ON THE ZEROS OF COSINE POLYNOMIALS: SOLUTION TO AN OLD PROBLEM OF LITTLEWOOD

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Abstract. Littlewood in his 1968 monograph “Some Problems in Real and Complex Analysis” [10, problem 22] poses the following research problem, which appears to still be open:

Problem. “If the $n_j$ are integral and all different, what is the lower bound on the number of real zeros of $\sum_{j=1}^{N} \cos(n_j\theta)$? Possibly $N-1$, or not much less.”

No progress appears to have been made on this in the last half-century. We show that this is false.

Theorem. There exists a cosine polynomial $\sum_{j=1}^{N} \cos(n_j\theta)$ with the $n_j$ integral and all different so that the number of its real zeros in the period is $O\left(N^{9/10}(\log N)^{1/5}\right)$.

1. Littlewood’s 22nd Problem

Problem. “If the $n_j$ are integral and all different, what is the lower bound on the number of real zeros of $\sum_{j=1}^{N} \cos(n_j\theta)$? Possibly $N-1$, or not much less.”

Here “real zeros” means “zeros in a period”. Denote the number of zeros of a trigonometric polynomial $T$ in the period $[-\pi, \pi]$ by $\mathcal{N}(T)$.

Note that if $T$ is a real trigonometric cosine polynomial of degree $n$, then it is of the form $T(t) = \exp(-int)P(\exp(it))$, $t \in \mathbb{R}$, where $P$ is a reciprocal algebraic polynomial of degree $2n$, and if $T$ has only real zeros, then $P$ has all its zeros on the unit circle. So in terms of reciprocal algebraic polynomials one is looking for a reciprocal algebraic polynomial with coefficients in $\{0, 1\}$, with $2N$ terms, and with $N-1$ or fewer zeros. Even achieving $N-1$ is fairly hard. An exhaustive search up to degree $2N = 32$ yields only 10

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example achieving $N-1$ and only one example with fewer. This first example disproving the “possibly $N-1$” part of the conjecture is

$$\sum_{j=0, j \notin \{9, 10, 11, 14\}}^{14} (x^j + x^{28-j})$$

which has 8 roots of modulus 1 and corresponds to a cosine sum of 11 terms with 8 roots in the period.

It is hard to see how one might generate infinitely many such examples or indeed why Littlewood made his conjecture.

The following is a reciprocal polynomial with 32 terms and exactly 14 zeros of modulus 1:

$$\sum_{j=0, j \notin \{10, 11, 17, 19\}}^{19} (x^j + x^{38-j})$$

So it corresponds to a cosine sum of 16 terms with 14 zeros in $[-\pi, \pi)$. In other words the sharp version of Littlewood’s conjecture is false again, though barely. The following is a reciprocal polynomial with 280 terms and 52 zeros of modulus 1:

$$\sum_{j=0, j \notin \{124, 125, 126, 127, 128, 134, 141, 143, 145, 147, 148, 151, 152\}}^{152} (x^j + x^{304-j})$$

So it corresponds to a cosine sum of 140 terms with 52 zeros in $[-\pi, \pi)$. In other words the sharp version of Littlewood’s conjecture is false, though this time by a margin. It was found by a version of the greedy algorithm (and some guessing). There is no reason to believe it is a minimal example.

The interesting feature of this example is how close it is to the Dirichlet kernel $(1 + x + x^2 + \ldots + x^{304})$. This is not accidental and suggests the approach that leads to our main result.

Littlewood explored many problems concerning polynomials with various restrictions on the coefficients. See [7], [8], and [9], and in particular Littlewood’s delightful monograph [10]. Related problems and results may be found in [2] and [3], for example. One of these is Littlewood’s well-known conjecture of around 1948 asking for the minimum $L_1$ norm of polynomials of the form

$$p(z) := \sum_{j=0}^{n} a_j z^{k_j},$$

where the coefficients $a_j$ are complex numbers of modulus at least 1 and the exponents $k_j$ are distinct nonnegative integers. It states that such polynomials have $L_1$ norms on the unit circle that grow at least like $c \log n$. This was proved by S. Konyagin [6] and independently by McGehee, Pigno, and Smith [11] in 1981. A short proof is available in [4]. It is believed that the minimum, for polynomials of degree $n$ with complex coefficients of modulus at least 1 is attained by $1 + z + z^2 + \ldots + z^n$, but this is open.
2. Auxiliary Functions

The key is to construct \( n \) term cosine sums that are large most of the time. This is the content of this section.

