setminus E_\delta) \geq \frac{1}{2} B\gamma^2.
\end{equation}

for all sufficiently small absolute constant \( \delta > 0 \). Observe that

\begin{equation}
|2Q'(t)R(t)| \geq 2\delta n^{3/2} \gamma n^{5/2} = 2\gamma \delta n^4, \quad t \in H_\gamma \setminus E_\delta,
\end{equation}

and

\begin{equation}
P'(t) = 2Q'(t)R(t).
\end{equation}
Combining (4.7), (4.8), and (4.9), we obtain

\[ m(\{t \in K : |P'(t)| \geq 2\gamma\delta n^4 \}) \geq \frac{1}{2} B\gamma^2, \]

and hence

\[ \int_K |P'(t)| \, dt \geq \frac{1}{2} B\gamma^2 (2\gamma\delta n^4) = B\gamma^3 \delta n^4. \tag{4.10} \]

Now let \( \tilde{P} := P - 2\pi\|P\|_1 \in T_{2n} \). Then (4.10) can be rewritten as

\[ \int_K |\tilde{P}'(t)| \, dt \geq B\gamma^3 \delta n^4, \]

and by Bernstein’s inequality in \( L_1 \) (see p. 390 of [5], for instance), we have

\[ 2\pi\|\tilde{P}\|_1 = \int_K |\tilde{P}(t)| \, dt \geq \frac{1}{2} B\gamma^3 \delta n^3. \tag{4.11} \]

Observe that

\[ 2\pi\|\tilde{P}\|_1 = \int_K |\tilde{P}(t)| \, dt = \int_K |P(t) - \|P\|_1| \, dt \leq \int_K \left| (P(t)^{1/2} - \|P\|_1^{1/2}) (P(t)^{1/2} + \|P\|_1^{1/2}) \right| \, dt \]

\[ \leq \left( \int_K \left| (P(t)^{1/2} - \|P\|_1^{1/2})^2 \right| \, dt \right)^{1/2} \left( \int_K \left| (P(t)^{1/2} + \|P\|_1^{1/2})^2 \right| \, dt \right)^{1/2} \]

\[ = 2\pi \left( 2\|P\|_1^{1/2} (\|P\|_1^{1/2} - \|P\|_1^{1/2}) \right)^{1/2} \left( 2\|P\|_1^{1/2} (\|P\|_1^{1/2} + \|P\|_1^{1/2}) \right)^{1/2} \]

\[ \leq 4\pi \|P\|_1^{1/2} (\|P\|_1 - \|P\|_1^{1/2})^{1/2} = 4\pi n^{3/2} (\|P\|_1 - \|P\|_1^{1/2})^{1/2}. \]

Combining (4.11), (4.12), and (4.1), we conclude

\[ \|P\|_1 - \|P\|_1^{1/2} \geq \left( \frac{2\pi\|\tilde{P}\|_1}{4\pi} \right)^2 \geq \delta^* n^3 \geq \left( \frac{B\gamma^3 \delta n^{3/2}}{8\pi} \right)^2 \geq \delta^* n^3 \]

with an absolute constant \( \delta^* > 0 \). \( \square \)

**Proof of Theorem 2.2.** By Theorem 2.1 there is an absolute constant \( \delta > 0 \) such that

\[ \|P\|_1 = 2\pi \int_K |P(t)| \, dt = 2\pi \int_K |P(t)|^{1/2} |P(t)|^{1/2} \, dt \leq 2\pi \int_K |P(t)|^{1/2} \|P\|_1^{1/2} \, dt \]

\[ \leq \|P\|_1^{1/2} \|P\|_\infty^{1/2} \leq (1 - \delta)^{1/2} \|P\|_1^{1/2} \|P\|_\infty^{1/2}. \]
Hence
\[ \|P\|_{1/2}^{1/2} \leq (1 - \delta)^{1/2}\|P\|_{\infty}^{1/2}, \]
and the result follows. □

**Proof of Theorem 2.3.** The Bernstein-Szegő inequality (see p. 232 of [5], for instance) yields
\[ Q'(t)^2 + n^2Q(t)^2 \leq n^2\|Q\|_{\infty}^2, \quad t \in \mathbb{R}, \quad Q \in A_n \subset T_n, \]
hence if \( P = (Q')^2 + n^2Q^2 \), then
\[ \|P\|_{\infty} \leq n^2\|Q\|_{\infty}. \]

