LAX-TYPE INEQUALITIES FOR POLYNOMIALS
ON SUBARCS OF THE UNIT CIRCLE

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Dedicated to Peter Lax on the occasion of his 86th birthday

ABSTRACT. We prove the right Lax-type inequality on subarcs of the unit circle of the complex plane for complex algebraic polynomials of degree $n$ having no zeros in the open unit disk. This is done by establishing the right Bernstein-Szegő-Videnskii type inequality for real trigonometric polynomials of degree at most $n$ on intervals shorter than the period. The paper is closely related to recent work by B. Nagy and V. Totik. In fact, their asymptotically sharp Bernstein-type inequality for complex algebraic polynomials of degree at most $n$ on subarcs of the unit circle is recaptured by using more elementary methods. Our discussion offers a somewhat new approach to see V.S. Videnskii's Bernstein-type inequalities for trigonometric polynomials of degree at most $n$ on intervals shorter than a period, a classical polynomial inequality published first in 1960.

1. Introduction

Let $D$ be the open unit disc of the complex plane. Let $\partial D$ be the unit circle of the complex plane. Let $\mathcal{T}_n$ be the collection of all real trigonometric polynomials $Q$ of degree at most $n$ of the form

$$Q(t) = a_0 + \sum_{j=1}^{n} (a_j \cos(jt) + b_j \sin(jt)), \quad a_j, b_j \in \mathbb{R}.$$ 

Let $\mathcal{T}_n^c$ be the collection of all complex trigonometric polynomials $Q$ of degree at most $n$ of the form

$$Q(t) = a_0 + \sum_{j=1}^{n} (a_j \cos(jt) + b_j \sin(jt)), \quad a_j, b_j \in \mathbb{C}.$$ 

Let $\mathcal{P}_n$ be the collection of all real algebraic polynomials $P$ of degree at most $n$ of the form

$$P(z) = \sum_{j=0}^{n} a_j z^j, \quad a_j \in \mathbb{R}.$$
Let $\mathcal{P}_n^c$ be the collection of all complex algebraic polynomials $P$ of degree at most $n$ of the form

$$P(z) = \sum_{j=0}^{n} a_j z^j, \quad a_j \in \mathbb{C}.$$ 

The following inequalities are due to Bernstein. We have

$$\max_{z \in \partial D} |P'(z)| \leq n \max_{z \in \partial D} |P(z)|, \quad P \in \mathcal{P}_n^c;$$

$$\max_{\tau \in [-\pi, \pi]} |Q'(\tau)| \leq n \max_{\tau \in [-\pi, \pi]} |Q(\tau)|, \quad Q \in \mathcal{T}_n^c;$$

$$\max_{x \in [-1, 1]} |P'(x)\sqrt{1-x^2}| \leq n \max_{x \in [-1, 1]} |P(x)|, \quad P \in \mathcal{P}_n^c.$$ 

The inequality

$$\max_{\tau \in [-\pi, \pi]} (|Q'(\tau)|^2 + n^2 |Q(\tau)|^2) \leq n^2 \max_{\tau \in [-\pi, \pi]} |Q(\tau)|^2, \quad Q \in \mathcal{T}_n,$$

is often referred to as the Bernstein-Szegő inequality. Note that it is valid only for all real trigonometric polynomials $Q \in \mathcal{T}_n$ and NOT for all complex trigonometric polynomials $Q \in \mathcal{T}_n^c$.

Books focusing on approximation theory contain these inequalities with various proofs. See [11],[14], or Section 5.1 of [1], for example. Inequalities on the size of gapped polynomials on subarcs and measurable closed subsets of the unit circle were studied by P. Turán [17], F. Nazarov [12]. The book [2] also discuss inequalities of this variety. A number of interesting polynomial inequalities, Remez-type inequalities in particular, on Jordan curves are studied in the survey [1] by V.V. Andrievskii.

