

SHARP EXTENSIONS OF BERNSTEIN'S INEQUALITY TO RATIONAL SPACES

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ABSTRACT. Sharp extensions of some classical polynomial inequalities of Bernstein are established for rational function spaces on the unit circle, on $K := \mathbb{R} \pmod{2\pi}$, on $[-1, 1]$ and on \mathbb{R} . The key result is the establishment of the inequality

$$|f'(z_0)| \leq \max \left\{ \sum_{\substack{j=1 \\ |a_j| > 1}} \frac{|a_j|^2 - 1}{|a_j - z_0|^2}, \sum_{\substack{j=1 \\ |a_j| < 1}} \frac{1 - |a_j|^2}{|a_j - z_0|^2} \right\} \|f\|_{\partial D}$$

for every rational function $f = p_n/q_n$, where p_n is a polynomial of degree at most n with complex coefficients and

$$q_n(z) = \prod_{j=1}^n (z - a_j)$$

with $|a_j| \neq 1$ for each j , and for every $z_0 \in \partial D$, where $\partial D := \{z \in \mathbb{C} : |z| = 1\}$. The above inequality is sharp at every $z_0 \in \partial D$.

1. Introduction, Notation.

We denote by \mathcal{P}_n^r and \mathcal{P}_n^c the sets of all algebraic polynomials of degree at most n with real or complex coefficients, respectively. The sets of all trigonometric polynomials of degree at most n with real or complex coefficients, respectively, are denoted by \mathcal{T}_n^r and \mathcal{T}_n^c . We will use the notation

$$\|f\|_A = \sup_{z \in A} |f(z)|$$

for continuous functions f defined on A . Let

$$D := \{z \in \mathbb{C} : |z| \leq 1\},$$

$$\partial D := \{z \in \mathbb{C} : |z| = 1\}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$

and

$$K := \mathbb{R} \pmod{2\pi}.$$

The classical inequalities of Bernstein [1] state that

$$\begin{aligned} |p'(z_0)| &\leq n \|p\|_{\partial D}, & p \in \mathcal{P}_n^c, & \quad z_0 \in \partial D, \\ |t'(\theta_0)| &\leq n \|t\|_K, & t \in \mathcal{T}_n^c, & \quad \theta_0 \in K, \\ |p'(x_0)| &\leq \frac{n}{\sqrt{1-x_0^2}} \|p\|_{[-1,1]}, & p \in \mathcal{P}_n^c, & \quad x_0 \in (-1, 1). \end{aligned}$$

Proofs of the above inequalities may be found in almost every book on approximation theory, see [4], [5], [6] or [8] for instance. An extensive study of Markov- and Bernstein-type inequalities is presented in [7].

In this paper we study the rational function spaces:

$$\mathcal{P}_n^c(a_1, a_2, \dots, a_n; \partial D) := \left\{ \frac{p_n(z)}{\prod_{j=1}^n (z - a_j)} : p_n \in \mathcal{P}_n^c \right\}$$

on ∂D with $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus \partial D$;

$$\mathcal{T}_n^c(a_1, a_2, \dots, a_{2n}; K) := \left\{ \frac{t_n(\theta)}{\prod_{j=1}^{2n} \sin((\theta - a_j)/2)} : t_n \in \mathcal{T}_n^c \right\}$$

on K with $\{a_1, a_2, \dots, a_{2n}\} \subset \mathbb{C} \setminus \mathbb{R}$;

$$\mathcal{P}_n^c(a_1, a_2, \dots, a_n; [-1, 1]) := \left\{ \frac{p_n(x)}{\prod_{j=1}^n (x - a_j)} : p_n \in \mathcal{P}_n^c \right\}$$

on $[-1, 1]$ with $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus [-1, 1]$;

$$\mathcal{P}_n^c(a_1, a_2, \dots, a_n; \mathbb{R}) := \left\{ \frac{p_n(x)}{\prod_{j=1}^n (x - a_j)} : p_n \in \mathcal{P}_n^c \right\}$$

on \mathbb{R} with $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}$, and

$$\mathcal{P}_n^r(a_1, a_2, \dots, a_n; \mathbb{R}) := \left\{ \frac{p_n(x)}{\prod_{j=1}^n |x - a_j|} : p_n \in \mathcal{P}_n^r \right\}$$

on \mathbb{R} with $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}$.