Hence, using Theorem 2.1 and Parseval’s formula we can deduce that
\[ \|Q\|_{\infty}^2 \geq n^{-2}\|P\|_{\infty} \geq n^{-2}(1 + \delta)\|P\|_1 = n^{-2}(1 + \delta)(\|Q'\|_2^2 + n^2\|Q\|_2^2) \]
\[ = \frac{n(n+1)(2n+1)}{12} + n^{-2}(1 + \delta)(\frac{n^3}{2}) \geq (1 + \delta)(4/3)(n/2) \]
with an absolute constant \( \delta > 0 \) and the theorem follows. □

The proofs of Theorems 2.1*, 2.2*, and 2.3* are similar to those of Theorems 2.1 and 2.2, respectively. The modifications required in the proofs of Theorems 2.1* and 2.2* are straightforward for the experts and we omit the details.

**Proof of Theorem 2.4.** First assume that \( m = 2n \) is even and \( f \in K_m \) is a conjugate reciprocal unimodular polynomial. Let \( f(z) = \sum_{j=0}^{n} a_j z^j \), where \( a_j \in \mathbb{C} \) and \( |a_j| = 1 \) for each \( j = 0, 1, \ldots, n \). As \( f \) is conjugate reciprocal, we have
\[ a_n - j = \overline{a_j}, \quad j = 0, 1, \ldots, n, \]
and \( a_n \in \{-1, 1\} \), in particular. Let \( Q \in A_n \) be defined by \( 2Q(t) = e^{-int}f(e^{it}) - a_n \). Then
\[ ie^{it}f'(e^{it}) = e^{int}(2Q'(t) + in(2Q(t) + a_n)) \]
hence the triangle inequality implies that
\[ |f'(e^{it})| = 2|e^{int}(Q'(t) + in(Q(t))| + e^{int}ina_n| \leq 2|(Q'(t) + in(Q(t))| + n \]
\[ \leq 2|P(t)|^{1/2} + n, \]
where \( P := (Q')^2 + n^2Q^2 \) is the same as in Theorem 2.1, and the theorem follows from Theorem 2.1 as
\[ M_1(f') \leq 2\|P\|_{1/2}^{1/2} + n \leq 2(1 - \delta)^{1/2}\|P\|_1^{1/2} + n \]
\[ \leq 2(1 - \delta)^{1/2} \left( \frac{n(n+1)(2n+1)}{12} \frac{n^3}{2} \right)^{1/2} + n \]
\[ \leq 2(1 - \delta)^{1/2} \left( \frac{2(n+1)^3}{3} \right)^{1/2} + n \leq \]
\[ \leq (1 - \delta)^{1/2} \sqrt{1/3}m^{3/2} + o(m^{3/2}). \]
Now assume that \( m = 2n + 1 \) is even and \( f \in \mathcal{K}_m \) is a conjugate reciprocal unimodular polynomial. Let \( Q \in \mathcal{B}_{m/2} \) be defined by \( Q(t) = e^{-imt/2}f(e^{it}) \). Then

\[
ie^{it}f'(e^{it}) = e^{imt/2}(Q'(t) + (im/2)Q(t))
\]

implies that

\[
|f'(e^{it})| = 2|e^{imt/2}(Q'(t) + im/2Q(t))| \leq 2|(Q'(t) + (im/2)Q(t))|
\]

where \( P := (Q')^2 + (n + 1/2)^2Q^2 \) is the same as in Theorem 2.1*, and the theorem follows from Theorem 2.1* as

\[
M_1(f') \leq 2\|P\|^{1/2} \leq 2(1 - \delta)^{1/2}\|P\|^{1/2}
\]

\[
\leq 2(1 - \delta)^{1/2} \left( \frac{(n+1)(n+2)(2n+3)}{12} \right)^{1/2} \leq 2(1 - \delta)^{1/2} \left( \frac{2(n+1)^3}{3} \right)^{1/2} \leq (1 - \delta)^{1/2} \sqrt{1/3m^{3/2} + o(m^{3/2})}.
\]

