LOWER BOUNDS FOR DERIVATIVES OF POLYNOMIALS AND REMEZ TYPE INEQUALITIES

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Abstract. P. Turán [!Tu] proved that if all the zeros of a polyniomial p lie in the unit interval $I \stackrel{\text{def}}{=} [-1,1]$, then $\|p'\|_{L^\infty(I)} \geq \sqrt{\deg(p)}/6\ \|p\|_{L^\infty(I)}$. Our goal is to study the feasibility of $\lim_{n\to\infty} \|{p_n}'\|_X/\|p_n\|_Y = \infty$ for sequences of polynomials $\{p_n\}_{n\in\mathbb{N}}$ whose zeros satisfy certain conditions, and to obtain lower bounds for derivatives of (generalized) polynomials and Remez type inequalities for generalized polynomials in various spaces.

1. Introduction

Markov and Bernstein type inequalities yield estimates of various norms of the derivatives of polynomials in terms of the (possibly different) norms of the polynomials themselves. For instance, the original A. A. Markov (cf. [!Na, p. 141]) inequality states that[1]

$$\|p'\|_{L^\infty(\Delta)} \leq \frac{2[\deg(p)]^2}{|\Delta|}\ \|p\|_{L^\infty(\Delta)} \tag{1.5}$$

for every algebraic polynomial p and interval $\Delta \subset \mathbb{R}$. Inequality (1.5) is sharp; if $n \in \mathbb{N}$ and p is the first kind Chebyshev polynomial of degree n associated with the interval Δ, then (1.5) turns into equality. Such inequalities play an essential role in proving inverse type theorems for approximation by polynomials; they are the crucial ingredient of *Bernstein's machinery*. It didn't take long to realize that (1.5) is no longer optimal for polynomials

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[1] In what follows, if A is a Lebesgue measurable subset of either $\mathbb{R}$ or $\mathbb{T} \stackrel{\text{def}}{=} \mathbb{R}/[0,2\pi)$ then the Lebesgue measure of A is denoted by either $m(A)$ or $|A|$.

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satisfying additional conditions. For instance, if p vanishes at the end points of Δ then according to a (sharp) inequality of I. Schur [!Sch, Theorem IV, p. 282]

$$\|p'\|_{L^\infty(\Delta)} \le \frac{2[\deg(p)]\cot\frac{\pi}{2\deg(p)}}{|\Delta|}\,\|p\|_{L^\infty(\Delta)}\,. \tag{1.10}$$

Under the even more restrictive condition that all the zeros of p are real and lie outside Δ, P. Erdős [!Er, Theorem, p. 310] and, according to P. Erdős, J. Erőd[2]proved the (asymptotically sharp) inequality

$$\|p'\|_{L^\infty(\Delta)} \le \frac{e\deg(p)}{|\Delta|}\,\|p\|_{L^\infty(\Delta)}\,. \tag{1.15}$$

It was their good friend P. Turán [!Tu] who asked the very natural question as to estimating polynomials in terms of their derivatives. He proved that, if all the zeros of p lie in the unit disk $D \overset{\text{def}}{=} \{\, z : |z| \le 1 \,\}$, then

$$\|p'\|_{L^\infty(D)} \ge \frac{\deg(p)}{2}\,\|p\|_{L^\infty(D)} \tag{1.20}$$

[!Tu, Satz I, p. 90], and, if all the zeros of p lie in the unit interval $I \overset{\text{def}}{=} [-1,1]$, then

$$\|p'\|_{L^\infty(I)} \ge \frac{\sqrt{\deg(p)}}{6}\,\|p\|_{L^\infty(I)} \tag{1.25}$$

[!Tu, Satz II, p. 91]. As a matter of fact, Turán proved

$$|p'(z)| \ge \frac{\deg(p)}{2}\,|p(z)|, \qquad z \in \partial D\,, \tag{1.30}$$

as long as all the zeros of p lie in D, from which not only (1.20) follows but also[3]

$$\|p'\|_{L^r_{d\mu}(\partial D)} \ge \frac{\deg(p)}{2}\,\|p\|_{L^r_{d\mu}(\partial D)}\,, \tag{1.35}$$

for all $0 \le r \le \infty$ and for all nonnegative Borel measures μ on the unit circle ∂D.

The proof of (1.30) is surprisingly natural and easy as opposed to the rather technical and counterintuitive proof of (1.25). Therefore, the analogue of (1.35) for polynomials with real zeros was and is a much more delicate problem. As a matter of fact, no such inequality is known in weighted L^q spaces.

[2]We thank József Szabados for finding out from Paul Erdős that Erőd never published his result.

[3]Here and in what follows, if $\mu \ge 0$ is a measure or $w \ge 0$ is a function in a set Ω then $\|\cdot\|_{L^p_{d\mu}(\Omega)} \overset{\text{def}}{=} \left(\int_\Omega |\cdot|^p\,d\mu\right)^{\frac{1}{p}}$ and $\|\cdot\|_{L^p_w(\Omega)} \overset{\text{def}}{=} \left(\int_\Omega |\cdot|^p w\right)^{\frac{1}{p}}$, respectively. For convenience, we will use these notations for all $0 < p < \infty$. If μ is the Lebesgue measure or $w \equiv 1$ then we will simply write $\|\cdot\|_{L^p(\Omega)}$.

The strongest result up-to-date is due the S. P. Zhou [!ZhIII, Theorem 4, p. 314] who proved that, if $0 < r \le q \le \infty$ and $1 \ge 1/r - 1/q$, then the following inverse Markov-Nikol′skiĭ type inequality

$$\|p'\|_{L^r(I)} \ge \mathrm{C}\,[\deg(p)]^{\frac{1}{2}-\frac{1}{2r}+\frac{1}{2q}}\,\|p\|_{L^q(I)}\ , \tag{1.40}$$

holds with a constant $\mathrm{C} = \mathrm{C}(r,q) > 0$ for every polynomial p whose zeros lie in the interval I. We refer to the references of [!ZhIII, p. 117–118] for a list of papers on this subject.

Our goal is to study a less ambitious analogue of (1.35). Namely, in Theorem 1.50 we investigate the feasibility of

$$\lim_{n\to\infty} \frac{\|{p_n}'\|_X}{\|p_n\|_Y} = \infty \tag{1.45}$$

for sequences of polynomials $\{p_n\}_{n\in\mathbb{N}}$ whose zeros satisfy certain conditions.

Our interest in this problem originates from the study of necessary conditions for the convergence of various approximation processes when one desperately needs limit relations such as (1.45). Luckily, in most applications one has the necessary information about the zeros of the polynomials.

Definition 1.40. Given a polynomial p and a set $\Omega \subset \mathbb{C}$, the number of zeros of p in Ω, counting multiplicities, is denoted by $Z_\Omega(p)$.

Our inverse Markov-Nikol′skiĭ type inequality is the following theorem.

Theorem 1.50. *Let $1 \le r \le \infty$, $0 < q < \infty$, $\ell \in \mathbb{N}$, and $m \in \mathbb{N}$. Let $\{\Delta_k \subset \mathbb{R}\}_{k=1}^m$ be a collection of disjoint bounded intervals of positive length, and let Ω be a bounded domain in $\mathbb{C}^+$.[4] Let $\Delta = \cup_{k=1}^m \Delta_k$. Let $u : \Delta \to \mathbb{R}^+$, $v : \Delta \to \mathbb{C}$, and $w : \Delta \to \mathbb{R}^+$ be such that $w \in L^1(\Delta)$, u is positive on a subset of Δ with positive Lebesgue measure, $(|v|u)^q w \in L^1(\Delta)$, $v \ne 0$ on a subset of Δ with positive Lebesgue measure, and $v^{(\ell-1)}$ is absolutely continuous in every closed subinterval of the interior of each Δ_k, $k = 1, 2, \dots, m$. Assume that for each $k = 1, 2, \dots, m$, either $u^{-1} \in L^\infty(\Delta_k)$ or else the function u is bounded and monotonic in Δ_k. Let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence of algebraic polynomials with zeros in $\mathbb{C}^+$ such that $\lim_{n\to\infty} Z_\Omega(p_n) = \infty$. Then*

$$\lim_{n\to\infty} \frac{\|(p_n v)^{(\ell)} u\|_{L^r(\Delta)}}{\|p_n v u\|_{L^q_w(\Delta)}} = \infty\ . \tag{1.50}$$

Remark 1.58. Without having some restrictions on the location of the zeros of $\{p_n\}_{n\in\mathbb{N}}$, formula (1.50) is not necessarily true. One of the many examples may be given by $p_n(x) \stackrel{\text{def}}{=} 1 + x^n$ with $r = 1$, $u \equiv 1$, $v \equiv 1$, and $\Delta = [0,1]$. Then all the zeros of p_n are in the unit disk but

$$\frac{\|{p_n}'\|_{L^1(\Delta)}}{\|p_n\|_{L^q_w(\Delta)}} \le \frac{1}{\left(\int_\Delta w\right)^{\frac{1}{q}}}\ .$$

[4] Here and in what follows, $\mathbb{R}^+ \stackrel{\text{def}}{=} \{\, x \in \mathbb{R} : x \ge 0 \,\}$, $\mathbb{C}^+ \stackrel{\text{def}}{=} \{\, z \in \mathbb{Z} : \Re z \ge 0 \,\}$, and $\mathbb{Z}^+ \stackrel{\text{def}}{=} \{\, n \in \mathbb{Z} : n \ge 0 \,\} \equiv \mathbb{N} \cup \{0\}$.

Other examples for

$$\limsup_{n\to\infty} \frac{\|(p_n v)^{(\ell)} u\|_{L^r(\Delta)}}{\|p_n v u\|_{L^q_w(\Delta)}} < \infty \, .$$

may be based on the inequality

$$\frac{|p'(x)|}{|p(x)|} \le \frac{\deg(p)}{\operatorname{dist}\{x, \operatorname{zeros}(p)\}}$$

which follows from the partial fraction decomposition of p'/p .

Remark 1.60. In general, the weight w in (1.50) cannot be replaced by positive Borel measures. For instance, if $\Delta = [a, b]$, u is bounded and nondecreasing on Δ, $v \equiv 1$, and $d\mu = w dx + J\delta(x - b)$, that is, we have a mass J at b, and if $p_n \ge 0$ and ${p_n}' \ge 0$ in Δ, then

$$\frac{\|{p_n}' u\|_{L^1(\Delta)}}{\|p_n u\|_{L^q_{d\mu}(\Delta)}} \le \frac{1}{J^{\frac{1}{q}}} \, .$$

Therefore, it may be interesting to find out what class of positive Borel measures could replace w in (1.50).

Remark 1.70. Clearly, $(|v|u)^q w \in L^1(\Delta)$ must hold. In addition, $w \in L^1(\Delta)$ cannot be omitted either. To see this, take, for instance, $r = 1$, $q = 1$, $p_n(x) = x^n$, $\Delta = [a, b]$ with $|a| < |b|$, $v(x) \equiv 1$, $u(x) \equiv (b - x)$, $v \equiv 1$, and $w(x) \equiv (b - x)^{-1}$. Then the limit in the right-hand side of (1.50) is equal to 1.