**Lemma 1.** There is an absolute constant \( c_1 \) such that for all \( n \) and \( \alpha > 1 \) there are coefficients \( a_0, a_1, \ldots, a_n \) with each \( a_j \in \{0, 1\} \) such that

\[
\text{meas}\{t \in [-\pi, \pi) : |P_n(t)| \leq \alpha\} \leq c_1 \alpha n^{-1/2},
\]

where

\[
P_n(t) = \sum_{j=0}^{n} a_j \cos(jt).
\]

**Proof.** We will prove the stronger result that there is an absolute constant \( c_1 \) such that for all \( \alpha > 0 \) and all \( n \)

\[
\lambda(\alpha) := 2^{-(n+1)} \sum_{\{a_0,a_1,\ldots,a_n\}} \text{meas}\{t \in [-\pi, \pi) : |P_n(t)| \leq \alpha\} \leq c_1 \alpha n^{-1/2}.
\]

If \( X_0, X_1, \ldots, X_n \) are independent Bernoulli random variables with

\[
P(X_j = 0) = P(X_j = 1) = \frac{1}{2}, \quad j = 0, 1, \ldots, n,
\]

then the indicated average is an expected value. Let

\[
R_n(t) = \sum_{j=0}^{n} X_j \cos(jt)
\]

and note that

\[
\lambda(\alpha) = \int_{-\pi}^{\pi} P(|R_n(t)| \leq \alpha) \, dt.
\]

Define

\[
D_n(t) := \sum_{j=0}^{n} \cos(jt).
\]

The expected value of \( R_n(t) \) is \( \mu_n(t) := D_n(t)/2 \); its variance is

\[
\sigma_n^2(t) := \frac{1}{4} \sum_{0}^{n} \cos^2(jt) = \frac{1}{8}(n + 1 + D_n(2t)).
\]

We now apply a uniform normal approximation to get the desired result. Define the cumulative normal distribution function by

\[
\Phi(x) := \int_{-\infty}^{x} \frac{e^{-u^2/2}}{\sqrt{2\pi}} \, du.
\]
Define
\[ \varrho_2 := \frac{1}{n+1} \sum_{j=0}^{n} \text{Var}(X_j \cos(jt)) = \]
\[ = \frac{1}{4(n+1)} \sum_{j=0}^{n} \cos^2(jt) = \frac{1}{8} \left( 1 + \frac{D_n(2t)}{n+1} \right), \]
\[ \varrho_3 = \frac{1}{n+1} \sum_{j=0}^{n} \mathbb{E} \left( \left| X_j - \frac{1}{2} \cos(jt) \right|^3 \right) \]

We suppress the dependence of each of these on \(n\) and \(u\). The Berry-Esseen bound in Bhattacharya and Ranga Rao [1, Theorem 12.4, page 104] is that
\[ \mathbb{P}\left( R_n(t) - \mu_n(t) \right) \leq \Phi \left( \frac{c - \mu_n(t)}{\sigma_n(t)} \right) \leq \frac{11\varrho_3}{4\sqrt{n} \varrho_2^{3/2}}. \]

It is elementary that \(\varrho_3 \leq 1/8\). Moreover there is an absolute constant \(c_2 > 0\) such that
\[ \varrho_2 > c_2 \]

for all \(t \in \mathbb{R}\) and all \(n = 1, 2, \ldots\). Finally the function \(\Phi\) has derivative bounded by \((2\pi)^{-1/2}\) so
\[ |\Phi(x) - \Phi(y)| \leq (2\pi)^{-1/2} |x - y|, \quad x, y \in \mathbb{R}. \]

It follows that there is an absolute constant \(c_1\) such that
\[ \mathbb{P}( -\alpha \leq R_n(u) \leq \alpha ) \leq c_1\alpha n^{-1/2}. \]

\[ \Box \]