The spaces

$$\mathcal{T}_n^r(a_1, a_2, \dots, a_{2n}; K) := \left\{ \frac{t_n(\theta)}{\prod_{j=1}^{2n} |\sin((\theta - a_j)/2)|} : t_n \in \mathcal{T}_n^r \right\}$$

on K with $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}$ and

$$\mathcal{P}_n^r(a_1, a_2, \dots, a_n; [-1, 1]) := \left\{ \frac{p_n(x)}{\prod_{j=1}^n |x - a_j|} : p_n \in \mathcal{P}_n^r \right\}$$

on $[-1, 1]$ with $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus [-1, 1]$ have been studied in [2] and [3], and the sharp Bernstein-Szegő type inequalities

$$f'(\theta_0)^2 + \tilde{B}_n(\theta_0)^2 f(\theta_0)^2 \leq \tilde{B}(\theta_0)^2 \|f\|_K^2, \quad \theta_0 \in K$$

for every $f \in \mathcal{T}_n^r(a_1, a_2, \dots, a_{2n}; K)$ with

$$(a_1, a_2, \dots, a_{2n}) \subset \mathbb{C} \setminus \mathbb{R}, \quad \text{Im}(a_j) > 0, \quad j = 1, 2, \dots, 2n$$

and

$$(1 - x_0^2) f'(x_0)^2 + B_n(x_0)^2 f(x_0)^2 \leq B_n(x_0)^2 \|f\|_{[-1, 1]}^2, \quad x_0 \in (-1, 1)$$

for every $f \in \mathcal{P}_n^r(a_1, a_2, \dots, a_n; [-1, 1])$ with

$$\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus [-1, 1]$$

have been proved, where

$$\tilde{B}_n(\theta) := \frac{1}{2} \sum_{j=1}^{2n} \frac{1 - |e^{ia_j}|^2}{|e^{ia_j} - e^{i\theta}|^2}, \quad \theta \in K$$

and

$$B_n(x) := \text{Re} \left(\sum_{j=1}^n \frac{\sqrt{a_j^2 - 1}}{a_j - x} \right), \quad x \in [-1, 1]$$

with the choice of $\sqrt{a_j^2 - 1}$ is determined by

$$\left| a_j - \sqrt{a_j^2 - 1} \right| < 1.$$

These inequalities give sharp upper bound for $|f'(\theta)|$ and $|f'(x_0)|$ only at n points in K and $[-1, 1]$, respectively. In this paper we establish Bernstein-type inequalities for the spaces

$$\mathcal{P}_n^c(a_1, a_2, \dots, a_n, \partial D) \quad \text{and} \quad \mathcal{T}_n^c(a_1, a_2, \dots, a_{2n}; K)$$

which are sharp at every $z \in \partial D$ and $\theta \in K$, respectively. An essentially sharp Bernstein-type inequality is also established for the space

$$\mathcal{P}_n^c(a_1, a_2, \dots, a_n; [-1, 1]).$$

A Bernstein-type inequality of Russak [7] is extended to the spaces

$$\mathcal{P}_n^c(a_1, a_2, \dots, a_n; \mathbb{R}),$$

and a Bernstein-Szegő type inequality is established for the spaces

$$\mathcal{P}_n^r(a_1, a_2, \dots, a_n; \mathbb{R}).$$

For a polynomial

$$q_n(z) = c \prod_{j=1}^n (z - a_j), \quad 0 \neq c \in \mathbb{C}, \quad a_j \in \mathbb{C}$$