For $n > 0$, $\omega \in (0, \pi)$, and $t \in (\omega, \omega)$, we define

$$B(n, \omega, t) := \frac{d}{dt} \left( 2n \arccos \left( \frac{\sin(t/2)}{\sin(\omega/2)} \right) \right) = 2n \left( 1 - \left( \frac{\sin(t/2)}{\sin(\omega/2)} \right)^2 \right)^{-1/2} \frac{1}{2} \frac{\cos(t/2)}{\sin(\omega/2)}$$

$$= \frac{n \cos(t/2)}{\sin(\omega/2)} \left( 1 - \left( \frac{\sin(t/2)}{\sin(\omega/2)} \right)^2 \right)^{-1/2}$$

$$= \sqrt{2n} \cos(t/2) \cos t - \cos \omega^{-1/2}.$$ 

Then

$$(B(n, \omega, t))^2 = \frac{2n^2 \cos^2(t/2)}{\cos t - \cos \omega} = n^2 \frac{1 + \cos t}{\cos t - \cos \omega} = n^2 \left( \frac{1 + \cos \omega}{\cos t - \cos \omega} + 1 \right).$$

The classical Bernstein inequality for trigonometric polynomials were extended by V.S. Videnskii, see e.g. [18] or E.19 of Section 5.1 on page 242 in [2]. He showed that

$$|Q'(t)| \leq B(n, \omega, t) \max_{\tau \in [-\omega, \omega]} |Q(\tau)|, \quad t \in (-\omega, \omega),$$

where $\mathcal{T}_n$ is the closure of the disk $|z| \leq 1$ in $\mathbb{C}$, and $\partial D$ is the boundary of $D$. 


for every \( Q \in \mathcal{T}_c^n \). There is an extension of this to “half-integer trigonometric polynomials” in [19] where it is shown that

\[
|Q'(t)| \leq B(n + 1/2, \omega, t) \max_{\tau \in [-\omega, \omega]} |Q(\tau)|, \quad t \in (-\omega, \omega),
\]

for every “half-integer trigonometric polynomials” \( Q \) of the form

\[
Q(t) = \sum_{j=1}^{n} \left( a_j \cos \left( \frac{2j-1}{2} t \right) + b_j \sin \left( \frac{2j-1}{2} t \right) \right), \quad a_j, b_j \in \mathbb{R}.
\]

Bernstein-type inequalities in \( L_p \) norms on subarcs of the unit circle were established by D. Lubinsky [10], C.K. Kobindarajah and D. Lubinsky [6] and T. Erdélyi [4].

In 1940 P. Lax [2] proved that

\[
\max_{t \in [-\pi, \pi]} |P'(t)| \leq \frac{n}{2} \max_{t \in [-\pi, \pi]} |P(t)|
\]

for all polynomials \( P \in \mathcal{P}_n^c \) having no zeros in the open unit disk \( D \).

Inequalities for polynomials with constraints are surveyed in [2,3,5].

In 1969 M.A. Malik [8] observed that

\[
|P_n'(e^{it})| + |P_n^{*'}(e^{it})| \leq n \max_{\tau \in [-\omega, \omega]} |P_n(e^{i\tau})|
\]

for every algebraic polynomial \( P_n \in \mathcal{P}_n^c \) and \( t \in (-\omega, \omega) \). See also [2,11].

It was observed by A. Kroó, see e.g. E.16 c] on p. 438 in [2], that if \( P \in \mathcal{P}_n^c \) has the property that \( 1/\alpha \) is a zero of \( P \) with multiplicity at least \( k \) whenever \( \alpha \in D \) is a zero of \( P \) with multiplicity \( k \) (there is no restriction on the zeros of \( P \) outside \( D \)), then

\[
\max_{\tau \in [-\pi, \pi]} |P'(e^{i\tau})| \leq \frac{n}{2} \max_{\tau \in [-\pi, \pi]} |P(e^{i\tau})|.
\]

Both of the above observations generalize of Lax’s inequality. We need to observe only that \( |P'(z)| \leq |P^{*'}(z)| \) for every \( z \in \mathbb{C} \) with \( |z| = 1 \). See e.g. page 438 of [2].

Lax-type inequalities for rational functions with fixed poles outside the closed unit circle were proved by X. Li, R.N. Mohapatra, R.Z. Rodriguez [9]. This was discovered independently by P. Borwein and T. Erdélyi [2] (Theorem 7.11, p. 329) by using similar methods.

Our first five theorems recapture some old results of Videnskii [18, 19] and some recent results of V. Totik [15,16], and B. Nagy and V. Totik [13]. Our methods of proof are somewhat different and some of them may be viewed as somewhat more elementary. Note that Nagy and Totik [13] and Totik [15,16] establish more general results on Jordan curves using potential theoretic tools.