3. The Main Theorem

**Theorem.** There exists a cosine polynomial \(\sum_{j=1}^{N} \cos(n_j \theta)\) with the \(n_j\) integral and all different so that the number of its real zeros in the period is
\[ O \left( N^{9/10} (\log N)^{1/5} \right). \]

The proof follows immediately from the following Lemma 2 stated below and Lemma 1. Namely, take \(m := N + 1\), \(n = m^{2/5} (\log m)^{-4/5}\), \(\alpha = n^{1/4}\) and \(\beta = c_1\alpha n^{-1/2} = c_1 n^{-1/4}\).

**Lemma 2.** Let \(m \leq n\),
\[ D_m(t) := \sum_{j=0}^{m} \cos(jt), \]
\[ P_n(t) := \sum_{j=0}^{n} a_j \cos(jt), \quad a_j \in \{0, 1\}. \]
Suppose $\alpha \geq 1$ and
\[
\meas\{t \in [-\pi, \pi] : |P_n(t)| \leq \alpha\} \leq \beta.
\]
Let $S_m := D_m - P_n$. Then the number of zeros of $S_m$ in $[-\pi, \pi]$ is at most
\[
\frac{c_3 m}{\alpha} + c_4 m \beta + c_5 n m^{1/2} \log m,
\]
where $c_3$, $c_4$, and $c_5$ are absolute constants.

To prove Lemma 2 we need the following consequence of the Erdős-Turan Theorem [12, p. 278], see also [5].

**Lemma 3.** Let
\[
S_m(t) = \sum_{j=0}^{n} a_j \cos(jt), \quad a_j \in \{0, 1\}.
\]
Denote the number of zeros of $S_m$ in $[\alpha, \beta] \subset [-\pi, \pi]$ by $N([\alpha, \beta])$. Then
\[
N([\alpha, \beta]) \leq c_6 m (\beta - \alpha) + c_6 \sqrt{m} \log m,
\]
where $c_6$ is an absolute constant.

Now we prove Lemma 2.

**Proof.** We write
\[
\{t \in [-\pi, \pi] : |P_n(t)| \leq \alpha\} = \bigcup_{j=1}^{k} I_j,
\]
where the intervals $I_j$ are disjoint and $k \leq 2n$. Let
\[
I_0 := \{t \in [-\pi, \pi] : |D_m(t)| \geq \alpha\}.
\]
Note that $I_0 \subset [-c/\alpha, c/\alpha]$. Then $S_m$ has all its zeros in $\bigcup_{j=0}^{k} I_j$. By Lemma 3 we have
\[
N(I_j) \leq c_6 m |I_j| + c_6 \sqrt{m} \log m, \quad j = 1, 2, \ldots, k,
\]
and
\[
N(I_0) \leq c_6 m |I_0| + c_6 \sqrt{m} \log m \leq \frac{c_7 m}{\alpha} + c_7 \sqrt{m} \log m
\]
with an absolute constant $c_7$. So
\[
N([\pi, \pi]) \leq \sum_{j=0}^{k} N(I_j)
\]
\[
\leq \frac{c_7 m}{\alpha} + c_7 \sqrt{m} \log m + c_6 \sum_{j=1}^{k} m |I_j| + kc_7 \sqrt{m} \log m
\]
\[
\leq \frac{c_7 m}{\alpha} + c_6 m \beta + 2nc_7 \sqrt{m} \log m
\]
and the proof is finished. \qed
4. Average Number of Zeros

Why did Littlewood make this conjecture? Perhaps because usually there are a lot of zeros. This is the point of this section.

Lemma 4. Suppose that $p$ is a polynomial of degree exactly $n$ and $p$ has $k$ zeros of modulus greater than 1 and $j$ zeros of modulus 1 then for any $m \geq 0$

$$z^m p(z) \pm p^*(z)$$

has degree $m + n$ and at least $m + n - 2k$ roots of modulus 1.

Here and throughout, as usual, $p^*(z) = z^{\deg(p)} p(1/z)$ is the reciprocal of $p$.