Remark 1.80. If $0 < r < 1$ then (1.50) is no longer valid for every $0 < q < \infty$. Take, for instance, $p_n(x) = x^n$, $\ell = 1$, and $\Delta = [0, 1]$, and let $u \equiv 1$, $v \equiv 1$, and $w \equiv 1$. Then the right-hand side of (1.50) is finite whenever $1 \le 1/r - 1/q$, that is, if $0 < r < 1$ and $q \ge r/(1 - r)$.

Corollary 1.100. *Let $1 \le r \le \infty$, $\ell \in \mathbb{N}$, and $m \in \mathbb{N}$. Let $\{\Delta_k \subset \mathbb{R}\}_{k=1}^m$ be a collection of disjoint bounded intervals of positive length, and let Ω be a bounded interval in $\mathbb{R}$. Let $\Delta = \cup_{k=1}^m \Delta_k$. Let $u(\ge 0) \in L^r(\Delta)$ be positive on a subset of Δ with positive Lebesgue measure. Assume that for each $k = 1, 2, \ldots, m$, either $u^{-1} \in L^\infty(\Delta_k)$ or else the function u is bounded and monotonic in Δ_k. Let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence of algebraic polynomials with zeros in $\mathbb{R}$ such that $\lim_{n\to\infty} Z_\Omega(p_n) = \infty$. Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of complex valued functions in Δ such that $g_n^{(\ell-1)}$ is absolutely continuous for every $n \in \mathbb{N}$ and*

$$\limsup_{n\in\mathbb{N}} \max_{1\le j\le\ell} \|g_n^{(j)}\|_{L^\infty(\Delta)} < \infty \quad \text{and} \quad \limsup_{n\in\mathbb{N}} \|g_n^{-1}\|_{L^\infty(\Delta)} < \infty \, . \tag{1.90}$$

Then

$$\liminf_{n\to\infty} \frac{\|\,(p_n g_n)^{(\ell)}\, u\|_{L^r(\Delta)}}{\|p_n^{(\ell)} g_n u\|_{L^r(\Delta)}} \ge 1 \quad \text{and} \quad \liminf_{n\to\infty} \frac{\|\,(p_n g_n)^{(\ell)}\, u\|_{L^r(\Delta)}}{\|p_n^{(\ell)} u\|_{L^r(\Delta)}} > 0 \, . \tag{1.100}$$

If $\ell = 1$ then (1.100) *remains valid even if the conditions that (i) Ω is a bounded interval in $\mathbb{R}$ and (ii) the zeros of $\{p_n\}_{n\in\mathbb{N}}$ are in $\mathbb{R}$, are replaced by (i) Ω is a bounded domain in $\mathbb{C}^+$ and (ii) the zeros of $\{p_n\}_{n\in\mathbb{N}}$ are in $\mathbb{C}^+$, respectively.*

Remark 1.15. Note that in Corollary 1.100 we assume that the zeros of $\{p_n\}_{n\in\mathbb{N}}$ are real (and $\Omega \subset \mathbb{R}$) because, in order to prove it, we need to use Theorem 1.50 not only with $\{p_n\}_{n\in\mathbb{N}}$ but with $\{p_n^{(j)}\}_{n\in\mathbb{N}}$ for $j = 1, 2, \ldots, \ell-1$ as well. Although, if the zeros of $\{p_n\}_{n\in\mathbb{N}}$ are real, then, by *Rolle's Theorem*, $\lim_{n\to\infty} Z_\Omega(p_n) = \infty$ guarantees $\lim_{n\to\infty} Z_\Omega(p_n^{(j)}) = \infty$ for $j > 0$, generally speaking

$$\lim_{n\to\infty} Z_\Omega(p_n) = \infty \quad \not\Longrightarrow \quad \lim_{n\to\infty} Z_\Omega({p_n}') = \infty\,, \qquad \Omega \subset \mathbb{C}\,.$$

For instance,[5]if $q_n(z) \stackrel{\text{def}}{=} \sum_{k=1}^n z^k/k!$ and $p_n(z) \stackrel{\text{def}}{=} q_n(a_n z)$ where a_n is the square root of the absolute value of the smallest (in absolute value) zero of ${q_n}'$, then, by *Hurwitz's Theorem* (cf. [!W, p. 6]), ∞ is the only limit point for the zeros of $\{{p_n}'\}_{n\in\mathbb{N}}$ since $\lim_{n\to\infty} {q_n}'(z) = \exp(z) \neq 0$ for $z \in \mathbb{C}$, whereas $\lim_{n\to\infty} Z_{\{z:|z|\le 1\}}(p_n) = \infty$ since $\lim_{n\to\infty} q_n(z) = \exp(z) - 1$ and the function $\exp(z) - 1$ vanishes infinitely many times on the imaginary line.

The trigonometric analogue of Theorem 1.50 is the following theorem.

Theorem 1.150. *Let $1 \le r \le \infty$, $0 < q < \infty$, $\ell \in \mathbb{N}$, and $m \in \mathbb{N}$. Let $\{\Delta_k \subset \mathbb{T}\}_{k=1}^m$ be a collection of disjoint bounded intervals of positive length. Let $\Delta = \cup_{k=1}^m \Delta_k$. Let $u : \Delta \to \mathbb{R}^+$, $v : \Delta \to \mathbb{C}$, and $w : \Delta \to \mathbb{R}^+$ be such that $w \in L^1(\Delta)$, u is positive on a subset of Δ with positive Lebesgue measure, $(|v|u)^q w \in L^1(\Delta)$, $v \neq 0$ on a subset of Δ with positive Lebesgue measure, and $v^{(\ell-1)}$ is absolutely continuous in every closed subinterval of the interior of each Δ_k, $k = 1, 2, \ldots, m$. Assume that for each $k = 1, 2, \ldots, m$, either $u^{-1} \in L^\infty(\Delta_k)$ or else the function u is bounded and monotonic in Δ_k. Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of trigonometric polynomials with real zeros such that $\lim_{n\to\infty} Z_\mathbb{T}(t_n) = \infty$.[6] Then*

$$\lim_{n\to\infty} \frac{\|(t_n v)^{(\ell)} u\|_{L^r(\Delta)}}{\|t_n v u\|_{L^q_w(\Delta)}} = \infty\,. \tag{1.150}$$

Remark 1.160. Note that in Theorem 1.150, in contrast to Theorem 1.50, we deal with trigonometric polynomials with *real* zeros only. It remains to be seen whether Theorem 1.50 can be extended to trigonometric polynomials with complex zeros.

Definition 1.303. The set of all polynomials, real polynomials, trigonometric polynomials, and real trigonometric polynomials is denoted by $\mathcal{P}$, $\mathcal{P}^r$, $\mathcal{T}$, and $\mathcal{T}^r$, respectively. Given $n \in \mathbb{Z}^+$, the set of all polynomials, real polynomials, trigonometric polynomials, and real trigonometric polynomials of degree at most n is denoted by $\mathcal{P}_n$, $\mathcal{P}_n^r$, $\mathcal{T}_n$, and $\mathcal{T}_n^r$, respectively.

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[5]We thank Peter Borwein for this example.

[6]In other words, $\lim_{n\to\infty} \deg(t_n) = \infty$.

Definition 1.305. The function

$$f(z)=\omega\prod_{j=1}^{k}|z-z_j|^{r_j}, \qquad \omega>0,\ z_j\in\mathbb{C},\ z\in\mathbb{C},\ r_j>0,$$

is called a *nonnegative generalized complex algebraic polynomial*, that is, $f\in\mathrm{NGAP}$, of degree $N=\sum_{j=1}^{k}r_j$, that is, $f\in\mathrm{NGAP}_N$. Similarly, the function

$$f(z)=\omega\prod_{j=1}^{k}\left|\sin\frac{z-z_j}{2}\right|^{r_j}, \qquad \omega>0,\ z_j\in\mathbb{C},\ z\in\mathbb{C},\ r_j>0,$$

is called a *nonnegative generalized complex trigonometric polynomial*, that is, $f\in\mathrm{NGTP}$, of degree $N=\frac{1}{2}\sum_{j=1}^{k}r_j$, that is, $f\in\mathrm{NGTP}_N$. If $f\in\mathrm{NGAP}$ or $f\in\mathrm{NGAP}$ and $z_0\in\mathbb{R}$ is a zero of f, then the greatest $r\in\mathbb{R}$ such that $f(\cdot)/|\cdot-z_0|^r\in\mathrm{NGAP}$ or $f(\cdot)/|\sin((\cdot-z_0)/2)|^r\in\mathrm{NGTP}$, respectively, is called the *multiplicity* of z_0.

Remark 1.307. In what follows we will consider both types of generalized polynomials for *real arguments* only, and then we can assume without loss of generality that each zero z_j appears in conjugate pairs. For instance, in the algebraic case, just write

$$f(x)=\omega\prod_{j=1}^{k}|x-z_j|^{r_j/2}|x-\overline{z_j}|^{r_j/2}\ , \qquad x\in\mathbb{R},$$

whereas in the trigonometric case, write

$$f(x)=\omega\prod_{j=1}^{k}\left|\sin\frac{x-z_j}{2}\right|^{r_j}\left|\sin\frac{x-\overline{z_j}}{2}\right|^{r_j}\ , \qquad x\in\mathbb{T}\ .$$

For real arguments both types of generalized polynomials have derivatives everywhere except at their real zeros with multiplicities at most 1. If $f\in\mathrm{NGAP}$ or $f\in\mathrm{NGTP}$ and $z_0\in\mathbb{R}$ is a zero of f with multiplicity at most 1 then both the (finite or infinite) left and right derivatives $f'_\pm$ of f are well defined at z_0 and they have the same absolute values, and in this case we set $|f'|\stackrel{\text{def}}{=}|f'_+|$. Therefore, if $f\in\mathrm{NGAP}$ or $f\in\mathrm{NGTP}$ then the (finite or infinite) $|f'|$ is well defined in $\mathbb{R}$.

The following two theorems with L^∞ and L^p Remez type inequalities for generalized polynomials are quite useful.