Our Theorem 7 offers an extension of Malik’s inequality to subarcs of the unit circle. This is based on our Theorem 6 that may be viewed as the special case of Theorem 7 dealing with conjugate reciprocal algebraic polynomials only. Our Theorem 8 is an extension of Lax’s inequality to subarcs. Moreover, in Theorem 8 we assume only that \( P \in \mathcal{P}_n^c \) satisfies the following: \( 1/\pi \) is a zero of \( P \) with multiplicity at least \( k \) whenever \( \alpha \in D \) is a zero of \( P \) with multiplicity \( k \) (there is no restriction on the zeros of \( P \) outside \( D \)).
2. Results and Proofs

**Theorem 1.** We have

\[ |Q'(t)|^2 + (B(n, \omega, t))^2|Q(t)|^2 \leq (B(n, \omega, t))^2 \max_{\tau \in [-\omega, \omega]} |Q(\tau)|^2, \quad t \in (-\omega, \omega), \]

for every \( Q \in T_n \).

**Proof.** Let \( t \in (-\omega, \omega) \) and \( n \in \mathbb{N} \) be fixed. A simple compactness argument shows that there is a trigonometric polynomial \( Q^* \in T_n \) such that

\[ |Q^*(t)|^2 + (B(n, \omega, t))^2|Q^*(t)|^2 = \sup_{Q \in T_n} (|Q'(t)|^2 + (B(n, \omega, t))^2|Q(t)|^2), \]

where the supremum is taken for all \( Q \in T_n \) with

\[ \max_{\tau \in [-\omega, \omega]} |Q(\tau)| = 1. \]

It can be shown by a standard variational method that \( Q^* \) equioscillates in \([-\omega, \omega]\) at least \( 2n \) times. That is, there are

\[-\omega \leq t_1 < t_2 < \cdots < t_{2n} \leq \omega\]

such that

\[ Q^*(t_j) = \pm (-1)^j, \quad j = 1, 2, \ldots, 2n. \]

To see this let \( m \) be the largest integer such that there are

\[-\omega \leq t_1 < t_2 < \cdots < t_m \leq \omega\]

for which

\[ Q^*(t_1) = \pm (-1)^j, \quad j = 1, 2, \ldots, m. \]

Since \( t \in (-\omega, \omega) \) and \( Q^*(t) \neq 0 \), we have \( t \notin \{t_1, t_2, \ldots, t_m\} \). If \( m < 2n \), then we can choose a trigonometric polynomial \( R \in T_n \) such that

\[ R(t_j) = Q(t_j) = \pm (-1)^j, \quad j = 1, 2, \ldots, m, \]

\[ Q^*(t) = 0 \quad \text{and} \quad Q''(t) = 0. \]

Let \( Q_\varepsilon(t) := c_\varepsilon(Q^* - \varepsilon R) \) where the constant \( c_\varepsilon \in \mathbb{R} \) is chosen so that

\[ \max_{\tau \in [-\omega, \omega]} |Q_\varepsilon(\tau)| = 1. \]

Then \( Q_\varepsilon \in T_n \) contradicts the extremal property of \( Q^* \). Hence \( Q^* \) equioscillates in \([-\omega, \omega]\) at least \( 2n \) times, as we stated.

Now it is easy to see that one of the two cases below holds.
Case 1. \( Q^* \) equioscillates \( 2n + 1 \) times on a larger interval \([-\tilde{\omega}, \omega]\) or \([-\omega, \tilde{\omega}]\) with some \( \tilde{\omega} \in [\omega, 2\pi - \omega] \).

Case 2. \( Q^* \) equioscillates \( 2n \) times on a period of length \( 2\pi \).