we define

$$q_n^*(z) = \bar{c} \prod_{j=1}^n (1 - \bar{a}_j z) = z^n \bar{q}_n(z^{-1}).$$

It is well-known, and simple to check, that

$$|q_n(z)| = |q_n^*(z)|, \quad z \in \partial D.$$

We also define the Blaschke products

$$S_n(z) := \prod_{j=1}^n \frac{1 - \bar{a}_j z}{z - a_j}$$

associated with $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus \partial D$, and

$$\tilde{S}_n(z) := \prod_{j=1}^n \frac{z - \bar{a}_j}{z - a_j}$$

associated with $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}$.

2. New Results.

Theorem 1. *Let $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus \partial D$. Then*

$$|f'(z_0)| \leq \max \left\{ \sum_{\substack{j=1 \\ |a_j| > 1}} \frac{|a_j|^2 - 1}{|a_j - z_0|^2}, \sum_{\substack{j=1 \\ |a_j| < 1}} \frac{1 - |a_j|^2}{|a_j - z_0|^2} \right\} \|f\|_{\partial D}$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \dots, a_n; \partial D)$ and $z_0 \in \partial D$. If the first sum is not less than the second sum for a fixed $z_0 \in \partial D$, then equality holds for $f = c S_n^+$, $c \in \mathbb{C}$, where S_n^+ is the Blaschke product associated with those a_j for which $|a_j| > 1$. If the first sum is not greater than the second sum for a fixed $z_0 \in \partial D$, then equality holds for $f = c S_n^-$, $c \in \mathbb{C}$, where S_n^- is the Blaschke product associated with those a_j for which $|a_j| < 1$.

Theorem 2. Let $\{a_1, a_2, \dots, a_{2n}\} \subset \mathbb{C} \setminus \mathbb{R}$. Then

$$|f'(\theta_0)| \leq \max \left\{ \sum_{\substack{j=1 \\ \operatorname{Im}(a_j) < 0}}^{2n} \frac{|e^{ia_j}|^2 - 1}{|e^{ia_j} - e^{i\theta_0}|^2}, \sum_{\substack{j=1 \\ \operatorname{Im}(a_j) > 0}}^{2n} \frac{1 - |e^{ia_j}|^2}{|e^{ia_j} - e^{i\theta_0}|^2} \right\} \|f\|_K$$

for every $f \in \mathcal{T}_n^c(a_1, a_2, \dots, a_{2n}; K)$ and $\theta_0 \in K$. If the first sum is not less than the second sum for a fixed $\theta_0 \in K$, then equality holds for $f(\theta) = cS_{2n}^+(e^{i\theta})$, $c \in \mathbb{C}$. If the first sum is not greater than the second sum for a fixed $\theta_0 \in K$, then equality holds for $f(\theta) = cS_{2n}^-(e^{i\theta})$, $c \in \mathbb{C}$. S_{2n}^+ and S_{2n}^- associated with $\{e^{ia_1}, e^{ia_2}, \dots, e^{ia_{2n}}\}$ are defined as in Theorem 1.

Theorem 3. Let $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C}/[-1, 1]$ and

$$c_j := a_j - \sqrt{a_j^2 - 1}, \quad |c_j| < 1$$

with the choice of root in $\sqrt{a_j^2 - 1}$ determined by $|c_j| < 1$. Then

$$|f'(x_0)| \leq \frac{1}{\sqrt{1-x_0^2}} \max \left\{ \sum_{j=1}^n \frac{|c_j|^{-2} - 1}{|c_j^{-1} - z_0|^2}, \sum_{j=1}^n \frac{1 - |c_j|^2}{|c_j - z_0|^2} \right\} \|f\|_{[-1,1]}$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \dots, a_n; [-1, 1])$ and $x_0 \in (-1, 1)$, where z_0 is defined by

$$z_0 := x_0 + i\sqrt{1-x_0^2}, \quad x_0 \in (-1, 1).$$

Note that

$$B_n(x_0) = \operatorname{Re} \left(\sum_{j=1}^n \frac{\sqrt{a_j^2 - 1}}{a_j - x_0} \right) = \sum_{j=1}^n \frac{1 - |c_j|^2}{|c_j - z_0|^2}, \quad x_0 \in (-1, 1).$$

Our next result extends an inequality established by Russak [7] to wider families of rational functions.