Theorem 1.3010. *There is an absolute constant* $\mathrm{c}\in\mathbb{R}^+$ *such that*

$$\|f\|_{L^\infty(\mathbb{T})}\le\begin{cases}\exp\left(\mathrm{c}\deg(f)\ s\right)\|f\|_{L^\infty(\mathbb{T}\setminus A)}\ , & 0<s<\pi/2\ ,\\ \exp\left(\mathrm{c}\deg(f)\ |\log(2\pi-s)|\right)\|f\|_{L^\infty(\mathbb{T}\setminus A)}\ , & \pi/2\le s<2\pi\ ,\end{cases} \tag{1.3000}$$

holds for every $f\in\mathrm{NGTP}$ *and* $A\subset\mathbb{T}$ *with* $m(A)\le s$. *In particular, given* $0<\epsilon<2\pi$, *there exists a constant* $\mathrm{c}_\epsilon\in\mathbb{R}^+$ *such that if* $m(A)\le2\pi-\epsilon$ *then the inequality*

$$\|f\|_{L^\infty(\mathbb{T})}\le\exp\left(\mathrm{c}_\epsilon\deg(f)\ m(A)\right)\|f\|_{L^\infty(\mathbb{T}\setminus A)} \tag{1.3005}$$

holds for every $f \in \mathrm{NGTP}$. *In other words,*

$$\|f\|_{L^\infty(\mathbb{T})} \le \exp\left(\mathrm{c}_\epsilon \deg(f)\, m(\, x \in \mathbb{T} : f(x) \ge 1 \,)\right) \tag{1.3010}$$

holds for every $f \in \mathrm{NGTP}$ *as long as the measure of the set* $\{\, x \in \mathbb{T} : f(x) \ge 1 \,\}$ *is less or equal* $2\pi - \epsilon$.

Remark 1.3015. The case $0 < s < \pi/2$ in (1.3000) was proved in [!ELond, Theorem 2, p. 257], whereas the case $\pi/2 \le s < 2\pi$ in (1.3000) was announced in [!ESurv, formula (3.10), p. 174].

Theorem 1.3020. *Let* χ *be a nonnegative and nondecreasing function in* $\mathbb{R}^+$ *such that* $\chi(\cdot)/(\cdot)$ *is nonincreasing. Let* $0 < p < \infty$. *Given* $0 < \delta < 2\pi$, *there exist two constants* $\mathrm{d}_\delta \in \mathbb{R}^+$ *and* $\mathrm{D}_\delta \in \mathbb{R}^+$ *such that the inequality*

$$\int_{\mathbb{T}} \chi^p(f) \le \left(1 + \mathrm{D}_\delta \exp\left(\mathrm{d}_\delta\, p \deg(f)\, m(A)\right)\right) \int_{\mathbb{T}\setminus A} \chi^p(f) \tag{1.3020}$$

holds for every $f \in \mathrm{NGTP}$ *and for every* $A \subset \mathbb{T}$ *with* $m(A) \le 2\pi - \delta$.

Remark 1.3025. This theorem is a slight extension of [!EMN, Theorem 8, p. 247] where 1.3020 was proved for $7\pi/4 < \delta < 2\pi$.

Generalized polynomials can be estimated in terms of their derivatives by the following

Theorem 1.3030. *Let* $1 \le p < \infty$, $0 < N < \infty$, $0 \le \Gamma < \infty$, $0 < \delta < 2\pi$, $f \in \mathrm{NGTP}_N$, *and* $u \in \mathrm{NGTP}_\Gamma$. *Assume that* f *has at least one real zero such that* u *does not vanish at that zero. Let* $\Delta = \cup \Delta_k$ *where (i)* $\Delta_k \subseteq \mathbb{T}$ *are disjoint intervals, (ii) the closure of each* Δ_k *contains at least one zero of* f, *(iii)* $\delta < m(\Delta) \le 2\pi$, *and (iv)*

$$\Gamma \max_k |\Delta_k| \sup_{\substack{x\in\Delta \\ u(\theta)=0}} |\cot(x-\theta)| \le \frac{1}{2} \,. \tag{1.3030}$$

Then there exist two constants $\mathrm{d}_\delta \in \mathbb{R}^+$ *and* $\mathrm{D}_\delta \in \mathbb{R}^+$ *such that*

$$\|f\|_{L^p_u(\mathbb{T})} \le 2p \max_k |\Delta_k| \left(1 + \mathrm{D}_\delta \exp\left(\mathrm{d}_\delta\, p(N+\Gamma)(2\pi - m(\Delta))\right)\right) \|f'\|_{L^p_u(\mathbb{T})} \,. \tag{1.3040}$$

The following two corollaries show that the weighted L^p type Bernstein and Markov inequalities (see, for instance, [!NMem, Theorems 6.16 and 6.19, pp. 163–164], [!NBer, Theorem 5, p. 242], and [!DT, Theorem 8.4.7, p. 107]) are sharp for polynomials with sufficiently uniformly distributed zeros.

Corollary 1.3100. *Let* $1 \le p < \infty$, $u \in \mathrm{NGTP}$, $m \in \mathbb{N}$, *and* $\ell \in \mathbb{N}$. *Let* $\{t_n\}_{n\in\mathbb{N}}$ *be a sequence of real trigonometric polynomials with* $\deg(t_n) \le mn$ *for each* $n \in \mathbb{N}$. *Assume that there is a constant* $\mathrm{d} \in \mathbb{R}^+$ *such that for each* $n \in \mathbb{N}$ *there is a partition* $\cup_i \tau_{in} = \mathbb{T}$ *where (i)* $\tau_{in} \subset \mathbb{T}$ *are disjoint intervals, (ii) the closure of each* τ_{in} *contains at least one zero of* t_n, *and (iii)* $m(\tau_{in}) \le d/n$. *Then there exists a constant* $\mathrm{C} \in \mathbb{R}^+$ *independent of* $\{t_n\}_{n\in\mathbb{N}}$ *such that*

$$\limsup_{n\to\infty} \frac{n^\ell \|t_n^{(j)}\|_{L^p_u(\mathbb{T})}}{n^j \|t_n^{(\ell)}\|_{L^p_u(\mathbb{T})}} \le \mathrm{C} \,, \qquad j = 0, 1, \ldots, \ell - 1 \,. \tag{1.3140}$$

Corollary 1.3200. *Let $\Delta \subset \mathbb{R}$ be an interval with endpoints, say, $a < b$, and let $w_\Delta(x) \stackrel{\text{def}}{=} \sqrt{(b-x)(x-a)}$ for $x \in \Delta$. Let $1 \le p < \infty$, $u \in \mathrm{NGAP}$, $m \in \mathbb{N}$, and $\ell \in \mathbb{N}$. Let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence of real algebraic polynomials with $\deg(p_n) \le mn$ for each $n \in \mathbb{N}$. Assume that there is a constant $\mathrm{d} \in \mathbb{R}^+$ such that for each $n \in \mathbb{N}$ there is a partition $\cup_i \tau_{in} = [0,\pi]$ where (i) $\tau_{in} \subset [0,\pi]$ are disjoint intervals, (ii) the closure of each $\frac{b-a}{2}\cos(\tau_{in}) + \frac{b+a}{2}$ contains at least one zero of p_n, and (iii) $m(\tau_{in}) \le d/n$. Then there exists a constant $\mathrm{C} \in \mathbb{R}^+$ independent of $\{p_n\}_{n\in\mathbb{N}}$ such that*

$$\limsup_{n\to\infty} \frac{n^\ell \|p_n^{(j)}\|_{L^p_u(\Delta)}}{n^j \|w_\Delta^{\ell-j} p_n^{(\ell)}\|_{L^p_u(\Delta)}} \le \mathrm{C}\,, \qquad j = 0,1,\dots,\ell-1\,. \tag{1.3240}$$

In Theorem 1.3030 and Corollaries 1.3100 and 1.3200 we apply Remez type inequalities to generalized polynomials to obtain estimates in the entire set $\mathbb{T}$ for trigonometric polynomials and in an interval, say Δ, for algebraic polynomials. In the following theorem and in two of its corollaries we prove similar inequalities for more general functions with sufficiently many zeros which are valid only on subsets of $\mathbb{T}$ or Δ, though these subsets, generally speaking, are close to $\mathbb{T}$ or Δ, and, thereby, they are useful in applications.

Theorem 1.3300. *Let*

$$u(z) \stackrel{\text{def}}{=} \prod_{j=1}^{k} \left|\sin\frac{z-z_j}{2}\right|^{\Gamma_j}, \qquad z_j \in \mathbb{T},\ z \in \mathbb{T},\ \Gamma_j \in \mathbb{R},\ 2\Gamma \stackrel{\text{def}}{=} \sum_j |\Gamma_j|\,, \tag{1.3301}$$

be fixed. Let $\tau = \cup_i \tau_i$ where $\tau_i \subset \mathbb{T}$ are disjoint intervals. Let $1 \le p < \infty$ and let $\epsilon > 0$. Let f be differentiable almost everywhere in each τ_i, and let $|f|^p$ be absolutely continuous in each τ_i. Let the closure of each τ_i contain at least one zero of f. Let τ_{i_k} be those intervals τ_{i_k} for which $\mathrm{dist}(\tau_{i_k}, \{z_j\}) \ge \epsilon$. Let $\Delta = \cup_k \tau_{i_k}$. Then

$$\|f\|_{L^p_u(\Delta)} \le p \sup_i m(\tau_i)\, \exp\left(\sup_i m(\tau_i)\, \Gamma\pi/\epsilon\right) \|f'\|_{L^p_u(\Delta)}\,. \tag{1.3305}$$

Corollary 1.3310. *Let u be given by (1.3301). Let $1 \le p < \infty$, $\ell \in \mathbb{N}$, and let $\{\epsilon_n\}_{n\in\mathbb{N}}$ be a positive sequence such that $\liminf_{n\to\infty} n\epsilon_n > 0$. Let $\{f_n\}_{n\in\mathbb{N}}$ be such that $f_n^{(\ell)}$ exists almost everywhere in $\mathbb{T}$, and $|f_n^{(\ell-1)}|^p$ is absolutely continuous in $\mathbb{T}$ for each $n \in \mathbb{N}$. If $\ell > 1$ then we also assume that each f_n is real valued. Assume that there is a constant $\mathrm{d} \in \mathbb{R}^+$ such that for each $n \in \mathbb{N}$ there is a partition $\cup_i \tau_{in} = \mathbb{T}$ where (i) $\tau_{in} \subset \mathbb{T}$ are disjoint intervals, (ii) the closure of each τ_{in} contains at least one zero of f_n, and (iii) $m(\tau_{in}) \le d/n$. Then there exist a sequence of open sets $\{\Delta_n \subset \mathbb{T}\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} m(\mathbb{T} \setminus \Delta_n) = 0$ and a constant $\mathrm{C} \in \mathbb{R}^+$ independent of $\{f_n\}_{n\in\mathbb{N}}$ such that*

$$\limsup_{n\to\infty} \frac{n^\ell \|f_n^{(j)}\|_{L^p_u(\Delta_n)}}{n^j \|f_n^{(\ell)}\|_{L^p_u(\Delta_n)}} < \mathrm{C}\,, \qquad j = 0,1,\dots,\ell-1\,. \tag{1.3310}$$

