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Titre français: Preuve de la conjecture de quasi-orthogonalité de Saffari pour les suites ultra-plates de polynômes unimodulaires.

**PROOF OF SAFFARI'S NEAR-ORTHOGONALITY
CONJECTURE FOR ULTRAFLAT SEQUENCES
OF UNIMODULAR POLYNOMIALS**

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Abstract. Let $P_n(z) = \sum_{k=0}^n a_{k,n} z^k \in \mathbb{C}[z]$ be a sequence of unimodular polynomials ($|a_{k,n}| = 1$ for all k, n) which is ultraflat in the sense of Kahane, i.e.,

$$\lim_{n \rightarrow \infty} \max_{|z|=1} \left| (n+1)^{-1/2} |P_n(z)| - 1 \right| = 0.$$

We prove the following conjecture of Saffari (1991): $\sum_{k=0}^n a_{k,n} a_{n-k,n} = o(n)$ as $n \rightarrow \infty$, that is, the polynomial $P_n(z)$ and its “conjugate reciprocal” $P_n^*(z) = \sum_{k=0}^n \bar{a}_{n-k,n} z^k$ become “nearly orthogonal” as $n \rightarrow \infty$. To this end we use results from [Er1] where (as well as in [Er3]) we studied the structure of ultraflat polynomials and proved several conjectures of Saffari.

PREUVE DE LA CONJECTURE DE QUASI-ORTHOGONALITÉ DE SAFFARI
POUR LES SUITES ULTRA-PLATES DE POLYNÔMES UNIMODULAIRES

Résumé. Soit $P_n(z) = \sum_{k=0}^n a_{k,n} z^k \in \mathbb{C}[z]$ une suite de polynômes unimodulaires ($|a_{k,n}| = 1$ pour tout k, n) supposée ultra-plate au sens de Kahane, c.à.d.

$$\lim_{n \rightarrow \infty} \max_{|z|=1} \left| (n+1)^{-1/2} |P_n(z)| - 1 \right| = 0.$$

Nous prouvons la conjecture suivante de Saffari (1991): $\sum_{k=0}^n a_{k,n} a_{n-k,n} = o(n)$ pour $n \rightarrow \infty$, c.à.d. que le polynôme $P_n(z)$ et son “reciproque conjugué” $P_n^*(z) = \sum_{k=0}^n \bar{a}_{n-k,n} z^k$ deviennent “quasi-orthogonaux” lorsque $n \rightarrow \infty$. Pour ce faire nous employons des résultats de [Er1] où (ainsi que dans [Er3]) nous avons étudié la structure des polynômes ultra-plats et avons prouvé plusieurs conjectures de Saffari.

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VERSION FRANÇAISE ABRÉGÉE

Une suite de polynômes $P_n(z) = \sum_{k=0}^n a_{k,n} z^k \in \mathbb{C}[z]$ à coefficients unimodulaires (appelés, pour abrégé, “polynômes unimodulaires”) est dite ultra-plate s’il existe une suite positive (ε_n) tendant vers zéro telle que, pour tout n , on ait

$$(1 - \varepsilon_n)\sqrt{n+1} \leq |P_n(e^{it})| \leq (1 + \varepsilon_n)\sqrt{n+1} \quad (\text{pour tout } t \in \mathbb{R}).$$

Le problème de l’existence de telles suites ultra-plates fut soulevé en 1966 par Littlewood [Li1] qui, selon ses collègues et selon des écrits ultérieurs, tantôt conjecturait leur existence et tantôt partageait l’opinion générale (laquelle penchait pour la conjecture d’inexistence). Cependant, en 1980, Kahane [Ka] prouva finalement leur *existence* par une méthode probabiliste (non constructive).

