ON THE REAL PART OF ULTRAFLAT SEQUENCES OF UNIMODULAR POLYNOMIALS

Tamás Erdélyi
Department of Mathematics, Texas A&M University
College Station, Texas 77843, USA
E-mail: terdelyi@math.tamu.edu

Abstract. Let $P_n(z) = \sum_{k=0}^n a_{k,n} z^k \in \mathbb{C}[z]$ be a sequence of unimodular polynomials ($|a_{k,n}| = 1$ for all k, n) which is ultraflat in the sense of Kahane, i.e.,

$$\lim_{n \to \infty} \max_{|z|=1} \left| (n+1)^{-1/2} |P_n(z)| - 1 \right| = 0\,.$$

For continuous functions f defined on $[0, 2\pi]$, and for $q \in (0, \infty)$, we define

$$\|f\|_q := \left(\int_0^{2\pi} |f(t)|^q \, dt \right)^{1/q}\,.$$

We also define

$$\|f\|_\infty := \lim_{q \to \infty} \|f\|_q = \max_{t \in [0, 2\pi]} |f(t)|\,.$$

We prove the following conjecture of Queffelec and Saffari, see (1.30) in [QS2]. If $q \in (0, \infty)$ and (P_n) is an ultraflat sequence of unimodular polynomials P_n of degree n, then for $f_n(t) := \text{Re}(P_n(e^{it}))$ we have

$$\|f_n\|_q \sim \left(\frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}} \right)^{1/q} n^{1/2}$$

and

$$\|f_n'\|_q \sim \left(\frac{\Gamma\left(\frac{q+1}{2}\right)}{(q+1)\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}} \right)^{1/q} n^{3/2}\,,$$

where Γ denotes the usual gamma function, and the $\sim$ symbol means that the ratio of the left and right hand sides converges to 1 as $n \to \infty$. To this end we use results from [Er1] where (as well as in [Er2], [Er3], and [Er4]) we studied the structure of ultraflat polynomials and proved several conjectures of Queffelec and Saffari.

1991 *Mathematics Subject Classification*. 41A17.
Key words and phrases. unimodular polynomials, ultraflat polynomials, angular derivatives.
Research supported in part by the NSF of the USA under Grant No. Grant No. DMS–9623156

1. Introduction and the New Result

Let D be the open unit disk of the complex plane. Its boundary, the unit circle of the complex plane, is denoted by ∂D. Let

$$\mathcal{K}_n := \left\{ p_n : p_n(z) = \sum_{k=0}^{n} a_k z^k, \;\; a_k \in \mathbb{C}, \;\; |a_k| = 1 \right\}.$$

The class $\mathcal{K}_n$ is often called the collection of all (complex) unimodular polynomials of degree n. Let

$$\mathcal{L}_n := \left\{ p_n : p_n(z) = \sum_{k=0}^{n} a_k z^k, \;\; a_k \in \{-1, 1\} \right\}.$$

The class $\mathcal{L}_n$ is often called the collection of all (real) unimodular polynomials of degree n. By Parseval's formula,

$$\int_0^{2\pi} |P_n(e^{it})|^2 \, dt = 2\pi(n+1)$$

for all $P_n \in \mathcal{K}_n$. Therefore

$$\min_{z \in \partial D} |P_n(z)| \leq \sqrt{n+1} \leq \max_{z \in \partial D} |P_n(z)|.$$

An old problem (or rather an old theme) is the following.

Problem 1.1 (Littlewood's Flatness Problem). *How close can a unimodular polynomial $P_n \in \mathcal{K}_n$ or $P_n \in \mathcal{L}_n$ come to satisfying*

$$|P_n(z)| = \sqrt{n+1}, \qquad z \in \partial D? \tag{1.1}$$

Obviously (1.1) is impossible if $n \geq 1$. So one must look for less than (1.1), but then there are various ways of seeking such an "approximate situation". One way is the following. In his paper [Li1] Littlewood had suggested that, conceivably, there might exist a sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ (possibly even $P_n \in \mathcal{L}_n$) such that $(n+1)^{-1/2}|P_n(e^{it})|$ converge to 1 uniformly in $t \in \mathbb{R}$. We shall call such sequences of unimodular polynomials "ultraflat". More precisely, we give the following definition.