Corollary 1.3320. *Let $\Delta \subset \mathbb{R}$ be an interval with endpoints, say, $a < b$, and let $w_\Delta(x) \stackrel{\text{def}}{=} \sqrt{(b-x)(x-a)}$ for $x \in \Delta$. Let*

$$u(z) \stackrel{\text{def}}{=} \prod_{j=1}^{k} |z - z_j|^{\Gamma_j}, \qquad z_j \in \Delta,\ z \in \Delta,\ \Gamma_j \in \mathbb{R}\,, \tag{1.3321}$$

be fixed. Let $1 \le p < \infty$, $\ell \in \mathbb{N}$, and let $\{\epsilon_n\}_{n\in\mathbb{N}}$ be a positive sequence such that $\liminf_{n\to\infty} n\epsilon_n > 0$. Let $\{f_n\}_{n\in\mathbb{N}}$ be such that $f_n^{(\ell)}$ exists almost everywhere in Δ, and $|f_n^{(\ell-1)}|^p$ is absolutely continuous in Δ for each $n \in \mathbb{N}$. If $\ell > 1$ then we also assume that each f_n is real valued. Assume that there is a constant $\mathrm{d} \in \mathbb{R}^+$ such that for each $n \in \mathbb{N}$ there is a partition $\cup_i \tau_{in} = [0,\pi]$ where (i) $\tau_{in} \subset [0,\pi]$ are disjoint intervals, (ii) the closure of each $\frac{b-a}{2}\cos(\tau_{in}) + \frac{b+a}{2}$ contains at least one zero of p_n, and (iii) $m(\tau_{in}) \le d/n$. Then there exist a sequence of open sets $\{\Delta_n \subset \Delta\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} m(\Delta \setminus \Delta_n) = 0$ and a constant $\mathrm{C} \in \mathbb{R}^+$ independent of $\{f_n\}_{n\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty} \frac{n^\ell \|f_n^{(j)}\|_{L^p_u(\Delta_n)}}{n^j \|w_\Delta^{\ell-j} f_n^{(\ell)}\|_{L^p_u(\Delta_n)}} < \mathrm{C}\,, \qquad j = 0, 1, \ldots, \ell - 1\,. \tag{1.3325}$$

2. Proofs of Theorem 1.50, Corollary 1.100, and Theorem 1.150

To prove Theorem 1.50, we need the following

Lemma 2.1010. *Given a number $0 < p \le \infty$, an interval $\Delta \subset \mathbb{R}$ of positive length, a function $\sigma \ge 0$ with $0 < \int_\Delta \sigma < \infty$, and an integer $k \ge 0$, then there exists a constant $\mathrm{B} \in \mathbb{R}^+$ such that if $f : \Delta \to \mathbb{R}$ is k-times continuously differentiable in the interior of Δ, then*[7]

$$\inf_{x\in\Delta} \left| f^{(k)}(x) \right| \le \mathrm{B}\, \|f\|_{L^p_\sigma(\Delta)}\,. \tag{2.1010}$$

Remark 2.1020. It would be interesting to find the optimal value of the constant B in (2.1010) for various classes of functions. For instance, if $\sigma \equiv 1$, then it is easy to show that the optimal value B_{opt} of the constant B satisfies

$$\mathrm{B}_{\text{opt}} \le (k+1)!\, \left(\frac{2k+1}{|\Delta|}\right)^{k+\frac{1}{p}} \|f\|_{L^p(\Delta)}\,.$$

Remark 2.1030. If $k > 0$, then inequality (2.1010) no longer holds for all *complex* valued functions. To see this, take, for instance, $f \stackrel{\text{def}}{=} \exp(i\lambda\cdot)$.

Proof of Lemma 2.1010. If $k = 0$ then for $0 < p < \infty$ inequality (2.1010) follows from the (first) mean value theorem for integrals, whereas for $p = \infty$ it is straightforward.

[7] We emphasize that B is independent of f.

Otherwise, we find $k+1$ disjoint *closed* intervals $\tau_j \subset \Delta^\circ$ such that $0 < \int_{\tau_j} \sigma < \infty$ for $j = 1, 2, \ldots, k+1$. By the case $k = 0$, we can pick $k+1$ points $x_j \in \tau_j$ such that

$$|f(x_j)| \overset{\text{def}}{=} \min_{x\in\tau_j} |f(x)| \le \mathrm{B}_{1,j} \, \|f\|_{L^p_\sigma(\tau_j)} , \qquad j = 1, 2, \ldots, k+1 ,$$

($\mathrm{B}_{1,j} \in \mathbb{R}^+$ is independent of f) so that the divided difference

$$[x_1, x_2, \ldots, x_{k+1}; f] \overset{\text{def}}{=} \sum_{j=1}^{k+1} \frac{f(x_j)}{\prod_{k\ne j}(x_j - x_k)}$$

(cf. [!MT, formula (1), p. 7]) satisfies

$$|[x_1, x_2, \ldots, x_{k+1}; f]| \le \mathrm{B}_2 \, \|f\|_{L^p_\sigma(\Delta)}$$

($\mathrm{B}_2 \in \mathbb{R}^+$ is independent of f) since $\min_j \prod_{k\ne j} |x_j - x_k| \ge \min_j \prod_{k\ne j} \operatorname{dist}(\tau_j, \tau_k) > 0$. Therefore, inequality (2.1010) follows from the mean value theorem for divided differences (cf. [!MT, formula (2), p. 6]). $\square$

Proof of Theorem 1.50. The basic idea of the proof is to show that if (1.50) were not true then there would be a sequence of algebraic polynomials which would converge to an entire function locally uniformly in $\mathbb{C}$ such that (i) this function would not vanish identically in $\mathbb{C}$, and (ii) it would have infinitely many zeros in a bounded domain. The details are given as follows.

In view of Hölder's inequality, it is sufficient to prove (1.50) for $r = 1$.

Step 1. We will prove (1.50) for $m = 1$ ($\Delta = \Delta_1$) when $u \equiv 1$. We can assume that $|v|^q w > 0$ on a set of positive Lebesgue measure.

If (1.50) were not true then there would be a sequence of algebraic polynomials $\{\tilde p_n\}_{n\in\mathbb{N}}$ with zeros in $\mathbb{C}^+$ satisfying

$$\text{(i)} \quad \lim_{n\to\infty} Z_\Omega(\tilde p_n) = \infty , \quad \text{(ii)} \quad \|\tilde p_n v\|_{L^q_w(\Delta)} = 1 , \quad \text{(iii)} \quad \sup_{n\in\mathbb{N}} \|(\tilde p_n v)^{(\ell)}\|_{L^1(\Delta)} < \infty . \tag{2.30}$$

We claim that the sequences $\{\tilde p_n v\}_{n\in\mathbb{N}}$ and $\{(\tilde p_n v)'\}_{n\in\mathbb{N}}$ are bounded in $L^\infty(\Delta)$ and $L^1(\Delta)$, respectively, that is,

$$\sup_{n\in\mathbb{N}} \|\tilde p_n v\|_{L^\infty(\Delta)} < \infty \qquad \text{and} \qquad \sup_{n\in\mathbb{N}} \|(\tilde p_n v)'\|_{L^1(\Delta)} < \infty . \tag{2.40}$$

Applying Lemma 2.1010 with $p = q$, $k = \ell - 1$, $\sigma \equiv w$, and $f \equiv \Re(\tilde p_n v)$, and using (ii) and (iii) in (2.30), we obtain

$$\sup_{n\in\mathbb{N}} \|\Re(\tilde p_n v)^{(\ell-1)}\|_{L^\infty(\Delta)} < \infty , \tag{2.45}$$

and, similarly,

$$\sup_{n\in\mathbb{N}} \|\Im(\tilde p_n v)^{(\ell-1)}\|_{L^\infty(\Delta)} < \infty , \tag{2.46}$$

that is, (2.40) holds if $\ell = 1$. If $\ell > 1$ then we can use Lemma 2.1010 repeatedly with $f \equiv \Re(\tilde{p}_n v)$ and $f \equiv \Im(\tilde{p}_n v)$ to show that (2.45) and (2.46) together with (ii) in (2.30) imply

$$\sup_{n\in\mathbb{N}} \|\Re(\tilde{p}_n v)^{(k)}\|_{L^\infty(\Delta)} < \infty \,, \qquad k = \ell-2, \ell-3, \dots, 0 \,,$$

and,

$$\sup_{n\in\mathbb{N}} \|\Im(\tilde{p}_n v)^{(k)}\|_{L^\infty(\Delta)} < \infty \,, \qquad k = \ell-2, \ell-3, \dots, 0 \,,$$

so that (2.40) holds for $\ell > 0$ as well.

Using (2.40), we can apply *Helly's Selection Theorem* (cf. [!Fr, p. 56]) to pick a subsequence $\{\tilde{p}_{n_k}\}_{k\in\mathbb{N}}$ such that $\lim_{n\to\infty} \tilde{p}_{n_k}(x)v(x)$ exists for every $x \in \Delta$. Let $f(x) \overset{\text{def}}{=} \lim_{n\to\infty} \tilde{p}_{n_k}(x)1_{v\neq 0}(x)$ for $x \in \Delta$.[8]By *Lebesgue's Dominated Convergence Theorem*,

$$1 = \|\tilde{p}_{n_k} v\|_{L^q_w(\Delta)} \xrightarrow[k\to\infty]{} \|fv\|_{L^q_w(\Delta)} \,, \tag{2.50}$$

so that f exists and $f \neq 0$ on a set $E \subset \{\, t : v(t) \neq 0 \,\} \subset \Delta$ with positive Lebesgue measure. Thus, by a theorem of B. Ya. Levin in [!Le, Theorem 2, p. 385] (cf. [!CK, Corollary 1.3, p. 110]), the limit $f_c(z) \overset{\text{def}}{=} \lim_{n\to\infty} \tilde{p}_{n_k}(z)$ exists locally uniformly in $\mathbb{C}$, where f_c is an entire function (from the Pólya-Obrechkoff class). Since $\lim_{n\to\infty} Z_\Omega(\tilde{p}_{n_k}) = \infty$ (cf. (i) in (2.30)), by *Hurwitz's Theorem* (cf. [!W, p. 6]), $f_c \equiv 0$. This contradicts (2.50) since $f_c v \equiv fv$ in Δ. Therefore,

$$\lim_{n\to\infty} \frac{\|(p_n v)^{(\ell)}\|_{L^1(\Delta)}}{\|p_n v\|_{L^q_w(\Delta)}} = \infty \tag{2.60}$$

holds.

Step 2. We will prove (1.50) for $m = 1$ ($\Delta = \Delta_1$) when $u^{-1} \in L^\infty(\Delta)$. This follows immediately from what we have already proved in Step 1 applied with $u^q w$ in place of w since

$$\|(p_n v)^{(\ell)} u\|_{L^1(\Delta)} \geq \|u^{-1}\|^{-1}_{L^\infty(\Delta)} \|(p_n v)^{(\ell)}\|_{L^1(\Delta)} \,.$$

Step 3. We will prove (1.50) for $m = 1$ ($\Delta = \Delta_1$) when u is bounded and monotonic in Δ. We can assume that u does not vanish inside Δ since otherwise u would vanish on a subinterval of Δ sharing an endpoint with Δ, and, then we could replace Δ by a smaller interval without changing any of the integrals in (1.50).