Theorem 4. Let $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}$. Then

$$|f'(x_0)| \leq \max \left\{ \sum_{\substack{j=1 \\ \operatorname{Im}(a_j) > 0}}^n \frac{2|\operatorname{Im}(a_j)|}{|x_0 - a_j|^2}, \sum_{\substack{j=1 \\ \operatorname{Im}(a_j) < 0}}^n \frac{2|\operatorname{Im}(a_j)|}{|x_0 - a_j|^2} \right\} \|f\|_{\mathbb{R}}$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \dots, a_n; \mathbb{R})$ and $x_0 \in \mathbb{R}$. If the first sum is not less than the second sum for a fixed $x_0 \in \mathbb{R}$, then equality holds for $f = c\tilde{S}_n^+$, $c \in \mathbb{C}$, where \tilde{S}_n^+ is the Blaschke product associated with the poles a_j lying in the upper half-plane

$$H^+ := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}.$$

If the first sum is not greater than the second sum for a fixed $x_0 \in \mathbb{R}$, then equality holds for $f = c\tilde{S}_n^-$, $c \in \mathbb{C}$, where \tilde{S}_n^- is the Blaschke product associated with the poles a_j lying in the lower half-plane

$$H^- := \{z \in \mathbb{C} : \text{Im}(z) < 0\}.$$

Our last result is a Bernstein-Szegő type inequality for

$$\mathcal{P}_n^r(a_1, a_2, \dots, a_{2n}; \mathbb{R})$$

which follows from the Bernstein-Szegő type inequality for

$$\mathcal{P}_n^r(a_1, a_2, \dots, a_n; [-1, 1])$$

mentioned in the introduction.

Theorem 5. *Let*

$$\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}, \quad \text{Im}(a_j) > 0, \quad j = 1, 2, \dots, n.$$

Then

$$f'(x_0)^2 + \widehat{B}_n(x_0)^2 f(x_0)^2 \leq \widehat{B}_n(x_0)^2 \|f\|_{\mathbb{R}}^2, \quad x_0 \in \mathbb{R},$$

for every $f \in \mathcal{P}_n^r(a_1, a_2, \dots, a_n; \mathbb{R})$, where

$$\widehat{B}_n(x) := \sum_{j=1}^n \frac{\text{Im}(a_j)}{|x - a_j|^2}, \quad x \in \mathbb{R}.$$

We remark that equality holds in Theorem 5 if and only if x_0 is a maximum point of f (i.e. $f(x_0) = \pm \|f\|_{\mathbb{R}}$) or f is a ‘‘Chebyshev polynomial’’ for the space $\mathcal{P}_n^r(a_1, a_2, \dots, a_n; \mathbb{R})$ which can be explicitly expressed by using the results of [2] and [3].

Note that Bernstein’s classical inequalities are contained in Theorem 1, 2, and 3 as limiting cases, by taking

$$\{a_1^{(k)}, a_2^{(k)}, \dots, a_n^{(k)}\} \subset \mathbb{C} \setminus D$$

in Theorems 1 and 3 so that $\lim_{k \rightarrow \infty} |a_j^{(k)}| = \infty$ for each $j = 1, 2, \dots, n$, and by taking

$$\{a_1^{(k)}, a_2^{(k)}, \dots, a_{2n}^{(k)}\} \subset \mathbb{C} \setminus \mathbb{R}$$

in Theorem 2 so that $a_{n+j}^{(k)} = \overline{a_j^{(k)}}$ and $\lim_{k \rightarrow \infty} |\text{Im}(a_j^{(k)})| = \infty$ for each $j = 1, 2, \dots, n$.