Definition 1.2. *Given a positive number ε, we say that a polynomial $P_n \in \mathcal{K}_n$ is ε-flat if*

$$(1-\varepsilon)\sqrt{n+1} \leq |P_n(z)| \leq (1+\varepsilon)\sqrt{n+1}, \qquad z \in \partial D.$$

Definition 1.3. *Given a sequence (ε_{n_k}) of positive numbers tending to 0, we say that a sequence (P_{n_k}) of unimodular polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is (ε_{n_k})-ultraflat if each P_{n_k} is (ε_{n_k})-flat. We simply say that a sequence (P_{n_k}) of unimodular polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is ultraflat if it is (ε_{n_k})-ultraflat with a suitable sequence (ε_{n_k}) of positive numbers tending to 0.*

The existence of an ultraflat sequence of unimodular polynomials seemed very unlikely, in view of a 1957 conjecture of P. Erdős (Problem 22 in [Er]) asserting that, for all $P_n \in \mathcal{K}_n$ with $n \geq 1$,

$$\max_{z \in \partial D} |P_n(z)| \geq (1+\varepsilon)\sqrt{n+1}\,, \tag{1.2}$$

where $\varepsilon > 0$ is an absolute constant (independent of n). Yet, refining a method of Körner [Kö], Kahane [Ka] proved that there exists a sequence (P_n) with $P_n \in \mathcal{K}_n$ which is (ε_n)-ultraflat, where $\varepsilon_n = O\left(n^{-1/17}\sqrt{\log n}\right)$. (Kahane's paper contained though a slight error which was corrected in [QS2].) Thus the Erdős conjecture (1.2) was disproved for the classes $\mathcal{K}_n$. For the more restricted class $\mathcal{L}_n$ the analogous Erdős conjecture is unsettled to this date. It is a common belief that the analogous Erdős conjecture for $\mathcal{L}_n$ is true, and consequently there is no ultraflat sequence of polynomials $P_n \in \mathcal{L}_n$. An interesting result related to Kahane's breakthrough is given in [Be]. For an account of some of the work done till the mid 1960's, see Littlewood's book [Li2] and [QS2].

Let (ε_n) be a sequence of positive numbers tending to 0. Let the sequence (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n$ be (ε_n)-ultraflat. We write

$$P_n(e^{it}) = R_n(t)e^{i\alpha_n(t)}\,, \qquad R_n(t) = |P_n(e^{it})|\,, \qquad t \in \mathbb{R}\,. \tag{1.3}$$

It is a simple exercise to show that α_n can be chosen so that it is differentiable on $\mathbb{R}$. This is going to be our understanding throughout the paper.

The structure of ultraflat sequences of unimodular polynomials is studied in [Er1], [Er2], [Er3], and [Er4] where several conjectures of Saffari are proved. Here, based on the results in [Er1], we prove yet another conjecture of Saffari and Queffelec, see (1.30) in [QS2].

For continuous functions f defined on $[0, 2\pi]$, and for $q \in (0, \infty)$, we define

$$\|f\|_q := \left(\int_0^{2\pi} |f(t)|^q \, dt \right)^{1/q} .$$

We also define

$$\|f\|_\infty := \lim_{q \to \infty} \|f\|_q = \max_{t \in [0,2\pi]} |f(t)|\,.$$

Theorem 1.4. *Let $q \in (0, \infty)$. If (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$, and $q \in (0, \infty)$, then for $f_n(t) := \operatorname{Re}(P_n(e^{it}))$ we have*

$$\|f_n\|_q \sim \left(\frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}} \right)^{1/q} n^{1/2}$$

and

$$\|f_n'\|_q \sim \left(\frac{\Gamma\left(\frac{q+1}{2}\right)}{(q+1)\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}} \right)^{1/q} n^{3/2} ,$$

where Γ denotes the usual gamma function, and the $\sim$ symbol means that the ratio of the left and right hand sides converges to 1 as $n \to \infty$.

We remark that trivial modifications of the proof of Theorem 1.4 yield that the statement of the above theorem remains true if the ultraflat sequence (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n$ is replaced by an ultraflat sequence (P_{n_k}) of unimodular polynomials $P_{n_k} \in \mathcal{K}_{n_k}$, $0 < n_1 < n_2 < \dots$.

2. Proof of Theorem 1.4

To prove the theorem we need a few lemmas. The first three are from [Er1].