Write Δ as the union of two intervals, $\Delta = \Delta' \cup \Delta''$, where $u \in L^\infty(\Delta')$ but $\inf_{\Delta'} u = 0$ and $u^{-1} \in L^\infty(\Delta'')$. If $|\Delta'| = 0$ then we are back to Step 2 and, therefore, (1.50) holds. Otherwise, we can always assume that $|\Delta''| > 0$, and then

$$\begin{aligned} \frac{\|(p_n v)^{(\ell)} u\|_{L^1(\Delta)}}{\|(p_n v u\|_{L^q_w(\Delta)}} &= \frac{\|(p_n v)^{(\ell)} u\|_{L^1(\Delta')} + \|(p_n v)^{(\ell)} u\|_{L^1(\Delta'')}}{\left(\|(p_n v u\|^q_{L^q_w(\Delta')} + \|(p_n v u\|^q_{L^q_w(\Delta'')}\right)^{\frac{1}{q}}} \\ &\geq \frac{1}{2^{\frac{1}{q}}} \min\left\{ \frac{\|(p_n v)^{(\ell)} u\|_{L^1(\Delta')}}{\|(p_n v u\|_{L^q_w(\Delta')}} , \frac{\|(p_n v)^{(\ell)} u\|_{L^1(\Delta'')}}{\|(p_n v u\|_{L^q_w(\Delta'')}} \right\} \,. \end{aligned} \tag{2.80}$$

[8]Here and in what follows, the characteristic function of a set $\mathcal{E}$ is denoted by $1_{\mathcal{E}}$.

We need to prove

$$\lim_{n\to\infty} \frac{\|(p_n v)^{(\ell)} u\|_{L^1(\Delta')}}{\|(p_n v u\|_{L^q_w(\Delta')}} = \infty \ , \tag{2.90}$$

since by Step 2

$$\lim_{n\to\infty} \frac{\|(p_n v)^{(\ell)} u\|_{L^1(\Delta'')}}{\|(p_n v u\|_{L^q_w(\Delta'')}} = \infty \ . \tag{2.100}$$

If $(|v|u)^q w = 0$ a.e. in Δ' then (2.90) is certainly true. Now assume that $(|v|u)^q w > 0$ on a set of positive Lebesgue measure in Δ'. If (2.90) were not true then, analogously to Step 1, there would be a sequence of algebraic polynomials $\{\tilde{p}_n\}_{n\in\mathbb{N}}$ with zeros in $\mathbb{C}^+$ satisfying

$$\text{(i)} \quad \lim_{n\to\infty} Z_\Omega(\tilde{p}_n) = \infty \ , \quad \text{(ii)} \quad \|\tilde{p}_n v u\|_{L^q_w(\Delta')} = 1 \ , \quad \text{(iii)} \quad \sup_{n\in\mathbb{N}} \|(\tilde{p}_n v)^{(\ell)} u\|_{L^1(\Delta')} < \infty \ . \tag{2.110}$$

As in Step 1, we claim that the sequences $\{\tilde{p}_n v u\}_{n\in\mathbb{N}}$ and $\{(\tilde{p}_n v u)'\}_{n\in\mathbb{N}}$ are bounded in $L^\infty(\Delta')$ and $L^1(\Delta')$, respectively, that is,

$$\sup_{n\in\mathbb{N}} \|\tilde{p}_n v u\|_{L^\infty(\Delta')} < \infty \qquad \text{and} \qquad \sup_{n\in\mathbb{N}} \|(\tilde{p}_n v)' u\|_{L^1(\Delta')} < \infty \ . \tag{2.120}$$

To this end we pick a closed interval $\Delta^* \subset (\Delta')^\circ$ such that $\int_{\Delta^*} u^q w > 0$. We use Lemma 2.1010 in the interval Δ^* with $p = q$, $k = \ell - 1$, $\sigma \equiv u^q w$, and with the functions $f \equiv \Re(\tilde{p}_n v)$ and $f \equiv \Im(\tilde{p}_n v)$, to show that for each $n \in \mathbb{N}$ there are $x_n \in \Delta^*$ and $y_n \in \Delta^*$ such that

$$\sup_{n\in\mathbb{N}} \left| \Re(\tilde{p}_n v)^{(\ell-1)}(x_n) \right| \le \mathrm{B}^* \, \|\tilde{p}_n v u\|_{L^q_w(\Delta^*)}$$

and

$$\sup_{n\in\mathbb{N}} \left| \Im(\tilde{p}_n v)^{(\ell-1)}(y_n) \right| \le \mathrm{B}^* \, \|\tilde{p}_n v u\|_{L^q_w(\Delta^*)}$$

with a constant $\mathrm{B}^* < \infty$, that is, by (ii) in (2.110),

$$\sup_{n\in\mathbb{N}} \left| \Re(\tilde{p}_n v)^{(\ell-1)}(x_n) \right| \le \mathrm{B}^* \qquad \text{and} \qquad \sup_{n\in\mathbb{N}} \left| \Im(\tilde{p}_n v)^{(\ell-1)}(y_n) \right| \le \mathrm{B}^* \ .$$

Since

$$\begin{aligned} |(\tilde{p}_n v)^{(\ell-1)}(x)| \le & |\Re(\tilde{p}_n v)^{(\ell-1)}(x_n)| + |\Im(\tilde{p}_n v)^{(\ell-1)}(y_n)| \\ & + \left| \int_x^{x_n} \left| (\tilde{p}_n v)^{(\ell)} \right| \right| + \left| \int_x^{y_n} \left| (\tilde{p}_n v)^{(\ell)} \right| \right| \ , \qquad x \in \Delta', \ n \in \mathbb{N}, \end{aligned}$$

we have

$$\begin{aligned} |(\tilde{p}_n v)^{(\ell-1)}(x) u(x)| \le & \, 2\,\mathrm{B}^* \, u(x) \\ & + \frac{u(x)}{\inf\limits_{t \text{ between } x \text{ and } x_n} u(t)} \left| \int_x^{x_n} \left| (\tilde{p}_n v)^{(\ell)} u \right| \right| \\ & + \frac{u(x)}{\inf\limits_{t \text{ between } x \text{ and } y_n} u(t)} \left| \int_x^{y_n} \left| (\tilde{p}_n v)^{(\ell)} u \right| \right| \ , \quad x \in \Delta', \ n \in \mathbb{N}, \end{aligned}$$

that is,

$$\left|(\tilde{p}_n v)^{(\ell-1)}(x)u(x)\right| \le 2\,\mathrm{B}^*\,u(x)$$
$$+\,2\sup_{t\in\Delta^*}\left(\frac{u(x)}{\min\{u(x),\ u(t)\}}\right)\left|\int_x^t \left|(\tilde{p}_n v)^{(\ell)}u\right|\right| , \qquad x\in\Delta',\ n\in\mathbb{N},$$

(u is monotone between the points x and x_n and the points x and y_n) so that

$$\sup_{n\in\mathbb{N}}\|(\tilde{p}_n v)^{(\ell-1)}u\|_{L^\infty(\Delta')} \le 2\,\mathrm{B}^*\,\|u\|_{L^\infty(\Delta')}$$
$$+\,2\left(1+\frac{\|u\|_{L^\infty(\Delta')}}{\inf_{t\in\Delta^*}u(t)}\right)\sup_{n\in\mathbb{N}}\int_{\Delta'}\left|(\tilde{p}_n v)^{(\ell)}u\right| < \infty\ .$$

Therefore, in view of (iii) in (2.110), the inequalities in (2.120) hold if $\ell = 1$. If $\ell > 1$ then we can use the above argument with Lemma 2.1010 repeatedly to show that inequality (ii) in (2.110) and $\sup_{n\in\mathbb{N}}\|(\tilde{p}_n v)^{(\ell-1)}u\|_{L^\infty(\Delta')} < \infty$ imply

$$\sup_{n\in\mathbb{N}}\|(\tilde{p}_n v)^{(k)}u\|_{L^\infty(\Delta')} < \infty\ , \qquad k = \ell-2, \ell-3, \dots, 0\ ,$$

and, therefore, (2.120) holds for $\ell > 1$ as well.

Now we pick a sequence of closed intervals $\{I_j\}_{j\in\mathbb{N}}$ such that $I_j \subset (I_{j+1})^\circ$ and $\cup_{j\in\mathbb{N}} I_j = (\Delta')^\circ$. Then, since $u^{\pm 1}\in L^\infty(I_j)$ for $j\in\mathbb{N}$, by (2.120),

$$\sup_{n\in\mathbb{N}}\|\tilde{p}_n v\|_{L^\infty(I_j)} < \infty \quad \text{and} \quad \sup_{n\in\mathbb{N}}\|(\tilde{p}_n v)'\|_{L^1(I_j)} < \infty\ , \qquad j\in\mathbb{N},$$

so that, by *Helly's Selection Theorem* (cf. [!Fr, p. 56]), for each $j\in\mathbb{N}$ we can pick an infinite subsequence $\{n_{k,j}\}$ such that $\{n_{k,j}\}\subset\{n_{k,j-1}\}$ where $\{n_{k,0}\}\overset{\text{def}}{=}\mathbb{N}$ and $\lim_{n\to\infty}\tilde{p}_{n_{k,j}}(x)v(x)$ exists for every $x\in I_j$. Let $f(x)\overset{\text{def}}{=}\lim_{n\to\infty}\tilde{p}_{n_{k,k}}(x)1_{v\neq 0}(x)$ for every $x\in(\Delta')^\circ$. Then, by (ii) in (2.110), (2.120), and by *Lebesgue's Dominated Convergence Theorem*,

$$1 = \|\tilde{p}_{n_{k,k}}vu\|_{L^q_w(\Delta')} \xrightarrow[k\to\infty]{} \|fvu\|_{L^q_w(\Delta')}\ , \tag{2.130}$$

so that f exists and $f\neq 0$ on a set $E\subset\Delta'\cap\{\,t : v(t)\neq 0\,\}$ with positive Lebesgue measure. Thus, just like in Step 1, by B. Ya. Levin's theorem in [!Le, Theorem 2, p. 385] (cf, [!CK, Corollary 1.3, p. 110]), the limit $f_c(z)\overset{\text{def}}{=}\lim_{n\to\infty}\tilde{p}_{n_{k,k}}(z)$ exists locally uniformly in $\mathbb{C}$, where f_c is an entire function (from the Pólya-Obrechkoff class). Since $\lim_{n\to\infty}Z_\Omega(\tilde{p}_{n_{k,k}}) = \infty$, by *Hurwitz's Theorem* (cf. [!W, p. 6]), $f\equiv 0$. This contradicts (2.130) since $f_c v\equiv fv$ in $(\Delta')^\circ$. Therefore,

$$\lim_{n\to\infty}\frac{\|(p_n v)^{(\ell)}u\|_{L^1(\Delta)}}{\|p_n vu\|_{L^q_w(\Delta)}} = \infty \tag{2.140}$$

holds.