En 1991 B. Saffari [Sa] étudia les polynômes ultra-plats (a priori quelconques, et pas seulement ceux obtenus par la méthode de Kahane [Ka]). Dans deux articles très récents [Er1] et [Er3] nous avons étudié la structure des polynômes ultra-plats (a priori quelconques) et prouvé plusieurs conjectures de Saffari [Sa]. Dans cette Note, nous prouvons le résultat suivant, également conjecturé par Saffari [Sa]:

Théorème. *Si la suite $P_n(z) = \sum_{k=0}^n a_{k,n} z^k \in \mathbb{C}[z]$ est ultra-plate, alors*

$$\sum_{k=0}^n a_{k,n} a_{n-k,n} = o(n)$$

ce qui signifie que $P_n(z)$ et $P_n^(z) = \sum_{k=0}^n \bar{a}_{n-k,n} z^k$, le polynôme “réciproque-conjugué” de $P_n(z)$, deviennent “quasi-orthogonaux” pour $n \rightarrow \infty$.*

La démonstration, donnée dans la version anglaise, est basée sur des techniques d’analyse réelle et sur des résultats que nous avons prouvés dans [Er3] par des techniques d’analyse complexe.

1. INTRODUCTION AND THE NEW RESULT

Let D be the open unit disk of the complex plane. Its boundary, the unit circle of the complex plane, is denoted by ∂D . Let

$$\mathcal{K}_n := \left\{ p_n : p_n(z) = \sum_{k=0}^n a_k z^k, \quad a_k \in \mathbb{C}, \quad |a_k| = 1 \right\}.$$

The class \mathcal{K}_n is often called the collection of all (complex) unimodular polynomials of degree n . Let

$$\mathcal{L}_n := \left\{ p_n : p_n(z) = \sum_{k=0}^n a_k z^k, \quad a_k \in \{-1, 1\} \right\}.$$

The class \mathcal{L}_n is often called the collection of all (real) unimodular polynomials of degree n . By Parseval’s formula,

$$\int_0^{2\pi} |P_n(e^{it})|^2 dt = 2\pi(n+1)$$

for all $P_n \in \mathcal{K}_n$. Therefore

$$\min_{z \in \partial D} |P_n(z)| \leq \sqrt{n+1} \leq \max_{z \in \partial D} |P_n(z)|.$$

An old problem (or rather an old theme) is the following.

Problem 1.1 (Littlewood's Flatness Problem). *How close can a unimodular polynomial $P_n \in \mathcal{K}_n$ or $P_n \in \mathcal{L}_n$ come to satisfying*

$$(1.1) \quad |P_n(z)| = \sqrt{n+1}, \quad z \in \partial D?$$

Obviously (1.1) is impossible if $n \geq 1$. So one must look for less than (1.1), but then there are various ways of seeking such an "approximate situation". One way is the following. In his paper [Li1] Littlewood had suggested that, conceivably, there might exist a sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ (possibly even $P_n \in \mathcal{L}_n$) such that $(n+1)^{-1/2}|P_n(e^{it})|$ converge to 1 uniformly in $t \in \mathbb{R}$. We shall call such sequences of unimodular polynomials "ultraflat". More precisely, we give the following definition.

Definition 1.2. *Given a positive number ε , we say that a polynomial $P_n \in \mathcal{K}_n$ is ε -flat if*

$$(1 - \varepsilon)\sqrt{n+1} \leq |P_n(z)| \leq (1 + \varepsilon)\sqrt{n+1}, \quad z \in \partial D.$$

Definition 1.3. *Given a sequence (ε_{n_k}) of positive numbers tending to 0, we say that a sequence (P_{n_k}) of unimodular polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is (ε_{n_k}) -ultraflat if each P_{n_k} is (ε_{n_k}) -flat. We simply say that a sequence (P_{n_k}) of unimodular polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is ultraflat if it is (ε_{n_k}) -ultraflat with a suitable sequence (ε_{n_k}) of positive numbers tending to 0.*