Lemma 2.1 (Uniform Distribution Theorem for the Angular Speed). *Suppose (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. Then, with the notation (1.3), in the interval $[0, 2\pi]$, the distribution of the normalized angular speed $\alpha_n'(t)/n$ converges to the uniform distribution as $n \to \infty$. More precisely, we have*

$$\operatorname{meas}(\{t \in [0, 2\pi] : 0 \le \alpha_n'(t) \le nx\}) = 2\pi x + \gamma_n(x)$$

for every $x \in [0, 1]$, where $\lim_{n\to\infty} \max_{x\in[0,1]} |\gamma_n(x)| = 0$.

Lemma 2.2 (Negligibility Theorem for Higher Derivatives). *Suppose (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. Then, with the notation (1.3), for every integer $r \ge 2$, we have*

$$\max_{0 \le t \le 2\pi} |\alpha_n^{(r)}(t)| \le \gamma_{n,r} n^r$$

with suitable constants $\gamma_{n,r} > 0$ converging to 0 for every fixed $r = 2, 3, \dots$.

Lemma 2.3. *Let $q > 0$. Suppose (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. Then we have*

$$\frac{1}{2\pi} \int_0^{2\pi} |\alpha_n'(t)|^q \, dt = \frac{n^q}{q+1} + \delta_{n,q} n^q ,$$

and as a limit case,

$$\max_{0 \le t \le 2\pi} |\alpha_n'(t)| = n + \delta_n n .$$

with suitable constants $\delta_{n,q}$ and δ_n converging to 0 as $n \to \infty$ for every fixed q.

Our next lemma is a special case of Lemma 4.2 from [Er1].

Lemma 2.4. *Suppose (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. Using notation (1.3), we have*

$$\max_{0 \le t \le 2\pi} |R_n'(t)| = \delta_n n^{3/2}\,, \qquad m = 1, 2, \dots\,,$$

with suitable constants δ_n converging to 0 as $n \to \infty$.

The next lemma follows from the ultraflatness property (see Definition 1.3) and Lemma 2.4.

Lemma 2.5. *Let $q \in (0, \infty)$. We have*

$$\|f_n\|_q^q = \int_0^{2\pi} |n^{1/2}(1+\delta_n(t))\cos(\alpha_n(t))|^q \, dt$$

and

$$\|f_n'\|_q^q = \int_0^{2\pi} |n^{1/2}(1+\eta_n(t))\sin(\alpha_n(t))\alpha_n'(t) + \eta_n^*(t) n^{3/2}|^q \, dt$$

with some numbers $\delta_n(t)$, $\eta_n(t)$, and $\eta_n^*(t)$ converging to 0 uniformly on $[0, 2\pi]$ as $n \to \infty$.

Finally we need the technical lemma below that follows by a simple calculation.

Lemma 2.6. *Assume that $A, B \in \mathbb{R}$, $q > 0$, and $I \subset [0, 2\pi]$ is an interval. Then*

$$\int_I |\cos(At+B)|^q \, dt = K(q)\mathrm{meas}(I) + \delta(I,q)$$

and

$$\int_I |\sin(At+B)|^q \, dt = K(q)\mathrm{meas}(I) + \delta(I,q)\,,$$

where, by (6.2.1), (6.2.2), and (6.1.8) from [AS] *(see pages 258 and 255), we have*

$$2\pi K(q) := \int_0^{2\pi} |\sin t|^q \, dt = \frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}}$$

and

$$|\delta(I,q)| \le \pi A^{-1}\,.$$

Proof of Theorem 1.4. By Lemma 2.5 it is sufficient to prove that

$$\int_0^{2\pi} |\cos(\alpha_n(t))|^q \, dt \sim 2\pi K(q) := \frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}} \tag{2.1}$$

and

$$\int_0^{2\pi} |\sin(\alpha_n(t)) n^{-1} \alpha_n'(t)|^q \, dt \sim \frac{2\pi K(q)}{q+1}\,. \tag{2.2}$$