Proof. Rouche’s theorem shows that $(1 + \epsilon) z^m p(z) \pm p^*(z)$ and $z^m p(z)$ have the same number of roots inside the unit disk. Note that $|p(z)| = |p^*(z)|$ for $|z| = 1$. So with $\epsilon = 0$, $z^m p(z) \pm p^*(z)$ has all but $k$ zeros in the closed unit disk. Now use the fact that $z^m p(z) \pm p^*(z)$ is reciprocal so has the same number of zeros of modulus less than 1 as of modulus greater than 1. □

Lemma 5. Suppose that $p$ is a polynomial of degree exactly $n$ and $p(0) \neq 0$. Consider

$$P(z) := z^m p(z) \pm p^*(z)$$

and

$$Q(z) := z^m p^*(z) \pm p(z)$$

with the same choice of sign (i.e. the cos case and the sin case). Suppose $P$ has $j_1$ zeros of modulus 1 and $Q$ has $j_2$ zeros of modulus 1. Then

$$j_1 + j_2 \geq 2m.$$

Proof. Use the previous lemma and note that if $p$ has $k$ zeros of modulus greater than 1 and $j$ zeros of modulus 1 then $p^*$ has $n - k - j$ zeros of modulus greater than 1 and $j$ zeros of modulus 1. □

Note that if $M := (m - N)/2 \geq 1$ with $M$ an integer then

$$C(t) := \sum_{j=M}^{n+M} a_j \cos(jt)$$

and

$$S(t) := \sum_{j=M}^{n+M} a_j \sin(jt)$$

correspond to

$$P(z) := (z^m p(z) \pm p^*(z)), \quad z = \exp(it),$$

with

$$p(z) = \sum_{j=0}^{n} a_j z^j, \quad a_j \in \mathbb{R}.$$
Also zeros of $P$ of modulus 1 correspond (with the same count) to zeros of the trigonometric polynomials $C$ and $S$ in the period $[0, \pi)$.

The next theorem explains why in any reasonable class one expects cosine sums with many real zeros in the period. The cosine sums naturally break into pairs, more-or-less by conjugation, with a large combined total number of real zeros.

**Theorem 1.** Suppose $a_{n+M} \neq 0$. Consider

$$C(t) := \sum_{j=M}^{n+M} a_j \cos(jt)$$

and

$$C^*(t) := \sum_{j=M}^{n+M} a_{n+2M-j} \cos(jt)$$

which reverses the coefficients. Let $w_1$ be the number of zeros of $C$ in $[0, \pi)$ and let $w_2$ be the number of zeros of $C^*$ in $[0, \pi)$ then

$$w_1 + w_2 \geq n + 1.$$ 

Furthermore $w_1 \geq M$ and $w_2 \geq M$.

**Proof.** Use the previous lemmas. \qed

Averaging gives results like:

**Theorem 2.** The average number of zeros of trigonometric polynomials in the classes

$$\left\{ \sum_{j=1}^{n} a_j \cos(jt), \ a_j \in \{-1, 1\} \right\}$$

and

$$\left\{ \sum_{j=1}^{n} a_j \cos(jt), \ a_j \in \{0, 1\} \right\},$$

respectively in $[0, \pi)$ is at least $n/2$.

**Proof.** Use the previous lemmas \qed

5. Conclusion

A cosine polynomial of the form $\sum_{j=1}^{N} \cos(n_j \theta)$ must have at least one real zero in a period. This is obvious if none of the integers $n_j$ is 0, since then the integral of the sum on a period is 0. The above statement is less obvious if one of the integers $n_j$ is 0, but it follows from Littlewood’s Conjecture simply. Here we mean the already mentioned Littlewood’s Conjecture proved by S. Konyagin [6] and independently by McGehee, Pigno, and Smith [11] in 1981. It seems likely that the number of zeros of the above sums in a period must tend to infinity with $N$. This does not appear to be easy. The case when the sequence $0 \leq n_0 \leq n_1 \leq \cdots$ is fixed will be handled in a forthcoming paper.
References

[9] J.E. Littlewood, On polynomials $\sum \pm z^n$ and $\sum e^{i\alpha n}z^n$, $z = e^{i\theta}$, J. London Math. Soc. 41 (1966), 367–376.

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