The existence of an ultraflat sequence of unimodular polynomials seemed very unlikely, in view of a 1957 conjecture of P. Erdős (Problem 22 in [Er]) asserting that, for all $P_n \in \mathcal{K}_n$ with $n \geq 1$,

$$(1.2) \quad \max_{z \in \partial D} |P_n(z)| \geq (1 + \varepsilon)\sqrt{n+1},$$

where $\varepsilon > 0$ is an absolute constant (independent of n). Yet, refining a method of Körner [Kö], Kahane [Ka] proved that there exists a sequence (P_n) with $P_n \in \mathcal{K}_n$ which is (ε_n) -ultraflat, where $\varepsilon_n = O(n^{-1/17}\sqrt{\log n})$. (Kahane's paper contained though a slight error which was corrected in [QS2].) Thus the Erdős conjecture (1.2) was disproved for the classes \mathcal{K}_n . For the more restricted class \mathcal{L}_n the analogous Erdős conjecture is unsettled to this date. It is a common belief that the analogous Erdős conjecture for \mathcal{L}_n is true, and consequently there is no ultraflat sequence of polynomials $P_n \in \mathcal{L}_n$. An interesting result related to Kahane's breakthrough is given in [Be]. For an account of some of the work done till the mid 1960's, see Littlewood's book [Li2] and [QS2].

Let (ε_n) be a sequence of positive numbers tending to 0. Let the sequence (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n$ be (ε_n) -ultraflat. We write

$$(1.3) \quad P_n(e^{it}) = R_n(t)e^{i\alpha_n(t)}, \quad R_n(t) = |P_n(e^{it})|, \quad t \in \mathbb{R}.$$

It is a simple exercise to show that α_n can be chosen so that it is differentiable on \mathbb{R} . This is going to be our understanding throughout the paper.

The structure of ultraflat sequences of unimodular polynomials is studied in [Er1] and [Er3] where several conjectures of Saffari are proved. Here, based on the results in [Er1], we prove yet another Saffari conjecture formulated in [Sa].

Theorem 1.4 (Saffari's Near-Orthogonality Conjecture). *Assume that (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. Let*

$$P_n(z) := \sum_{k=0}^n a_{k,n} z^k.$$

Then

$$\sum_{k=0}^n a_{k,n} a_{n-k,n} = o(n).$$

Here, as usual, $o(n)$ denotes a quantity for which $\lim_{n \rightarrow \infty} o(n)/n = 0$. The statement remains true if the ultraflat sequence (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n$ is replaced by an ultraflat sequence (P_{n_k}) of unimodular polynomials $P_{n_k} \in \mathcal{K}_{n_k}$, $0 < n_1 < n_2 < \dots$.

If Q_n is a polynomial of degree n of the form $Q_n(z) = \sum_{k=0}^n a_k z^k$, $a_k \in \mathbb{C}$, then its conjugate reciprocal polynomial is defined by $Q_n^*(z) := z^n \overline{Q_n(1/z)} := \sum_{k=0}^n \overline{a_{n-k}} z^k$. In terms of the above definition Theorem 1.4 may be rewritten as

Corollary 1.5. *Assume that (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. Then*

$$\int_{\partial D} |P_n(z) - P_n^*(z)|^2 |dz| = 2n + o(n).$$

2. PROOF OF THEOREM 1.4

To prove the theorem we need a few lemmas. The first two are from [Er1].

Lemma 2.1 (Uniform Distribution Theorem for the Angular Speed).

Suppose (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. Then, with the notation (1.3), in the interval $[0, 2\pi]$, the distribution of the normalized angular speed $\alpha'_n(t)/n$ converges to the uniform distribution as $n \rightarrow \infty$. More precisely, we have

$$\text{meas}(\{t \in [0, 2\pi] : 0 \leq \alpha'_n(t) \leq nx\}) = 2\pi x + \gamma_n(x)$$

for every $x \in [0, 1]$, where $\lim_{n \rightarrow \infty} \max_{x \in [0, 1]} |\gamma_n(x)| = 0$.