Further results can be obtained as limiting cases by fixing a_1, a_2, \dots, a_m , $1 \leq m \leq n$, in Theorems 1 and 3, and by taking

$$\{a_1, a_2, \dots, a_m, a_{m+1}^{(k)}, a_{m+2}^{(k)}, \dots, a_n^{(k)}\} \subset \mathbb{C} \setminus D$$

so that $\lim_{k \rightarrow \infty} |a_j^{(k)}| = \infty$ for each $j = m+1, m+2, \dots, n$. One may also fix the poles $a_1, a_2, \dots, a_m, a_{n+1}, a_{n+2}, \dots, a_{n+m}$, $1 \leq m \leq n$, in Theorem 2 and take

$$\{a_1, \dots, a_m, a_{m+1}^{(k)}, \dots, a_n^{(k)}, a_{n+1}, \dots, a_{n+m}, a_{n+m+1}^{(k)}, \dots, a_{2n}^{(k)}\} \subset \mathbb{C} \setminus \mathbb{R}$$

so that $a_{n+j}^{(k)} = \overline{a_j^{(k)}}$ and $\lim_{k \rightarrow \infty} |\text{Im}(a_j^{(k)})| = \infty$ for each $j = m+1, m+2, \dots, n$.

Several interesting corollaries of the above three theorems can be obtained. We formulate only one of these.

Corollary 6. *Suppose $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C}$ and*

$$1 < R \leq |a_j|, \quad j = 1, 2, \dots, n.$$

Then

$$|f'(z_0)| \leq \frac{R+1}{R-1} n \|f\|_{\partial D}, \quad z \in \partial D,$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \dots, a_n; \partial D)$. For a fixed $z_0 \in \partial D$ equality holds if and only if

$$a_1 = a_2 = \dots = a_n = Rz_0$$

and $f = cS_n$, $c \in \mathbb{C}$, where S_n is the Blaschke product associated with the poles a_j , $j = 1, 2, \dots, n$.

3. Proofs.

To prove Theorem 1 we need the following result (see [9, p. 38] for instance).

Interpolation Theorem. Let V be an $n + 1$ dimensional subspace over \mathbb{C} of $C(Q)$, the linear space of complex-valued continuous functions defined on a compact Hausdorff space Q , and let $L \neq 0$ be a linear functional on V . Then there exists distinct points x_1, x_2, \dots, x_r in Q , where $1 \leq r \leq 2n + 1$, and nonzero real numbers c_1, c_2, \dots, c_r so that

$$L(f) = \sum_{j=1}^r c_j f(x_j), \quad f \in V$$

and

$$\|L\| := \max_{0 \neq f \in V} \frac{|L(f)|}{\|f\|_Q} = \sum_{j=1}^r |c_j|.$$

Proof of Theorem 1. For the reason of symmetry it is sufficient to prove the theorem when $z = 1$. Without loss of generality we may assume that

$$(1) \quad \operatorname{Re} \left(\sum_{j=1}^n \frac{1}{1 - a_j} \right) \neq \frac{n}{2}$$

the other cases follow from this by a limiting argument. Let $Q := \partial D$ (with the usual metric topology),

$$V := \mathcal{P}_n^c(a_1, a_2, \dots, a_n; \partial D)$$

and

$$L(f) := f'(1), \quad f \in V.$$

We show in this situation that $n + 1 \leq r$ in the Interpolation Theorem. Suppose to the contrary that $r \leq n$. By the Interpolation Theorem there are r distinct points x_1, x_2, \dots, x_r on ∂D so that

$$(2) \quad \frac{p'_n(1)q_n(1) - q'_n(1)p_n(1)}{q_n(1)^2} = \sum_{j=1}^r c_j \frac{p_n(x_j)}{q_n(x_j)}, \quad p_n \in \mathcal{P}_n^c,$$