In Case 1 without loss of generality we may assume that \( Q^* \) equioscillates \( 2n + 1 \) times on \([-\omega, \tilde{\omega}]\) with some \( \tilde{\omega} \in [\omega, 2\pi - \omega] \), the other case is analogous. Thus, there are

\[
-\omega = t_0 < t_1 < \ldots < t_{2n} = \tilde{\omega}
\]
such that

\[
Q^*(y_j) = \pm (-1)^j, \quad j = 1, 2, \ldots, 2n.
\]

Then it is a routine argument to identify \( Q^* \) as

\[
Q^*(t) = \pm \cos \left( 2n \arccos \left( \frac{\sin((t - \alpha)/2)}{\sin(\beta/2)} \right) \right)
\]

with

\[
\alpha := \frac{\tilde{\omega} - \omega}{2} \quad \text{and} \quad \beta := \frac{\tilde{\omega} - \omega}{2} = \omega + \alpha < \pi.
\]

Therefore

(1) \[
|Q''(t)|^2 + (B(n, \beta, t - \alpha))^2 |Q^*(t)|^2 = (B(n, \beta, t - \alpha))^2.
\]

Elementary calculus shows that

(2) \[
(B(n, \beta, t - \alpha))^2 = (B(n, \omega + \alpha, t - \alpha))^2 \leq (B(n, \omega, t))^2, \quad \alpha \in [0, \pi - \omega).
\]

To see this we have to show that

(3) \[
\frac{1 + \cos(\omega + \alpha)}{\cos(t - \alpha) - \cos(\omega + \alpha)} \leq \frac{1 + \cos \omega}{\cos t - \cos \omega},
\]

that is,

(4) \[
\frac{1 + \cos(\omega + \alpha)}{1 + \cos \omega} \leq \frac{\cos(t - \alpha) - \cos(\omega + \alpha)}{\cos t - \cos \omega}.
\]

However,

(5) \[
\frac{\cos(t - \alpha) - \cos(\omega + \alpha)}{\cos t - \cos \omega} = \frac{2 \sin \left( \frac{1}{2}(\omega + t) \right) \sin \left( \frac{1}{2}(\omega - t) + \alpha \right)}{2 \sin \left( \frac{1}{2}(\omega + t) \right) \sin \left( \frac{1}{2}(\omega - t) \right)}
\]

\[
= \frac{\sin \left( \frac{1}{2}(\omega - t) + \alpha \right)}{\sin \left( \frac{1}{2}(\omega - t) \right)}.
\]

Let

\[
f(x) := \frac{\sin(x + \alpha)}{\sin x}, \quad x \in (0, \pi).
\]
Then
\[ f'(x) = \frac{-\sin \alpha}{\sin^2 x}, \]
hence \( f \) is decreasing on \((0, \pi)\). Therefore \( t \in (-\omega, \omega) \) implies \( f(\omega) \leq f(\frac{1}{2} t) \), that is,
\[ \frac{\sin(\omega + \alpha)}{\sin \omega} \leq \frac{\sin \left( \frac{1}{2}(\omega - t) + \alpha \right)}{\sin \left( \frac{1}{2}(\omega - t) \right)}. \]
Combining this with (5) we deduce that in order to prove (4) we have to show only
\[ \frac{1 + \cos(\omega + \alpha)}{1 + \cos \omega} \leq \frac{\sin(\omega + \alpha)}{\sin \omega}. \]
However, this is equivalent to
\[ \tan \left( \frac{1}{2}(\omega + \alpha) \right) \geq \tan \left( \frac{1}{2}\omega \right), \]
which obviously holds since \( \omega/2 \leq (\omega + \alpha)/2 < \pi/2 \). So (3) is justified.

Now (1) and (2) give that
\[
|Q''(t)|^2 + (B(n, \omega, t))^2|Q'(t)|^2
=|Q''(t)|^2 + (B(n, \beta, t - \alpha))^2|Q'(t)|^2 + ((B(n, \omega, t))^2 - (B(n, \beta, t - \alpha))^2)|Q'(t)|^2
\leq(B(n, \beta, t - \alpha))^2 + ((B(n, \omega, t))^2 - (B(n, \beta, t - \alpha))^2)
=(B(n, \omega, t))^2.
\]
This finishes the proof in Case 1.

In Case 2, observe first that
\[
(B(n, \omega, t))^2 = n^2 \left( \frac{1 + \cos \omega}{\cos t - \cos \omega} + 1 \right) \geq n^2.
\]
Hence
\[
|Q''(t)|^2 + (B(n, \omega, t))^2|Q'(t)|^2
=|Q''(t)|^2 + n^2|Q'(t)|^2 + ((B(n, \omega, t))^2 - n^2)|Q'(t)|^2
\leq n^2 + ((B(n, \omega, t))^2 - n^2)
=(B(n, \omega, t))^2.
\]
This finishes the proof in Case 2.