Lemma 2.2 (Negligibility Theorem for Higher Derivatives). *Suppose (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. Then, with the notation (1.3), for every integer $r \geq 2$, we have*

$$\max_{0 \leq t \leq 2\pi} |\alpha_n^{(r)}(t)| \leq \gamma_{n,r} n^r$$

with suitable constants $\gamma_{n,r} > 0$ converging to 0 for every fixed $r = 2, 3, \dots$

Lemma 2.3. *Suppose (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. Let*

$$P_n(z) := \sum_{k=0}^n a_{k,n} z^k.$$

Then, with the notation (1.3),

$$\sum_{k=0}^n a_{k,n} a_{n-k,n} - \frac{n}{2\pi} \int_0^{2\pi} \exp(i(2\alpha_n(t) - nt)) = o(n).$$

Proof of Lemma 2.3. This follows easily by using the formula

$$\sum_{k=0}^n a_{k,n} a_{n-k,n} = \frac{1}{2\pi} \int_0^{2\pi} P_n(e^{it})^2 e^{-int} dt$$

and the ultraflatness inequalities

$$(1 - \varepsilon_n) \sqrt{n+1} \leq |P_n(e^{it})| \leq (1 + \varepsilon_n) \sqrt{n+1}, \quad n = 1, 2, \dots,$$

(cf. Definitions 1.2 and 1.3), where (ε_n) is a sequence of positive numbers tending to 0.

Proof of Theorem 1.4. By Lemma 2.3 it is sufficient to prove that

$$\int_0^{2\pi} \exp(i\beta_n(t)) dt = \eta_n, \quad \text{with} \quad \beta_n(t) := 2\alpha_n(t) - nt,$$

where (η_n) is a sequence tending to 0. To see this let $\varepsilon > 0$ be fixed. Let $K_n := \gamma_{n,2}^{-1/4}$, where $\gamma_{n,2}$ is defined in Lemma 2.2. We divide the interval $[0, 2\pi]$ into subintervals

$$I_m := [a_{m-1}, a_m] := \left[\frac{(m-1)K_n}{n}, \frac{mK_n}{n} \right], \quad m = 1, 2, \dots, N-1 := \left\lfloor \frac{2\pi n}{K_n} \right\rfloor,$$

and

$$I_N := [a_{N-1}, a_N] := \left[\frac{(N-1)K_n}{n}, 2\pi \right].$$

For the sake of brevity let

$$A_{m-1} := \beta_n(a_{m-1}), \quad m = 1, 2, \dots, N,$$

and

$$B_{m-1} := \beta'_n(a_{m-1}), \quad m = 1, 2, \dots, N.$$

Then by Taylor's Theorem

$$|\beta_n(t) - (A_{m-1} + B_{m-1}(t - a_{m-1}))| \leq \gamma_{n,2} n^2 (K_n/n)^2 \leq \gamma_{n,2} \gamma_{n,2}^{-1/2} \leq \gamma_{n,2}^{1/2}$$

for every $t \in I_m$, where $\lim_{n \rightarrow \infty} \gamma_{n,2}^{1/2} = 0$ by Lemma 2.2. Hence

$$\int_{I_m} \exp(i\beta_n(t)) dt = \int_{I_m} \exp(i(A_{m-1} + B_{m-1}(t - a_{m-1}))) dt + \int_{I_m} \delta_n(t) dt,$$

with functions $\delta_n(t)$ satisfying

$$\lim_{n \rightarrow \infty} \max_{0 \leq t \leq 2\pi} |\delta_n(t)| = 0.$$

Hence for $|B_{m-1}| \geq n\varepsilon$ we have

$$\left| \int_{I_m} \exp(i\beta_n(t)) dt \right| \leq \frac{2}{|B_{m-1}|} + \Delta_n \text{meas}(I_m) \leq \frac{2}{n\varepsilon} + \Delta_n \text{meas}(I_m),$$

where

$$\Delta_n := \max_{0 \leq t \leq 2\pi} |\delta_n(t)| > 0 \quad \text{with} \quad \lim_{n \rightarrow \infty} \Delta_n = 0.$$