where

$$(3) \quad q_n(z) := \prod_{j=1}^n (z - a_j).$$

We claim that $x_j \neq 1$ for each $j = 1, 2, \dots, r$. Indeed, if there is an index j so that $x_j = 1$, then the Interpolation Theorem implies that

$$p_n(z) := (z + 1)^{n-r} \prod_{j=1}^r (z - x_j) \in \mathcal{P}_n^c$$

has a zero at 1 with multiplicity at least 2, a contradiction. Applying (2) to the above p_n , we obtain

$$p_n'(1)q_n(1) - q_n'(1)p_n(1) = 0,$$

and since $p_n(1) \neq 0$ and $q_n(1) \neq 0$, this is equivalent to

$$\frac{q_n'(1)}{q_n(1)} = \frac{p_n'(1)}{p_n(1)}$$

or in terms of the zeros of p_n and q_n

$$(4) \quad \sum_{j=1}^n \frac{1}{1 - a_j} = \frac{n-r}{2} + \sum_{j=1}^r \frac{1}{1 - x_j}.$$

Since $x_j \in \partial D$ and $x_j \neq 1$, $j = 1, 2, \dots, r$, we have

$$(5) \quad \operatorname{Re} \left(\frac{1}{1 - x_j} \right) = \frac{1}{2}, \quad j = 1, 2, \dots, r.$$

It follows from (4) and (5) that

$$\operatorname{Re} \left(\sum_{j=1}^n \frac{1}{1 - a_j} \right) = \frac{n}{2}$$

which contradicts assumption (1). So $n + 1 \leq r$, indeed.

A simple compactness argument shows that there is a function $\tilde{f} \in V$ so that $\|\tilde{f}\|_{\partial D} = 1$ and $|L(\tilde{f})| = \|L\|$. The interpolation Theorem implies

$$|\tilde{f}(x_j)| = 1, \quad j = 1, 2, \dots, r.$$

Hence, if

$$\tilde{f} = \frac{\tilde{p}_n}{q_n}, \quad \tilde{p}_n \in \mathcal{P}_n^c, \quad q_n(z) = \prod_{j=1}^n (z - a_j),$$

then

$$(6) \quad h(z) = |\tilde{p}_n(z)|^2 - |q_n(z)|^2 \leq 0, \quad z \in \partial D$$

and

$$(7) \quad h(x_j) = 0, \quad j = 1, 2, \dots, r.$$

Note that $t(\theta) := h(e^{i\theta}) \in \mathcal{T}_n^r$ vanishes at each θ_j , where $\theta_j \in [0, 2\pi)$ is defined by $x_j = e^{i\theta_j}$, $j = 1, 2, \dots, r$. Because of (6), each of these zeros is of even multiplicity. Hence, $n + 1 \leq r$ implies that $t \in \mathcal{T}_n$ has at least $2n + 2$ zeros with multiplicities, therefore $t(\theta) \equiv 0$. From this we can deduce that $h(z) = 0$ for every $z \in \partial D$, so

$$(8) \quad |\tilde{p}_n(z)| = |q_n(z)|, \quad z \in \partial D.$$

We have

$$z^{-n} \tilde{p}_n(z) \tilde{p}_n^*(z) = |\tilde{p}_n(z)|^2 = |q_n(z)|^2 = z^{-n} q_n(z) q_n^*(z), \quad z \in \partial D,$$

so by the Unicity Theorem of analytic functions

$$\tilde{p}_n \tilde{p}_n^* = q_n q_n^*.$$

From this it follows that there is a constant $0 \neq c \in \mathbb{C}$ so that

$$\tilde{f}(z) = \frac{\tilde{p}_n(z)}{q_n(z)} = c \prod_{j=1}^m \frac{z - 1/\bar{\alpha}_j}{z - \alpha_j}$$

with some $m \leq n$ and

$$\alpha_j := a_{k_j}, \quad j = 1, 2, \dots, m, \quad 1 \leq k_1 < k_2 < \dots < k_m \leq n.$$