Step 4. The case when $m > 1$ follows from the case when $m = 1$ since

$$\frac{\|(p_n v)^{(\ell)} u\|_{L^1(\Delta)}}{\|(p_n v u\|_{L^q_w(\Delta)}} = \frac{\sum_{k=1}^m \|(p_n v)^{(\ell)} u\|_{L^1(\Delta_k)}}{\left(\sum_{k=1}^m \|(p_n v u\|^q_{L^q_w(\Delta_k)}\right)^{\frac{1}{q}}}$$
$$\geq \frac{1}{m^{\frac{1}{q}}} \min_{\substack{1 \le k \le m \\ \int_{\Delta_k} |v| > 0}} \left\{ \frac{\|(p_n v)^{(\ell)} u\|_{L^1(\Delta_k)}}{\|(p_n v u\|_{L^q_w(\Delta_k)}} \right\} .$$

Therefore, we have completed the proof of (1.50) in its entire generality. □

Proof of Corollary 1.100. By Leibniz's rule, we have

$$\frac{\| (p_n g_n)^{(\ell)} u\|_{L^r(\Delta)}}{\|p_n^{(\ell)} u\|_{L^r(\Delta)} \|g_n^{-1}\|^{-1}_{L^\infty(\Delta)}} \geq \frac{\| (p_n g_n)^{(\ell)} u\|_{L^r(\Delta)}}{\|p_n^{(\ell)} g_n u\|_{L^r(\Delta)}}$$
$$\geq 1 - \sum_{j=1}^{\ell} \binom{\ell}{j} \frac{\|p_n^{(\ell-j)} u\|_{L^r(\Delta)} \|g_n^{(j)}\|_{L^\infty(\Delta)}}{\|p_n^{(\ell)} u\|_{L^r(\Delta)} \|g_n^{-1}\|^{-1}_{L^\infty(\Delta)}}$$

so that, in view of (1.90), Theorem 1.50, applied to $\{p_n^{(\ell-j)}\}_{n\in\mathbb{N}}$ for $j = 1, 2, \ldots, \ell$ with $v \equiv 1$ and $w \equiv 1$, immediately yields (1.100).[9] □

Proof of Theorem 1.150. The proof is essentially the same as that of Theorem 1.50 with some modifications. Instead of giving a complete proof we will only concentrate on the differences between the two proofs.

In view of Hölder's inequality, it is sufficient to prove (1.150) for $r = 1$. Since the zeros of each t_n are real, we can assume that each t_n itself is real for $n \in \mathbb{Z}^+$ as well. Moreover, since

$$\frac{\|(t_n v)^{(\ell)} u\|_{L^1(\Delta)}}{\|t_n v u\|_{L^q_w(\Delta)}} \geq \frac{1}{2^{\max\{1, 1/q\}}} \min \left\{ \frac{\|(t_n \Re v)^{(\ell)} u\|_{L^1(\Delta)}}{\|t_n(\Re v) u\|_{L^q_w(\Delta)}}, \frac{\|(t_n \Im v)^{(\ell)} u\|_{L^1(\Delta)}}{\|t_n(\Im v) u\|_{L^q_w(\Delta)}} \right\} ,$$

we can also assume that $v : \Delta \to \mathbb{R}$.

Step 1. We will outline the proof of (1.150) for $m = 1$ ($\Delta = \Delta_1$) when $u \equiv 1$. We can assume that $|v|^q w > 0$ on a set of positive Lebesgue measure.

If (1.150) were not true then there would be a sequence of real trigonometric polynomials $\{\tilde{t}_n\}_{n\in\mathbb{N}}$ with real zeros satisfying

$$\text{(i)} \quad \lim_{n\to\infty} Z_{\mathbb{T}}(\tilde{t}_n) = \infty , \quad \text{(ii)} \quad \|\tilde{t}_n v\|_{L^q_w(\Delta)} = 1 , \quad \text{(iii)} \quad \sup_{n\in\mathbb{N}} \|(\tilde{t}_n v)^{(\ell)}\|_{L^1(\Delta)} < \infty . \tag{2.230}$$

[9]Note that if $\ell > 1$ then, since the zeros of $\{p_n\}_{n\in\mathbb{N}}$ are real and $\lim_{n\to\infty} Z_\Omega(p_n) = \infty$ ($\Omega \subset \mathbb{R}$), *Rolle's Theorem* yields $\lim_{n\to\infty} Z_\Omega(p_n^{(\ell-j)}) = \infty$ for $j = 1, 2, \ldots, \ell - 1$, so that Theorem 1.50 may indeed be used.

We claim that

$$\sup_{n\in\mathbb{N}} \|\tilde{t}_n v\|_{L^\infty(\Delta)} < \infty \qquad \text{and} \qquad \sup_{n\in\mathbb{N}} \|(\tilde{t}_n v)'\|_{L^1(\Delta)} < \infty \,. \tag{2.240}$$

Applying Lemma 2.1010 with $p = q$, $k = \ell - 1$, $\sigma \equiv w$, and $f \equiv \tilde{t}_n v$, and using (ii) and (iii) in (2.230), we obtain

$$\sup_{n\in\mathbb{N}} \|(\tilde{t}_n v)^{(\ell-1)}\|_{L^\infty(\Delta)} < \infty \,, \tag{2.245}$$

that is, (2.240) holds if $\ell = 1$. If $\ell > 1$ then we can use Lemma 2.1010 repeatedly with $f \equiv \tilde{t}_n v$ to show that (2.245) together with (ii) in (2.230) imply

$$\sup_{n\in\mathbb{N}} \|(\tilde{t}_n v)^{(k)}\|_{L^\infty(\Delta)} < \infty \,, \qquad k = \ell-2, \ell-3, \dots, 0 \,,$$

so that (2.240) holds for $\ell > 0$ as well.

Using (2.240), we can apply *Helly's Selection Theorem* (cf. [!Fr, p. 56]) to pick a subsequence $\{\tilde{t}_{n_k}\}_{k\in\mathbb{N}}$ such that $\lim_{n\to\infty} \tilde{t}_{n_k}(x)v(x)$ exists for every $x \in \Delta$. Let $f(x) \overset{\text{def}}{=} \lim_{n\to\infty} \tilde{t}_{n_k}(x) 1_{v\neq 0}(x)$ for $x \in \Delta$. By *Lebesgue's Dominated Convergence Theorem*,

$$1 = \|\tilde{t}_{n_k} v\|_{L^q_w(\Delta)} \xrightarrow[k\to\infty]{} \|fv\|_{L^q_w(\Delta)} \,, \tag{2.250}$$

so that f exists and $f \neq 0$ on a set $E \subset \{\ t : v(t) \neq 0\ \} \subset \Delta$ with positive Lebesgue measure. Let $x_{\mathcal{L}} \in E$ be a *Lebesgue density point* of E, that is, let

$$\lim_{h\downarrow 0} \frac{|(x_{\mathcal{L}} - h, x_{\mathcal{L}} + h) \cap E|}{2h} = 1$$

(cf. [!RS, Theorem, p. 13]). Fix $0 < \epsilon < 1$ and pick $0 < h < \pi$ such that

$$|(x_{\mathcal{L}} - h, x_{\mathcal{L}} + h) \cap E| > (2 - \epsilon)h \,.$$

Let $E_h \overset{\text{def}}{=} (E - x_{\mathcal{L}}) \cap (-h, h) \cap (x_{\mathcal{L}} - E)$.[10]Then $E_h \equiv E_{-h}$ and

$$|E_h| \geq |E - x_{\mathcal{L}}| + |x_{\mathcal{L}} - E| - 2h \geq (2-\epsilon)h + (2-\epsilon)h - 2h = 2h(1-\epsilon) > 0 \,,$$

and, therefore, $|\arccos(E_h \cap (0,h))| > 0$ as well. Let the sequence of algebraic polynomials $\{\tilde{p}_{n_k}\}_{k\in\mathbb{N}}$ be defined by $\tilde{p}_{n_k}(\cos x) \overset{\text{def}}{=} t_{n_k}(x_{\mathcal{L}} + x) t_{n_k}(x_{\mathcal{L}} - x)$. Then $\lim_{n\to\infty} \tilde{p}_{n_k}(\cos x) = f(x_{\mathcal{L}} + x) f(x_{\mathcal{L}} - x) \neq 0$ for $x \in E_h \cap (0,h)$, that is, $f_r \overset{\text{def}}{=} \lim_{n\to\infty} \tilde{p}_{n_k} \neq 0$ on the set $\arccos(E_h \cap (0,h)) \subset [-1,1]$ which is of positive measure. Thus, by a theorem of B. Ya. Levin in [!Le, Theorem 2, p. 385] (cf. [!CK, Corollary 1.3, p. 110]), the limit $f_c(z) \overset{\text{def}}{=} \lim_{n\to\infty} \tilde{t}_{n_k}(z)$ exists locally uniformly in $\mathbb{C}$, where f_c is an entire function (from the

[10]Here and in what follows, given a set Θ and numbers $\rho_1 \in \mathbb{C}$ and $\rho_2 \in \mathbb{C}$, we use the notation $\rho_1 + \rho_2\Theta \overset{\text{def}}{=} \{\rho_1 + \rho_2 t : t \in \Theta\}$.

Pólya-Obrechkoff class). Since $\lim_{n\to\infty} Z_{[-1,1]}(\tilde{p}_{n_k}) = \infty$ (cf. (i) in (2.230)), by *Hurwitz's Theorem* (cf. [!W, p. 6]), $f_c \equiv 0$. This is a contradiction since $f_c \equiv f_r \neq 0$ in $\arccos(E_h \cap (0,h))$. Therefore,

$$\lim_{n\to\infty} \frac{\|(t_n v)^{(\ell)}\|_{L^1(\Delta)}}{\|t_n v\|_{L^q_w(\Delta)}} = \infty \tag{2.260}$$

holds.
Step 2. The case when $m = 1$ and $u^{-1} \in L^\infty(\Delta)$ follows from Step 1 the same way as it does in Step 2 in the proof of Theorem 1.50.
Step 3. The case when $m = 1$ and u is bounded and monotonic in Δ is proved analogously to Step 3 in the proof of Theorem 1.50 using the same modifications as in Step 1, that is, by considering the subsequence

$$\{\tilde{p}_{n_k}(\cos\cdot) \stackrel{\text{def}}{=} t_{n_k}(x_{\mathcal{L}} + \cdot)t_{n_k}(x_{\mathcal{L}} - \cdot)\}$$

as opposed to the choice of $\{\tilde{p}_{n_k}\}$ in Step 3 in the proof of Theorem 1.50.
Step 4. The case when $m > 1$ follows from the case when $m = 1$ in the same way as it does in Theorem 1.50.

Therefore, we have completed the (outline of the) proof of (1.150) in its entire generality. □

3. Proofs of Results on Generalized Polynomials

The proof of Theorem 1.3010 is based on the following

Lemma 3.15. *There exists a constant* $\mathrm{c} \in \mathbb{R}^+$ *such that for every closed interval* $\tau \subset \mathbb{T}$ *of length* $\pi/4$ *and for every* $7\pi/4 \le s < 2\pi$ *the inequality*

$$\|t\|_{L^\infty(\tau - \pi/4)} \le \exp(\mathrm{c}\deg(t)\,|\log(2\pi - s)|)\,\|t\|_{L^\infty(A)}, \qquad t \in \mathcal{T}^r\,, \tag{3.1050}$$

holds for every set $A \subset \tau$ *with* $m(A) \ge 2\pi - s$.