A straightforward modification of the proof of Theorem 1 gives the following result.
Theorem 2. We have

\[ |Q'(t)|^2 + (B(n - \frac{1}{2}, \omega, t))^2 |Q(t)|^2 \leq (B(n - \frac{1}{2}, \omega, t))^2 \max_{\tau \in [-\omega, \omega]} |Q(\tau)|^2, \quad t \in (-\omega, \omega), \]

for all functions \( Q \) of the form

\[ Q(t) = \sum_{j=1}^{n} \left( a_j \cos \left( \frac{2j-1}{2} t \right) + b_j \sin \left( \frac{2j-1}{2} t \right) \right), \quad a_j, b_j \in \mathbb{R}. \]

It is routine now to derive the following Bernstein-type inequality from Theorems 1 and 2.

Theorem 3. We have

\[ |R'(t)| \leq B(n, \omega, t) \max_{\tau \in [-\omega, \omega]} |R(\tau)|, \quad t \in (-\omega, \omega), \]

for all real trigonometric polynomials \( R \in T_n^c \). Furthermore, we have

\[ |R'(t)| \leq B(n - \frac{1}{2}, \omega, t) \max_{\tau \in [-\omega, \omega]} |R(\tau)|, \quad t \in (-\omega, \omega), \]

for all functions \( R \) of the form

\[ R(t) = \sum_{j=1}^{n} \left( a_j \cos \left( \frac{2j-1}{2} t \right) + b_j \sin \left( \frac{2j-1}{2} t \right) \right), \quad a_j, b_j \in \mathbb{C}. \]

Proof. We prove only the first statement, the second one can be verified in the same way. If \( R \in T_n^c \) is a trigonometric polynomial of degree at most \( n \) and \( t \in (-\omega, \omega) \), then pick \( \alpha \in \mathbb{C} \) with \( |\alpha| = 1 \) so that \( \alpha R'(t) = |R'(t)| \). Now if we apply Theorem 1 to the real trigonometric polynomial \( Q \in T_n \) defined as \( Q(\tau) := \text{Re}(\alpha R(\tau)) \), we get theorem. \( \square \)

Now we can easily prove the following Bernstein-type inequality for complex polynomials on a subarc of the unit circle. This is stated as Theorem 1 in [10].

Theorem 4. We have

\[ |P'(e^{it})| \leq (n/2 + B(n/2, \omega, t)) \max_{\tau \in [-\omega, \omega]} |P(e^{i\tau})|, \quad t \in (-\omega, \omega), \]

for every algebraic polynomial \( P \in \mathcal{P}_n^c \).

Proof. Let \( P \in \mathcal{P}_n^c \). We introduce \( R \) as

\[ R(t) = e^{-int/2} P(e^{it}). \]
If \( n = 2m \) is even, then \( R \in T_m^c \), while if \( n = 2m - 1 \) is odd, then \( R \) is a function of the form

\[
R(t) = \sum_{j=1}^{n} \left( a_j \cos \left( \frac{(2j-1)t}{2} \right) + b_j \sin \left( \frac{(2j-1)t}{2} \right) \right), \quad a_j, b_j \in \mathbb{C}.
\]

In both cases we have

\[
R'(t) = e^{-int/2}(-int/2)P(e^{it}) + e^{-int/2}P'(e^{it})e^{it},
\]

and hence

\[
|P'(e^{it})| \leq |R'(t)| + \frac{n}{2} |P(e^{it})|.
\]

The theorem now follows from Theorem 3. \( \square \)

The next theorem is stated as Theorem 2 in [10] and it shows that Theorem 4 is sharp.

**Theorem 5.** For every \( \omega \in (0, \pi) \), \( t \in (-\omega, \omega) \) and \( n \in \mathbb{N} \) there are nonzero polynomials \( P_n \in P_n^c \) such that

\[
|P_n(e^{it})| \geq (1 - o(1))(n/2 + B(n/2, \omega, t)) \max_{\tau \in [-\omega, \omega]} |P_n(e^{i\tau})|, \quad t \in (-\omega, \omega).
\]

**Proof.** Let

\[
U_k(\tau) = \cos \left( 2k \arccos \left( \frac{\sin(\tau/2)}{\sin(\omega/2)} \right) \right), \quad t \in (-\omega, \omega),
\]

and

\[
V_k(\tau) = \sin \left( 2k \arcsin \left( \frac{\sin(\tau/2)}{\sin(\omega/2)} \right) \right), \quad t \in (-\omega, \omega).
\]