A straightforward calculation gives

$$\begin{aligned} |\tilde{f}'(1)| &= \left| \frac{\tilde{f}'(1)}{\tilde{f}(1)} \right| = \left| \sum_{j=1}^m \left(\frac{1}{1 - 1/\bar{\alpha}_j} - \frac{1}{1 - \alpha_j} \right) \right| \\ &= \left| \sum_{j=1}^m \frac{|\alpha_j|^2 - 1}{|\alpha_j - 1|^2} \right| \leq \max \left\{ \sum_{\substack{j=1 \\ |a_j| > 1}} \frac{|a_j|^2 - 1}{|a_j - 1|^2}, \sum_{\substack{j=1 \\ |a_j| < 1}} \frac{1 - |a_j|^2}{|a_j - 1|^2} \right\} \end{aligned}$$

which finishes the proof. \square

Proof of Theorem 2. Observe that if

$$h_n(\theta) := \prod_{j=1}^{2n} \sin((\theta - a_j)/2) \in \mathcal{T}_n^c$$

and $t_n \in \mathcal{T}_n^c$, then there are $p_{2n} \in \mathcal{P}_{2n}^c$ and $q_{2n} \in \mathcal{P}_{2n}^c$ so that

$$\frac{t_n(\theta)}{h_n(\theta)} = \frac{p_{2n}(e^{i\theta})e^{-in\theta}}{q_{2n}(e^{i\theta})e^{-in\theta}} = \frac{p_{2n}(e^{i\theta})}{q_{2n}(e^{i\theta})},$$

where

$$q_{2n}(z) = c \prod_{j=1}^{2n} (z - e^{ia_j})$$

with some $0 \neq c \in \mathbb{C}$. Therefore the theorem follows from Theorem 1. \square

Proof of Theorem 3. The result follows from Theorem 1 by the substitution

$$x = \frac{1}{2}(z + z^{-1}).$$

\square

Proof of Theorem 4. The function

$$x = i \frac{z + 1}{z - 1}$$

maps $\partial D \setminus \{1\} = \{z \in \mathbb{C} : |z| = 1, z \neq 1\}$ onto the real line. A straightforward calculation shows that the inequality of the theorem follows from Theorem 1 by the above substitution. \square

Proof of Theorem 5. By Corollary 3.3. of [2] we have

$$(9) \quad (1 - y_0^2)g'(y_0)^2 + B_n(y_0)^2 g(y_0)^2 \leq B_n(y_0)^2 \|g\|_{[-1,1]}^2$$

for every $g \in \mathcal{P}_n^r(b_1, b_2, \dots, b_n; [-1, 1])$ and $y_0 \in [-1, 1]$, where

$$\{b_1, b_2, \dots, b_n\} \subset \mathbb{C} \setminus [-1, 1]$$

and

$$B_n(y_0) := \operatorname{Re} \left(\sum_{j=1}^n \frac{\sqrt{b_j^2 - 1}}{b_j - y_0} \right), \quad y_0 \in [-1, 1],$$

with the choice of root in $\sqrt{b_j^2 - 1}$ determined by

$$|b_j - \sqrt{b_j^2 - 1}| < 1.$$

Let $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}$, $x_0 \in \mathbb{R}$, and

$$f \in \mathcal{P}_n(a_1, a_2, \dots, a_n; \mathbb{R})$$

be fixed. Let $a \in \mathbb{R}$ be chosen so that $|x_0| < a$, let $y_0 := x_0/a \in (-1, 1)$, $b_j := a_j/a$, $j = 1, 2, \dots, n$, and

$$g(x) := f(ax) \in \mathcal{P}_n^r(b_1, b_2, \dots, b_n; [-1, 1]).$$

Applying (9) with the above g and y_0 , we obtain

$$(1 - y_0)^2 a^2 f'(x_0)^2 + B_n(y_0)^2 f(x_0)^2 \leq B_n(y_0)^2 \|f\|_{[-a, a]}^2$$