First we show (2.1). Let $\varepsilon > 0$ be fixed. Let $K_n := \gamma_{n,2}^{-1/4}$, where $\gamma_{n,2}$ is defined in Lemma 2.2. We divide the interval $[0, 2\pi]$ into subintervals

$$I_m := [a_{m-1}, a_m) := \left[\frac{(m-1)K_n}{n}, \frac{mK_n}{n} \right), \qquad m = 1, 2, \ldots, N-1 := \left\lfloor \frac{2\pi n}{K_n} \right\rfloor ,$$

and

$$I_N := [a_{N-1}, a_N) := \left[\frac{(N-1)K_n}{n}, 2\pi \right) .$$

For the sake of brevity let

$$A_{m-1} := \alpha_n(a_{m-1}) , \qquad m = 1, 2, \ldots, N ,$$

and

$$B_{m-1} := \alpha_n'(a_{m-1}) , \qquad m = 1, 2, \ldots, N .$$

Then by Taylor's Theorem

$$|\alpha_n(t) - (A_{m-1} + B_{m-1}(t - a_{m-1})| \leq \gamma_{n,2} n^2 (K_n/n)^2 \leq \gamma_{n,2}\gamma_{n,2}^{-1/2} \leq \gamma_{n,2}^{1/2}$$

for every $t \in I_m$, where $\lim_{n\to\infty} \gamma_{n,2}^{1/2} = 0$ by Lemma 2.2. Hence the functions

$$G_{n,q}(t) := \begin{cases} |\cos(A_0 + B_0(t-a_0))|^q, & t \in I_1 , \\ |\cos(A_1 + B_1(t-a_0))|^q, & t \in I_2 , \\ \quad \vdots & \quad \vdots \\ |\cos(A_{N-1} + B_{N-1}(t-a_{N-1}))|^q, & t \in I_N , \end{cases}$$

and

$$F_{n,q}(t) := |\cos(\alpha_n(t)|^q$$

satisfy

$$\lim_{n\to\infty} \sup_{t\in[0,2\pi)} |G_{n,q}(t) - F_{n,q}(t)| = 0 . \tag{2.3}$$

Therefore, in order to prove (2.1), it is sufficient to prove that

$$\int_0^{2\pi} G_{n,q}(t)\, dt \sim 2\pi K(q) . \tag{2.4}$$

By using Lemma 2.5, if $|B_{m-1}| \geq n\varepsilon$, then

$$\left| \int_{I_m} G_{n,q}(t)\, dt - K(q)\mathrm{meas}(I_m) \right| \leq \frac{\pi}{n\varepsilon} .$$

Therefore $\lim_{n\to\infty} K_n = \infty$ implies

(2.5)

$$\left| \sum_m \int_{I_m} G_{n,q}(t)\, dt - K(q) \sum_m \mathrm{meas}(I_m) \right| \leq N \frac{\pi}{n\varepsilon} \leq \left(\frac{2\pi n}{K_n} + 1 \right) \frac{\pi}{n\varepsilon}$$
$$\leq \eta_n^*(\varepsilon) ,$$

where the summation is taken over all $m = 1, 2, \dots, N$ for which $|B_{m-1}| \geq n\varepsilon$, and where $(\eta_n^*(\varepsilon))$ is a sequence tending to 0 as $n \to \infty$. Now let

$$E_{n,\varepsilon} := \bigcup_{m:\, |B_{m-1}| \leq n\varepsilon} I_m \,.$$

If $|B_{m-1}| \leq n\varepsilon$, then we obtain by Lemma 2.2 that

$$|\alpha_n'(t)| \leq |B_{m-1}| + \frac{K_n}{n} \max_{t \in I_m} |\alpha_n''(t)| \leq |B_{m-1}| + \frac{\gamma_{n,2}^{-1/4}}{n} \gamma_{n,2} n^2 \leq 2n\varepsilon$$

for every $t \in I_m$ if n is sufficiently large n. So

$$E_{n,\varepsilon} \subset \{t \in [0, 2\pi] : |\alpha_n'(t)| \leq 2n\varepsilon\}$$

for every sufficiently large n. Hence we obtain by Lemma 2.1 that

$$\text{meas}(E_{n,\varepsilon}) \leq 4\pi\varepsilon + \eta_n^{**}(\varepsilon)\,,$$

where $(\eta_n^{**}(\varepsilon))$ is a sequence tending to 0 as $n \to \infty$. Combining this with $0 \leq G_{n,q}(t) \leq 1$, $t \in [0, 2\pi)$, we obtain

$$\left| \sum_m \int_{I_m} G_{n,q}(t)\, dt - K(q) \sum_m \text{meas}(I_m) \right| \leq (4\pi\varepsilon + \eta_n^{**}(\varepsilon))(1 + K(q))\,, \tag{2.6}$$

where n is sufficiently large and the summation is taken over all $m = 1, 2, \dots, N$ for which $|B_{m-1}| < n\varepsilon$. Since $\varepsilon > 0$ is arbitrary, (2.4) follows from (2.5) and (2.6). The proof of (2.1) is now finished.