Let \( n = 2m \), \( m = k + u \), \( u = o(m) \),

\[
e^{-im\tau} P_n(e^{i\tau}) = Q_m(\tau) + iR_m(\tau)
\]

with \( Q_m := U_k \in T_m, R_m := V_k S_u \in T_m \), where \( S_u \) is defined as

\[
S_u(\tau) := H_u \left( \frac{\sin(\tau/2)}{\sin(\omega/2)} \right), \quad \tau \in [-\omega, \omega],
\]

with an odd \( H_u \in P_{2u-1} \). It is easy to see that the fact that \( H_u \in P_{2u-1} \) is odd ensures that \( R_m \in T_m \), indeed. Let \( t \in (-\omega, \omega) \) be fixed. Pick an \( \varepsilon > 0 \). Since the theorem claims only asymptotic sharpness, it is sufficient to prove the theorem in the case when \( |t| \geq \varepsilon \) and \( U_k(t) = 0 \). Using the Weierstrass Theorem we can pick the odd polynomial \( H_u \in P_{2u-1} \) so that

\[
S_u(t) = -V_k(t) \in \{-1, 1\} \quad \text{and} \quad \max_{\tau \in [-\omega, \omega]} |S_u(\tau)| \leq 1 + \varepsilon.
\]

Then

\[
|P_n(e^{i\tau})| = |Q_m(\tau)|^2 + |R_m(\tau)|^2 = |U_k(\tau)|^2 + |V_k(\tau)|^2 |S_u(\tau)|^2 \\
\leq (1 + \varepsilon)(|U_k(\tau)|^2 + |V_k(\tau)|^2) \leq 1 + \varepsilon, \quad \tau \in [-\omega, \omega],
\]

\[
(1 - o(1))(n/2 + B(n/2, \omega, t)) \max_{\tau \in [-\omega, \omega]} |P_n(e^{i\tau})|.
\]
that is,
\[
(6) \quad \max_{\tau \in [-\omega, \omega]} |P_n(e^{i\tau})| \leq 1 + \varepsilon.
\]

Also,
\[
ie^{i\tau} P'_n(e^{i\tau}) = \frac{d}{d\tau} (e^{im\tau} P_n(e^{i\tau})) = \frac{d}{d\tau} (e^{im\tau} (Q_m(\tau) + iR_m(\tau)))
\]
\[
= e^{im\tau} (Q'_m(\tau) + iR'_m(\tau) + im(Q_m(\tau) + iR_m(\tau))),
\]
and hence
\[
|P'_n(e^{it})| = |e^{-imt} i e^{it} P'_n(e^{it})| \geq |\text{Re}(e^{-imt} i e^{it} P'_n(e^{it}))|
\]
\[
\geq |Q'_m(t) - mR_m(t)| = |B(m, t, \omega) V_k(t) + mV_k(t)| = (B(m, t, \omega) + m)|V_k(t)| = B(m, t, \omega) + m.
\]

Combining this with (6) and recalling that \(n = 2m\), we get the theorem. □

Associated with an algebraic polynomial \(P_n\) of the form
\[
P_n(z) = \sum_{j=0}^{n} a_j z^j, \quad a_j \in \mathbb{C},
\]
we introduce the polynomial \(P^*_n\) defined by
\[
P^*_n(z) := \sum_{j=0}^{n} a_{n-j} z^j.
\]
The algebraic polynomial \(P_n\) of the above form is called conjugate reciprocal if \(P_n = P^*_n\).

**Theorem 6.** We have
\[
|P'(e^{it})| \leq \frac{1}{2} B(n, \omega, t) \max_{\tau \in [-\omega, \omega]} |P(e^{i\tau})|
\]
for every conjugate reciprocal algebraic polynomial \(P\) of degree \(n\), \(t \in (-\omega, \omega)\), and \(\omega \in (0, \pi)\).