so

$$(10) \quad \frac{a^2 - x_0^2}{a^2} f'(x_0)^2 + (a^{-1} B_n(y_0))^2 f(x_0)^2 \leq (a^{-1} B_n(y_0))^2 \|f\|_{\mathbb{R}}^2$$

where

$$(11) \quad \begin{aligned} \lim_{a \rightarrow +\infty} a^{-1} B_n(y_0) &= \lim_{a \rightarrow +\infty} \operatorname{Re} \left(\sum_{j=1}^n \frac{\sqrt{b_j^2 - 1}}{a(b_j - y_0)} \right) \\ &= \lim_{a \rightarrow +\infty} \operatorname{Re} \left(\sum_{j=1}^n \frac{\sqrt{(a_j/a)^2 - 1}}{a_j - x_0} \right) \\ &= \lim_{a \rightarrow +\infty} \operatorname{Re} \left(\sum_{j=1}^n \frac{\sqrt{(a_j/a)^2 - 1} - a_j/a}{a_j - x_0} \right) \\ &= \operatorname{Re} \left(\sum_{j=1}^n \frac{i \operatorname{sign} \left(\operatorname{Im} \left(\sqrt{a_j^2 - 1} - a_j \right) \right) (\bar{a}_j - x_0)}{|\bar{a}_j - x_0|^2} \right) \\ &= \sum_{j=1}^n \frac{\operatorname{Im}(a_j)}{|a_j - x_0|^2} = \widehat{B}_n(x_0) \end{aligned}$$

(note that the map $a \rightarrow \sqrt{(a_j/a)^2 - 1} - a_j/a$ is a continuous map on $(0, \infty)$ taking only nonreal values, and

$$\operatorname{Im} \left(\sqrt{a_j^2 - 1} - a_j \right) < 0$$

follows from $|a_j - \sqrt{a_j^2 - 1}| < 1$ and $\operatorname{Im}(a_j) > 0$.) Therefore, taking the limit on (10) when $a \rightarrow +\infty$, we obtain the theorem by (11).

Proof of Corollary 6. The inequality follows from Theorem 1 since $R \leq |a_j|$ and $|z_0| = 1$ imply

$$\frac{|a_j|^2 - 1}{|a_j - z_0|^2} \leq \frac{R + 1}{R - 1}, \quad j = 1, 2, \dots, n.$$

Now assume that $\tilde{f} \neq 0$ satisfies

$$|\tilde{f}'(z_0)| = \frac{R + 1}{R - 1} n, \quad \|\tilde{f}\|_{\partial D} = 1,$$

for some $z_0 \in \partial D$. Then we obtain from Theorem 1 that

$$\frac{|a_j|^2 - 1}{|a_j - z_0|^2} = \frac{R + 1}{R - 1}, \quad j = 1, 2, \dots, n,$$

therefore

$$a_j = R z_0, \quad j = 1, 2, \dots, n.$$

Now observe that $1 < R \leq |a_j|$, $j = 1, 2, \dots, n$, implies

$$\operatorname{Re} \left(\sum_{j=1}^n \frac{1}{1 - a_j} \right) < \sum_{j=1}^n \frac{1}{2} = \frac{n}{2},$$

so the proof of Theorem 1 yields that $\tilde{f} = cS_n$, $|c| = 1$, where S_n is the Blaschke product associated with $\{a_1, a_2, \dots, a_n\}$.

On the other hand, if $z_0 \in \partial D$, $a_1 = a_2 = \dots = a_n = Rz_0$, S_n is the Blaschke product associated with $\{a_1, a_2, \dots, a_n\}$ and $f = cS_n$, $c \in \mathbb{C}$, then

$$|f'(z_0)| = \frac{R+1}{R-1} \|f\|_{\partial D} = c \frac{R+1}{R-1}$$

and the proof is finished. \square

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