Proof of Lemma 3.15. Without loss of generality we assume $\tau = [\pi/4, \pi/2]$. Fix $n \in \mathbb{N}$, $7\pi/4 \le s < 2\pi$, and $y \in [0, \pi/4)$. Let

$$\mathcal{T}_n^r(s) \stackrel{\text{def}}{=} \{\, t \in \mathcal{T}_n^r : m(\{\, x \in [\pi/4, \pi/2] : |t(x)| \le 1 \,\}) \ge 2\pi - s \,\}\,.$$

A standard compactness argument shows that there exists $t^* \in \mathcal{T}_n^r$ such that

$$|t^*(y)| = \sup_{t \in \mathcal{T}_n^r(s)} |t(y)|\,. \tag{3.1055}$$

We will show that the set

$$M(t^*) \stackrel{\text{def}}{=} \{\, x \in [\pi/4, \pi/2] : |t(x)| \le 1 \,\} \tag{3.1060}$$

is an interval. Clearly, there is $m \in \mathbb{N}$ such that

$$M(t^*) = \bigcup_{k=1}^{m} M_k(t^*) \,, \tag{3.1070}$$

where the sets $M_k(t^*)$ are disjoint closed subintervals of $[\pi/4, \pi/2]$. Hence, our job is to show that $m = 1$ in (3.1070).

First, we claim that all the zeros of t^* are real. If not then $t^*(\zeta) = 0$ for some $\zeta \in \mathbb{C} \setminus \mathbb{R}$, and then for sufficiently small $\eta_1 > 0$ and $\eta_2 > 0$ the polynomial

$$(1+\eta_1)t^*(z)\left(1 - \frac{\eta_2 \sin^2 \frac{z-y}{2}}{\sin \frac{z-\zeta}{2} \sin \frac{z-\overline{\zeta}}{2}}\right)$$

also belongs to $\mathcal{T}_n^r(s)$ which contradicts the extremality of t^*.

Second, we claim that all the zeros of t^* (more precisely, those which are in $[0, 2\pi)$) lie in $[\pi/4, \pi/2]$. If not then $t^*(\zeta) = 0$ for some $\zeta \in \mathbb{T} \setminus [\pi/4, \pi/2]$, and then for sufficiently small $\eta_1 > 0$ and $\eta_2 > 0$ the polynomial

$$(1+\eta_1)t^*(z)\left(1 - \frac{\eta_2 \operatorname{sgn}\left(\sin \frac{\pi - 2\zeta}{4}\right) \sin \frac{z-y}{2}}{\sin \frac{z-\zeta}{2}}\right)$$

also belongs to $\mathcal{T}_n^r(s)$ which contradicts the extremality of t^*.

Third, we claim that each *component* $M_k(t^*)$ in (3.1070) contains at least one zero of t^*. This follows from the observation that if there is more then one component then between consecutive components t^* has an extremal point so that $(t^*)'$ vanishes there, and, therefore, the the zeros of t^* and $(t^*)'$ can interlace only if each component contains at least one zero of t^*.

Finally, we claim that m in (3.1070) is indeed equal to 1. If not, then let $M_1(t^*)$ and $M_2(t^*)$ be the closest and the second closest components of $M(t^*)$ to $\pi/4$, and let ζ_j denote the zeros of t^* in $M_1(t^*)$. Then the polynomial

$$t^*(z) \prod_j \frac{\sin \frac{z - \zeta_j - \operatorname{dist}(M_1(t^*), M_2(t^*))}{2}}{\sin \frac{z-\zeta_j}{2}}$$

also belongs to $\mathcal{T}_n^r(s)$ which contradicts the extremality of t^*.

Hence, $M(t^*)$ in (3.1060) is an interval so that $|t^*| \leq 1$ in an interval of length at least $2\pi - s$, and then, by [!ELond, Lemma 3, p. 257],

$$\|t^*\|_{L^\infty(\mathbb{T})} \leq T_{2n}\left(1/\sin(\pi/2 - s/4)\right) \,,$$

where T_{2n} denotes the first kind Chebysev polynomial of degree $2n$, that is, $T_{2n}(x) \stackrel{\text{def}}{=} \cos(2n \arccos x)$. Therefore, by routine estimates,

$$\|t^*\|_{L^\infty(\mathbb{T})} \leq \exp(c\, n|\log(2\pi - s)|) \,,$$

with an appropriate absolute constant $c > 0$. By the extremality of t^* (cf. (3.1055)), we obtain

$$|t(y)| \leq \exp(c\, n|\log(2\pi - s)|)\ , \qquad t \in \mathcal{T}_n^r(s)\ . \tag{3.1080}$$

Finally, pick $t \in \mathcal{T}^r$ and $A \subset [\pi/4, \pi/2]$ with $m(A) \geq 2\pi - s$. Let $n = \deg(t)$. Then $t/\|t\|_{L^\infty(A)} \in \mathcal{T}_n^r(s)$ so that (3.1050) in Lemma 3.15 follows from (3.1080) applied with $t/\|t\|_{L^\infty(A)}$ in place of t. $\square$

Proof of Theorem 1.3010. The case $0 < s < \pi/2$ in (1.3000) was proved in [!ELond, Theorem 2, p. 257], and, therefore, we will consider only the case $\pi/2 \leq s < 2\pi$ in (1.3000). Let $A \subset \mathbb{T}$ with $m(A) \leq s$. Then there is a closed interval $\tau \subset \mathbb{T}$ such that $m(\tau) = \pi/4$ and $m((\mathbb{T} \setminus A) \cap \tau) \geq (2\pi - s)/8$. By Lemma 3.15, applied with $7\pi/4 + s/8$ in place of s,

$$\|t\|_{L^\infty(\tau - \pi/4)} \leq \exp(c \deg(t)\ |\log(\pi/4 - s/8)|)\ \|t\|_{L^\infty(\mathbb{T}\setminus A)}\ , \qquad t \in \mathcal{T}^r\ .$$

On the other hand, by [!ELond, Lemma 3, p. 257],

$$\|t\|_{L^\infty(\mathbb{T})} \leq T_{2\deg(t)}\,(1/\sin(\pi/16))\ \|t\|_{L^\infty(\tau-\pi/4)}\ , \qquad t \in \mathcal{T}^r\ ,$$

where T_n denotes the first kind Chebysev polynomial of degree n, that is, $T_n(x) \stackrel{\text{def}}{=} \cos(n \arccos x)$. Combining the last two inequalities, we obtain

$$\|t\|_{L^\infty(\mathbb{T})} \leq T_{2\deg(t)}\,(1/\sin(\pi/16))\ \exp(c \deg(t)\ |\log(\pi/4 - s/8)|)\ \|t\|_{L^\infty(\mathbb{T}\setminus A)}\ , \qquad t \in \mathcal{T}^r\ ,$$

that is,

$$\|t\|_{L^\infty(\mathbb{T})} \leq \exp(c^* \deg(t)|\log(\pi/4 - s/8)|)\ \|t\|_{L^\infty(\mathbb{T}\setminus A)}\ , \qquad t \in \mathcal{T}^r\ , \tag{3.1090}$$

with an appropriate constant $c^* \in \mathbb{R}^+$.

If $f \in$ NGAP is of the form $f(x) = \omega \prod_{j=1}^k |z - z_j|^{r_j}$ with rational exponents r_j, say $r_j = q_j/q$, where $q_j \in \mathbb{N}$ and $q \in \mathbb{N}$, then applying (3.1090) with $t \stackrel{\text{def}}{=} f^{2q}$, the case $\pi/2 \leq s < 2\pi$ in (1.3000) follows with the same constant $c \stackrel{\text{def}}{=} c^*$. Once there is an absolute constant $c \in \mathbb{R}^+$ such that (1.3000) holds for all $f \in$ NGAP with rational exponents, it can be extended to all $f \in$ NGAP by approximating the exponents with rational numbers. Therefore, inequality (1.3000) has been completely proved. $\square$

Proof of Theorem 1.3020. We will expand and extend the proof of [!EMN, Theorem 8, p. 251]. Let $0 < p < \infty$, $0 < \delta < 2\pi$, $f \in$ NGTP, and $A \subset \mathbb{T}$ with $m(A) \leq 2\pi - \delta$. Let

$$s = s_{\delta,A} \stackrel{\text{def}}{=} \frac{2\pi - \delta/2}{2\pi - \delta}\ m(A) \tag{3.22}$$

and let $c_{\delta/2}$ be defined by (1.3010) applied with $\epsilon = \delta/2$. Define

$$I(f) \stackrel{\text{def}}{=} \left\{\ x \in \mathbb{T} : \chi(f(x)) \geq \exp\left(-\,c_{\delta/2} \deg(f)\ s\right) \|\chi(f)\|_{L^\infty(\mathbb{T})}\right\}\ .$$

If $f(x) \geq \exp(-\mathrm{c}_{\delta/2} \deg(f)\ s)\|f\|_{L^\infty(\mathbb{T})}$ then, since χ is nondecreasing,

$$\chi(f(x)) \geq \chi\left(\exp\left(-\mathrm{c}_{\delta/2} \deg(f)\ s\right) \|f\|_{L^\infty(\mathbb{T})}\right),$$

and then, since $\chi(\cdot)/(\cdot)$ is nonincreasing as well,

$$\chi(f(x)) \geq \exp\left(-\mathrm{c}_{\delta/2} \deg(f)\ s\right) \chi\left(\|f\|_{L^\infty(\mathbb{T})}\right).$$

Hence $J(f) \subseteq I(f)$ where

$$J(f) \stackrel{\text{def}}{=} \left\{\ x \in \mathbb{T} : f(x) \geq \exp\left(-\mathrm{c}_{\delta/2} \deg(f)\ s\right) \|f\|_{L^\infty(\mathbb{T})}\right\}\ .$$

We claim that $m(J(f)) \geq s$. If not, then there is $\eta > 0$ such that

$$m\left(x \in \mathbb{T} : g(x) \geq 1\right) \leq s = \frac{2\pi - \delta/2}{2\pi - \delta}\ m(A) \leq 2\pi - \delta/2$$

where

$$g \stackrel{\text{def}}{=} (1+\eta) \exp\left(\mathrm{c}_{\delta/2} \deg(f)\ s\right) \|f\|^{-1}_{L^\infty(\mathbb{T})}\ f,$$

so that, by (1.3010) in Theorem 1.3010,

$$\begin{aligned}(1+\eta) \exp\left(\mathrm{c}_{\delta/2} \deg(f)\ s\right) &= \|g\|_{L^\infty(\mathbb{T})} \\ &\leq \exp\left(\mathrm{c}_{\delta/2} \deg(f)\ \ m(\ x \in \mathbb{T} : g(x) \geq 1\ )\right) \leq \exp\left(\mathrm{c}_{\delta/2} \deg(f)\ s\right)\end{aligned}$$

which leads to the contradiction that $\eta \leq 0$. Thus, $m(J(f)) \geq s$, and then $m(I(f)) \geq s$ as well since $J(f) \subseteq I(f)$.