Now we prove (2.2). Let $\varepsilon > 0$ be fixed. Let the intervals I_m and the numbers A_m and B_m, $m = 1, 2, \dots, N$, as in the proof of (2.1). We define

$$G_{n,q}(t) := \begin{cases} |\sin(A_0 + B_0(t - a_0))|^q, & t \in I_1\,, \\ |\sin(A_1 + B_1(t - a_0))|^q, & t \in I_2\,, \\ \quad \vdots & \quad \vdots \\ |\sin(A_{N-1} + B_{N-1}(t - a_{N-1}))|^q, & t \in I_N \end{cases}$$

and

$$F_{n,q}(t) := |\sin(\alpha_n(t)|^q$$

Similarly to the corresponding argument in the proof of (2.1), we obtain (2.3). Let

$$G_{n,q}^*(t) := \begin{cases} |\sin(A_0 + B_0(t - a_0))|^q |n^{-1}B_0|^q, & t \in I_1\,, \\ |\sin(A_1 + B_1(t - a_0))|^q |n^{-1}B_1|^q, & t \in I_2\,, \\ \quad \vdots & \quad \vdots \\ |\sin(A_{N-1} + B_{N-1}(t - a_{N-1}))|^q |n^{-1}B_{N-1}|^q, & t \in I_N \end{cases}$$

and

$$F^*_{n,q}(t) := |\sin(\alpha_n(t))|^q |n^{-1}\alpha'_n(t)|^q \, .$$

We have

$$G^*_{n,q}(t) = G_{n,q}(t) H_{n,q}(t) \, , \tag{2.7}$$

where

$$H_{n,q}(t) := \begin{cases} |n^{-1}B_0|^q, & t \in I_1 \, , \\ |n^{-1}B_1|^q, & t \in I_2 \, , \\ \quad \vdots & \quad \vdots \\ |n^{-1}B_{N-1}|^q, & t \in I_N \, . \end{cases}$$

It follows from Lemma 2.2 that

$$\begin{aligned} |n^{-1}\alpha'_n(t)| - |n^{-1}B_{m-1}| &= |n^{-1}\alpha'_n(t)| - |n^{-1}\alpha'_n(a_{m-1})| \\ &\le \frac{K_n}{n} \max_{t \in I_m} |n^{-1}\alpha''_n(t)| \le \frac{\gamma_{n,2}^{-1/4}}{n} n^{-1}\gamma_{n,2} n^2 = \gamma_{n,2}^{3/4} \end{aligned}$$

for every $t \in I_m$. Since $\lim_{n\to\infty} \gamma_{n,2}^{3/4} = 0$, we obtain that

$$\lim_{n\to\infty} \sup_{t\in[0,2\pi)} |H_{n,q}(t) - |n^{-1}\alpha'_n(t)|^q| = 0 \, . \tag{2.8}$$

Now observe that

$$\sup_{t\in[0,2\pi)} |\sin(\alpha_n(t)|^q \le 1 \tag{2.9}$$

and by Lemma 2.1 we have

$$\sup_{t\in[0,2\pi)} |n^{-1}\alpha'_n(t)|^q \le 2^q \tag{2.10}$$

for all sufficiently large n. Now (2.3), (2.8), (2.9), (2.10), and (2.7) imply

$$\lim_{n\to\infty} \sup_{t\in[0,2\pi)} |G^*_{n,q}(t) - F^*_{n,q}(t)| = 0 \, .$$

Therefore, in order to prove (2.2), it is sufficient to prove that

$$\int_0^{2\pi} G^*_{n,q}(t)\, dt \sim \frac{2\pi K(q)}{q+1} \, . \tag{2.11}$$

As a special case of (2.10), we have

$$|n^{-1}B_{m-1}|^q \le 2^q \, , \qquad m = 1, 2, \ldots, N \, ,$$

for all sufficiently large n. Hence, if n is sufficiently large and $|B_{m-1}| \ge n\varepsilon$, then, with the help of Lemma 2.6, we obtain that

$$\left| \int_{I_m} G^*_{n,q}(t)\, dt - K(q) \text{meas}(I_m) |n^{-1}B_{m-1}|^q \right| \le 2^q \frac{\pi}{n\varepsilon} \, .$$