□

**Proof of Theorem 2.5.** Let \( f \in \mathcal{K}_m \) be a conjugate reciprocal unimodular polynomial. By Theorem 2.4 there is an absolute constant \( \delta > 0 \) such that

\[
M_1(f') = 2\pi \int_K |f'(e^{it})| dt = 2\pi \int_K |f'(e^{it})|^{1/2} |f'(e^{it})|^{1/2} dt
\]

\[
\leq 2\pi \int_K |f'(e^{it})|^{1/2} \max_{t \in K} |f'(e^{it})|^{1/2} dt
\]

\[
\leq (M_{1/2}(f'))^{1/2} (M_{\infty}(f'))^{1/2}
\]

\[
\leq (1 - \delta)^{1/2} (M_1(f'))^{1/2} (M_{\infty}(f'))^{1/2}.
\]

Hence

\[
(M_1(f'))^{1/2} \leq (1 - \delta)^{1/2}(\|P\|_\infty)^{1/2},
\]

and the theorem follows. □

**Proof 1 of Theorem 2.6.** First assume that \( m = 2n \) is even and \( f \in \mathcal{K}_m \) is a conjugate reciprocal unimodular polynomial. Let \( f(z) = \sum_{j=0}^n a_j z^j \), where \( a_j \in \mathbb{C} \) and \( |a_j| = 1 \) for each \( j = 0, 1, \ldots, n \). As \( f \) is conjugate reciprocal, we have

\[
a_{n-j} = \overline{a}_j, \quad j = 0, 1, \ldots, n,
\]
and $a_n \in -1, 1$, in particular. Let $Q \in A_n$ be defined by $Q(t) = e^{-int} f(e^{it}) - a_n$. Observe that
\[
\max_{z \in \partial D} |f(z)| - \|Q\|_\infty \leq 1,
\]
hence the theorem follows from Theorem 2.3. Now assume that $m = 2n + 1$ is even and $f \in \mathcal{K}_m$ is a conjugate reciprocal unimodular polynomial. Let $Q \in B_{n/2}$ be defined by $Q(t) = e^{-int/2} f(e^{it})$. Observe that
\[
\max_{z \in \partial D} |f(z)| = \|Q\|_\infty,
\]
hence the theorem follows from Theorem 2.3*. □

Proof 2 of Theorem 2.6. It is well known that $f$ is a conjugate reciprocal unimodular polynomial of degree $m$ then $\|f'(\|_\infty = (m/2) \|f\|_\infty$. Hence the theorem follows from a combination of this and Theorem 2.3. □

Proof of Theorem 2.7. Let $f \in \mathcal{K}_m$ be conjugate reciprocal. Observe that by the Parseval’s formula we have
\[
(4.13) \quad M_2(f') = \left( \frac{m(m + 1)(2m + 1)}{6} \right)^{1/2} = (1 + o(1)) \frac{m^{3/2}}{\sqrt{3}}.
\]
As we will see, both inequalities of the theorem follows from Theorem 2.4 (4.13), and the following convexity property if the function $h(q) := q \log(M_q(f))$ on $(0, \infty)$. Let $f$ be a continuous function on $\partial D$ and let
\[
I_q(f) := M_q(f)^q = \frac{1}{2\pi} \int_K |f(e^{it})|^q \, dt.
\]
Then $h(q) := \log(I_q(f)) = q \log(M_q(f))$ is a convex function of $q$ on $(0, \infty)$. This is a simple consequence of Hölder’s inequality. For the sake of completeness, before we apply it, we present the short proof of this fact. We need to see that if $q < r < p$, then
\[
I_r(f) \leq I_p(f) \frac{r-q}{p-q} I_q(f)^{\frac{p-r}{p-q}},
\]
that is,
\[
(4.6) \quad \left( \frac{1}{2\pi} \int_K |f(e^{it})|^r \, dt \right)^{p-q} \leq \left( \frac{1}{2\pi} \int_K |f(e^{it})|^p \, dt \right)^{r-q} \left( \frac{1}{2\pi} \int_K |f(e^{it})|^q \, dt \right)^{q-r}.
\]
To see this let
\[
\alpha := \frac{p-q}{r-q}, \quad \beta := \frac{p-q}{p-r}, \quad \gamma := \frac{p}{\alpha}, \quad \delta := \frac{q}{\beta},
\]
hence $1/\alpha + 1/\beta = 1$ and $\gamma + \delta = r$. Let
\[
F(t) := |f(e^{it})|^\gamma = |f(e^{it})|^\frac{p(r-q)}{p-q},
\]
and
\[ G(t) := |f(e^{it})|^\delta = |f(e^{it})|^{q(p-r)\gamma}. \]

Then by Hölder’s inequality we conclude
\[
\int_K F(t)G(t) \, dt \leq \left( \int_K F(t)^\alpha \, dt \right)^{1/\alpha} \left( \int_K G(t)^\beta \, dt \right)^{1/\beta},
\]
and (4.6) follows.