Let

$$K \stackrel{\text{def}}{=} (\mathbb{T} \setminus A) \bigcap I(f)\ .$$

Since $m(I(f)) \geq s$ and $m(\mathbb{T} \setminus A) = 2\pi - m(A)$, we have $2m(K) \geq s + 2\pi - m(A) - 2\pi = (\delta/2)m(A)/(2\pi - \delta)$ (cf. (3.22)).[11] Therefore,

$$\begin{aligned}\int_A \chi^p(f) \leq \int_A \|\chi(f)\|^p_{L^\infty(\mathbb{T})} &= m(A)\|\chi(f)\|^p_\infty \\ &\leq \frac{4(2\pi - \delta)}{\delta} m(K)\|\chi(f)\|^p_{L^\infty(\mathbb{T})} = \frac{4(2\pi - \delta)}{\delta} \int_K \|\chi(f)\|^p_{L^\infty(\mathbb{T})}\ .\end{aligned}$$

Hence, since $K \subseteq I(f)$,

$$\int_A \chi^p(f) \leq \frac{4(2\pi - \delta)}{\delta} \exp\left(\mathrm{c}_{\delta/2}\ \frac{2\pi - \frac{\delta}{2}}{2\pi - \delta}\ p \deg(f)\ m(A)\right) \int_K \chi(f)\ ,$$

[11] If S_1 and S_2 are two sets then $S_1 = (S_1 \cap S_2) \cup (S_1 \setminus S_2)$ and $S_2 = (S_1 \cap S_2) \cup (S_2 \setminus S_1)$. Hence, $m(S_1) + m(S_2) = 2m(S_1 \cap S_2) + m(S_1 \setminus S_2) + m(S_2 \setminus S_1)$ so that $2m(S_1 \cap S_2) \geq m(S_1) + m(S_2) - m(S_1 \cup S_2)$.

and, finally, since $K \subseteq \mathbb{T} \setminus A$,

$$\int_A \chi^p(f) \le \frac{4(2\pi - \delta)}{\delta} \exp\left(\mathrm{c}_{\delta/2} \, \frac{2\pi - \frac{\delta}{2}}{2\pi - \delta} \, p \deg(f) \, m(A) \right) \int_{\mathbb{T}\setminus A} \chi(f) \, .$$

Theorem 1.3020 has completely been proved. $\square$

Proof of Theorem 1.3030. First assume that $p = 1$. Fix one of the component intervals in $\Delta = \cup \Delta_k$, say, Δ_{k_0}. Then

$$f(x)u(x) \le \int_{\Delta_{k_0}} |(fu)'| \, , \qquad x \in \Delta_{k_0} \, ,$$

so that

$$\int_{\Delta_{k_0}} fu \le |\Delta_{k_0}| \int_{\Delta_{k_0}} |(fu)'| \, ,$$

that is,

$$\int_{\Delta_{k_0}} fu \le |\Delta_{k_0}| \int_{\Delta_{k_0}} |f'| u + |\Delta_{k_0}| \int_{\Delta_{k_0}} f |u'| \, ,$$

where the second term on the right-hand side vanishes if $\Gamma = 0$. Hence, for $\Gamma > 0$,

$$\int_{\Delta_{k_0}} fu \le |\Delta_{k_0}| \int_{\Delta_{k_0}} |f'| u + |\Delta_{k_0}| \sup_{t \in \Delta_{k_0}} \frac{|u'(t)|}{u(t)} \int_{\Delta_{k_0}} fu \, .$$

If u is given by

$$u(z) = \omega \prod_j \left| \sin \frac{z - z_j}{2} \right|^{r_j} , \qquad \omega > 0, \; z_j \in \mathbb{C}, \; z \in \mathbb{C}, \; r_j > 0, \; 2\Gamma \stackrel{\mathrm{def}}{=} \sum_j r_j \, ,$$

then, by (1.3030),

$$|\Delta_{k_0}| \sup_{t \in \Delta_{k_0}} \frac{|u'(t)|}{u(t)} \le |\Delta_{k_0}| \frac{\sum_j r_j}{2} \sup_{\substack{t \in \Delta_{k_0} \\ z_j}} |\cot(t - z_j)| \le \frac{1}{2} \, .$$

Therefore, for all $\Gamma \ge 0$,

$$\int_{\Delta_{k_0}} fu \le 2 |\Delta_{k_0}| \int_{\Delta_{k_0}} |f'| u \, .$$

Since this is true for each component interval of Δ, we obtain

$$\int_{\Delta} fu \le 2 \max_k |\Delta_k| \int_{\Delta} |f'| u \le 2 \max_k |\Delta_k| \int_{\mathbb{T}} |f'| u \, ,$$

and then (1.3040) follows immediately from Theorem 1.3020 applied with $p = 1$ and $\chi \equiv 1$.

Once (1.3040) holds with $p = 1$, we can use it with f replaced by $f^p \in \mathrm{NGTP}_{pN}$, and then (1.3040) follows for all $1 < p < \infty$ from Hölder's inequality. $\square$

Proof of Corollary 1.3100.
Step 1. If $\ell = 1$ and $j = 0$ in (1.3140) then we can apply Theorem 1.3030 after making the following two observations. First, if $t_n \in \mathcal{T}^r$ then $f \stackrel{\mathrm{def}}{=} |t_n| \in \mathrm{NGTP}_{\deg(t_n)}$ and $|f'| = |(|t_n|)'| \equiv |t'_n|$ everywhere except at the zeros of f (cf. Remark 1.307). Second, we can choose a constant $\mathrm{c}_1 \in \mathbb{R}^+$ such that, if for each $n \in \mathbb{N}$ we pick only those intervals $\tau_{i_k,n}$ for which $\mathrm{dist}(\tau_{i_k,n}, \{\text{zeros of } u\}) > \mathrm{c}_1/n$, then $\Delta_k \stackrel{\mathrm{def}}{=} \tau_{i_k,n}$ satisfy (1.3030), and $2\pi - \mathrm{c}_2/n < m(\cup\Delta_k) \le 2\pi$ with an appropriate constant $\mathrm{c}_2 \in \mathbb{R}^+$ independently of n.
Step 2. If $\ell > 1$ and $j = \ell - 1$ in (1.3140), then we can apply Step 1 with $t_n^{(\ell-1)}$ in place of t_n after observing that if the closure of each τ_{in} contains at least one zero of t_n then, by *Rolle's Theorem*, the closure of each $\cup_{i^*=i}^{i+2\ell-2}\tau_{i^*,n}$ contains at least one zero of $t_n^{(\ell-1)}$, so that a new set of intervals $\{\tau^*_{in}\}$ can be formed by joining sufficiently many adjacent ones (or their closures) from the original set $\{\tau_{in}\}$ in such a way that for the new set (i) $\tau^*_{in} \subset \mathbb{T}$ are disjoint intervals, (ii) the closure of each τ^*_{in} contains at least one zero of $t_n^{(\ell-1)}$, and (iii) $m(\tau_{in}) \le \mathrm{d}^*/n$ with an appropriate constant $\mathrm{d}^* \in \mathbb{R}^+$.
Step 3. If $\ell > 1$ and $0 \le j < \ell - 1$ in (1.3140) then we just write the fraction in (1.3140) as the product of ratios with consecutive derivatives, and use Step 2 with each term. $\square$

Proof of Corollary 1.3200. If $j = \ell - 1$ in (1.3240) then it follows immediately from Corollary 1.3100 applied with $t_n(\cdot) \stackrel{\mathrm{def}}{=} p_n\left(\frac{b-a}{2}\cos(\cdot) + \frac{b+a}{2}\right)$. Otherwise, we just write the fraction in (1.3240) as the product of ratios with consecutive derivatives, and apply what we have proved already to each term. $\square$

The following is very simple and very useful.

Lemma 3.100. *Let $\Delta = \cup_k\Delta_k$ where Δ_k are disjoint intervals either in $\mathbb{R}$ or in $\mathbb{T}$. Let $1 \le p < \infty$. Let f be differentiable almost everywhere in each Δ_k, and let $|f|^p$ be absolutely continuous in each Δ_k. Let the closure of each Δ_k contain at least one zero of f. Let $u \ge 0$ be Lebesgue measurable in Δ. Then*

$$\|f\|_{L^p_u(\Delta)} \le p \sup_k m(\Delta_k) \ \sup_k \left(\|u\|_{L^\infty(\Delta_k)} \ \|u^{-1}\|_{L^\infty(\Delta_k)}\right) \|f'\|_{L^p_u(\Delta)} \ . \tag{3.100}$$

Proof of Lemma 3.100. Fix one of the component intervals in $\Delta = \cup\Delta_k$, say, Δ_{k_0}. Then

$$|f(x)|^p \le p \int_{\Delta_{k_0}} |f'||f|^{p-1} \ , \qquad x \in \Delta_{k_0},$$

so that

$$\int_{\Delta_{k_0}} |f|^p u \le p\, m(\Delta_{k_0}) \|u\|_{L^\infty(\Delta_{k_0})} \|u^{-1}\|_{L^\infty(\Delta_{k_0})} \int_{\Delta_{k_0}} |f'||f|^{p-1} u \ .$$

Now add up these inequalities, and then use Hölder's inequality to obtain (3.100). $\square$

Proof of Theorem 1.3300. Inequality (1.3305) follows directly from Lemma 3.100 by observing that

$$\frac{u(y)}{u(x)} = \exp\left(\frac{1}{2}\sum_j \pm\Gamma_j \int_x^y \cot\frac{z-z_j}{2}\,dz\right) \le \exp\left(|x-y|\Gamma\pi/\operatorname{dist}(\tau_{i_k},\{z_j\})\right)$$

for $x, y \in \tau_{i_k}$. $\square$

Proof of Corollary 1.3310. If $\ell = 1$ and $j = 0$ in (1.3310), then it follows directly from Theorem 1.3300. If $\ell > 1$ and $j = \ell - 1$ in (1.3310), then we can use *Rolle's Theorem* just like in Step 2 in the proof of Corollary 1.3100. If $\ell > 1$ and $0 \le j < \ell - 1$ in (1.3310), then we can proceed again as in Step 3 in the proof of Corollary 1.3100. $\square$

Proof of Corollary 1.3320. We just repeat the proof of Corollary 1.3200, that is, if $j = \ell - 1$ in (1.3325) then it follows immediately from Corollary 1.3310 applied with $t_n(\cdot) \stackrel{\text{def}}{=} p_n\left(\frac{b-a}{2}\cos(\cdot) + \frac{b+a}{2}\right)$, and, otherwise, we write the fraction in (1.3325) as the product of ratios with consecutive derivatives, and apply what we have proved already to each term. $\square$

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