**Proof.** If the algebraic polynomial \(P\) of degree \(n\) is conjugate reciprocal then \(Q\) defined by
\[
Q(\tau) = e^{-in\tau/2} P(e^{i\tau})
\]
is a real trigonometric polynomial of degree \(n/2\). Hence we can apply the Bernstein-Szegö inequality of Theorem 1 (when \(n\) is even) or Theorem 2 (when \(n\) is odd) to \(Q_n\) to obtain
\[
|P'(e^{i\tau})| = |ie^{i\tau} P'(e^{i\tau})| = |Q'(\tau)e^{in\tau/2} + (in/2)e^{in\tau/2}Q(\tau)| = |Q'(\tau) + (in/2)Q(\tau)|
\]
\[
\leq B(n/2, \omega, \tau) \max_{\tau \in [-\omega, \omega]} |Q(\tau)| = \frac{1}{2} B(n, \omega, \tau) \max_{\tau \in [-\omega, \omega]} |Q(\tau)|
\]
\]
□
Theorem 7. We have
\[
|P'(e^{it})| + |P^{*'}(e^{it})| \leq B(n, \omega, t) \max_{\tau \in [-\omega, \omega]} |P(e^{i\tau})|
\]
for every \(P \in \mathcal{P}_n^c\), \(t \in (-\omega, \omega)\), and \(\omega \in (0, \pi)\).

Proof. Let \(P \in \mathcal{P}_n^c\). Let \(c = e^{i\gamma}\), where \(\gamma \in \mathbb{R}\) will be chosen later. We define \(R := cP\). Then \(R^* = \overline{c}P_n\). Observe that \(S := R + R^*\) satisfies \(S = S^*\), and hence it is a conjugate reciprocal algebraic polynomial of degree at most \(n\). Observe that \(R^*(z) = z^{-n}R(z)\) and hence \(|R^*(z)| = |R(z)|\) for all \(z \in \mathbb{C}\) with \(|c| = 1\). Therefore
\[
\max_{\tau \in [-\omega, \omega]} |S(e^{i\tau})| \leq 2 \max_{\tau \in [-\omega, \omega]} |R(e^{i\tau})| = 2 \max_{\tau \in [-\omega, \omega]} |P(e^{i\tau})|.
\]
Using Theorem 6 with \(S\) we conclude that
\[
|cP' + \overline{c}P^{*'}(e^{it})| = |R' + R^{*'}(e^{it})| = |S'(e^{it})| \leq (1/2) B(n, \omega, t) \max_{\tau \in [-\omega, \omega]} |S(e^{i\tau})|
\leq B(n, \omega, t) \max_{\tau \in [-\omega, \omega]} |P(e^{i\tau})|
\]
Now we choose \(c = e^{i\gamma}\) so that
\[
e^{2i\gamma} \frac{P'(e^{it})}{P^{*'}(e^{it})} \in \mathbb{R},
\]
that is,
\[
\gamma := \frac{1}{2} \arg \left( \frac{P^{*'}(e^{it})}{P'(e^{it})} \right)
\]
if
\[
P'(e^{it}) \neq 0.
\]
If \(P'(e^{it}) \neq 0\), then \(\gamma \in \mathbb{R}\) can be arbitrary. We conclude that
\[
|P'(e^{it})| + |P^{*'}(e^{it})| = |(cP' + \overline{c}P^{*'})(e^{it})| = |(R' + R^{*'})(e^{it})| = |S'(e^{it})| \leq B(n, \omega, t) \max_{\tau \in [-\omega, \omega]} |S(e^{i\tau})|
\leq B(n, \omega, t) \max_{\tau \in [-\omega, \omega]} |P(e^{i\tau})|
\]
\[
\square
\]

Theorem 8. Suppose \(P \in \mathcal{P}_n^c\) has the property that \(1/\overline{a}\) is a zero of \(P\) with multiplicity at least \(k\) whenever \(a \in D\) is a zero of \(P\) with multiplicity \(k\) (there is no restriction on the zeros of \(P\) outside \(D\),
\[
|P'(e^{it})| \leq \frac{1}{2} B(n, \omega, t) \max_{\tau \in [-\omega, \omega]} |P(e^{i\tau})|
\]
for every \(t \in (-\omega, \omega)\), and \(\omega \in (0, \pi)\).

Proof. This follows from Theorem 7. We need to observe only that \(|P_n'(z)| \leq |P_n^{*'}(z)|\) for every \(z \in \mathbb{C}\) with \(|z| = 1\). See page 438 of [1], for instance. \(\square\)


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