Therefore $\lim_{n \rightarrow \infty} K_n = \infty$ implies

$$(2.1) \quad \sum_m \left| \int_{I_m} \exp(i\beta_n(t)) dt \right| \leq \frac{2}{n\varepsilon} N + 2\pi \Delta_n \\ \leq \frac{2}{n\varepsilon} \left(\frac{2\pi n}{K_n} + 1 \right) + 2\pi \Delta_n \leq \eta_n^*(\varepsilon),$$

where the summation is taken over all $m = 1, 2, \dots, N$ for which $|B_{m-1}| \geq n\varepsilon$, and where $(\eta_n^*(\varepsilon))$ is a sequence tending to 0. Now let

$$E_{n,\varepsilon} := \bigcup_{m: |B_{m-1}| \leq n\varepsilon} I_m.$$

For $|B_{m-1}| \leq n\varepsilon$ we deduce by Lemma 2.2 that

$$|\beta'_n(t)| \leq |B_{m-1}| + \frac{K_n}{n} \max_{t \in I_m} |\beta''_n(t)| = \\ = |B_{m-1}| + \frac{K_n}{n} \max_{t \in I_m} |2\alpha''_n(t)| = |B_{m-1}| + \frac{\gamma_{n,2}^{-1/4}}{n} 2\gamma_{n,2} n^2 \leq 2n\varepsilon$$

for every $t \in I_m$ and for every sufficiently large n (independent of m). So

$$E_{n,\varepsilon} \subset \{t \in [0, 2\pi] : |\beta'_n(t)| \leq 2n\varepsilon\} \subset \{t \in [0, 2\pi] : |\alpha'_n(t) - n/2| \leq n\varepsilon\}$$

for every $t \in I_m$ and every sufficiently large n . Hence we obtain by Lemma 2.1 that

$$\text{meas}(E_{n,\varepsilon}) \leq 4\pi\varepsilon + \eta_n^{**}(\varepsilon),$$

where $(\eta_n^{**}(\varepsilon))$ is a sequence tending to 0. Therefore

$$(2.2) \quad \sum_m \left| \int_{I_m} \exp(i\beta_n(t)) dt \right| \leq \text{meas}(E_{n,\varepsilon}) \leq 4\pi\varepsilon + \eta_n^{**}(\varepsilon),$$

where the summation is taken over all $m = 1, 2, \dots, N$ for which $|B_{m-1}| \leq n\varepsilon$. Since $\varepsilon > 0$ is arbitrary, the result follows from (2.1) and (2.2). \square

3. REMARKS

In [Sa] another “near orthogonality” relation has been conjectured. Namely it was suspected that if (P_{n_m}) is an ultraflat sequence of unimodular polynomials $P_{n_m} \in \mathcal{K}_{n_m}$ and

$$P_n(z) := \sum_{k=0}^n a_{k,n} z^k, \quad n = n_m, \quad m = 1, 2, \dots,$$

then

$$\sum_{k=0}^n a_{k,n} \bar{a}_{n-k,n} = o(n), \quad n = n_m, \quad m = 1, 2, \dots,$$

where, as usual, $o(n_m)$ denotes a quantity for which $\lim_{n_m \rightarrow \infty} o(n_m)/n_m = 0$. However, it was Saffari himself, together with Queffelec [QS2], who showed that this could not be any farther away from being true. Namely they constructed an ultraflat sequence (P_{n_m}) of plain-reciprocal unimodular polynomials $P_{n_m} \in \mathcal{K}_{n_m}$ such that

$$P_n(z) := \sum_{k=0}^n a_{k,n} z^k, \quad a_{k,n} = a_{n-k,n}, \quad k = 0, 1, 2, \dots, n,$$

and hence

$$\sum_{k=0}^n a_{k,n} \bar{a}_{n-k,n} = n + 1$$

for the values $n = n_m$, $m = 1, 2, \dots$.

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