Let \( q \in [1, 2) \). Then, using the convexity property if the function \( h(q) := q \log(M_q(f)) \) on \((0, \infty)\), we obtain
\[
\frac{2 \log M_2 - q \log M_q}{2 - q} \geq \frac{2 \log M_2 - \log M_1}{2 - 1}.
\]

Combining this with the Theorem 2.4 and (4.13) gives theorem.

Now let \( q \in (2, \infty) \). Then, using the convexity property if the function \( h(q) := q \log(M_q(f)) \) on \((0, \infty)\), we obtain
\[
\frac{q \log M_q - 2 \log M_2}{q - 2} \geq \frac{2 \log M_2 - \log M_1}{2 - 1}.
\]

Combining this with the Theorem 2.4 and (4.13) gives theorem. \( \square \)

**Proof of Remark 2.1.** Let \( (f_n) \) be an ultraflat sequence of unimodular polynomials \( f_n \in \mathcal{K}_n \) satisfying \( M_\infty(f_n) \leq (1 + \varepsilon_n)\sqrt{n} \) with a sequence \( \varepsilon_n \) of numbers \( \varepsilon_n > 0 \) converging to 0. It is shown in [7] that such a sequence \( (f_n) \) exists. Let \( g_n(z) = zf_{n-1}(z) \). Let \( Q_n \in \mathcal{A}_n \) be defined by \( Q_n(t) := \text{Re}(g_n(e^{it})) \). Then the Bernstein-Szegő inequality (see page 232 in [5], for instance) gives that \( P_n := (Q_n')^2 + n^2Q_n^2 \) satisfy
\[
\|P_n\|_\infty \leq n^2\|Q_n\|_\infty^2 \leq (1 + \varepsilon_n)^2n^3,
\]
while by Parseval’s formula we have
\[
\|P_n\|_1 = \left( \frac{n^3}{2} + \frac{n(n+1)(2n+1)}{12} \right) \geq \frac{2n^3}{3}.
\]
\( \square \)

**Proof of Remark 2.2.** Let \( Q_n \in \mathcal{A}_n \) be in the proof of Remark 2.1. Then
\[
\|Q_n\|_\infty \leq \|P_n\|_\infty^{1/2} \leq (1 + \varepsilon_n)^2n^{3/2},
\]
\( \square \)

**Proof of Remark 2.4.** Let \( f_n \in \mathcal{K}_n \) and \( g_n(z) = zf_{n-1}(z) \) be the same as in the proof of Remark 2.1. For \( m = 2n \) we define \( h_m \in \mathcal{K}_m \) by
\[
h_m(z) := z^n(g_n(z) + \mathcal{g}_n(1/z) + 1).
\]
We have
\[ M_\infty(h_m) \leq 2(1 + \varepsilon_n)\sqrt{n} \leq (1 + \varepsilon_n)\sqrt{2} \sqrt{m} \]

\[ \Box \]

**Proof of Remark 2.3.** For \( m = 2n \) let \( h_m \in \mathcal{K}_m \) be the same as in the proof of Lemma 2.4. Then using the well-known Bernstein-type inequality for conjugate reciprocal polynomials (see p. 438 in [5], for instance), we have
\[ M_\infty(h'_m) \leq \frac{m}{2} (1 + \varepsilon_n)\sqrt{2} \sqrt{m} \leq \frac{1}{\sqrt{2}} (1 + \varepsilon_n)m^{3/2} . \]

\[ \Box \]

**References**


36. J.E. Littlewood, *On polynomials $\sum_0^\infty (\pm z^m, \sum_0^\infty \exp(\alpha_m)z^{m}, z = e^{i\theta}$, J. London Math. Soc. **41** (1966), 367–376.


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