Therefore $\lim_{n\to\infty} K_n = \infty$ implies

(2.12)
$$\left|\sum_m \int_{I_m} G_{n,q}(t)\,dt - K(q)\sum_m \text{meas}(I_m)|n^{-1}B_{m-1}|^q\right| \le 2^q N\frac{\pi}{n\varepsilon} \le 2^q\left(\frac{2\pi n}{K_n}+1\right)\frac{\pi}{n\varepsilon} \le \eta^*_{n,q}(\varepsilon)\,,$$

where the summation is taken over all $m = 1, 2, \dots, N$ for which $|B_{m-1}| \ge n\varepsilon$, and where $(\eta^*_{n,q}(\varepsilon))$ is a sequence tending to 0 as $n \to \infty$. Now let

$$E_{n,\varepsilon} := \bigcup_{m:\,|B_{m-1}|\le n\varepsilon} I_m\,.$$

As in the proof of (2.1) we have

$$\text{meas}(E_{n,\varepsilon}) \le 4\pi\varepsilon + \eta^{**}_n(\varepsilon)\,,$$

where $(\eta^{**}_n(\varepsilon))$ is a sequence tending to 0 as $n \to \infty$. Combining this with (2.9) and (2.10), and recalling the definition of $G^*_{n,q}$, we obtain

$$\left|\sum_m \int_{I_m} G^*_{n,q}(t)\,dt - K(q)\sum_m \text{meas}(I_m)|n^{-1}B_{m-1}|^q\right| \le (4\pi\varepsilon + \eta^{**}_n(\varepsilon))2^q(1+K(q))\,, \tag{2.13}$$

where n is sufficiently large and the summation is taken over all $m = 1, 2, \dots, N$ for which $|B_{m-1}| < n\varepsilon$. Since $\varepsilon > 0$ is arbitrary, from (2.12) and (2.13) we obtain that

$$\int_0^{2\pi} G^*_{n,q}(t)\,dt \sim K(q)\int_0^{2\pi} H_{n,q}(t)\,dt \tag{2.14}$$

However (2.8) and Lemma 2.3 imply that

$$\int_0^{2\pi} H_{n,q}(t)\,dt \sim n^{-q}\int_0^{2\pi} |\alpha'_n(t)|^q\,dt \sim \frac{2\pi}{q+1} \tag{2.15}$$

The statement under (2.11) now follows by combining (2.14), and (2.15). As we have remarked before, (2.11) implies (2.2). □

References

[AS] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, Inc., New York, 1972.

[Be] J. Beck, *"Flat" polynomials on the unit circle – note on a problem of Littlewood*, Bull. London Math. Soc. (1991), 269–277.

[Er1] T. Erdélyi, *The phase problem of ultraflat unimodular polynomials: the resolution of the conjecture of Saffari*, Math. Annalen (to appear).

[Er2] T. Erdélyi, *How far is a sequence of ultraflat unimodular polynomials from being conjugate reciprocal*, Michigan Math. J. (to appear).

[Er3] T. Erdélyi, *The resolution of Saffari's Phase Problem*, C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), 803–808.

[Er4] T. Erdélyi, *Proof of Saffari's near-orthogonality conjecture for ultraflat sequences of unimodular polynomials*, C. R. Acad. Sci. Paris Sér. I Math. (to appear).

[Er] P. Erdős, *Some unsolved problems*, Michigan Math. J. **4** (1957), 291–300.

[Ka] J.P. Kahane, *Sur les polynomes a coefficient unimodulaires*, Bull. London Math. Soc. **12** (1980), 321–342.

[Kö] T. Körner, *On a polynomial of J.S. Byrnes*, Bull. London Math. Soc. **12** (1980), 219–224.

[Li1] J.E. Littlewood, *On polynomials* $\sum \pm z^m, \sum \exp(\alpha_m i) z^m, z = e^{i\theta}$, J. London Math. Soc. **41**, 367–376, yr 1966.

[Li2] J.E. Littlewood, *Some Problems in Real and Complex Analysis*, Heath Mathematical Monographs, Lexington, Massachusetts, 1968.

[QS1] H. Queffelec and B. Saffari, *Unimodular polynomials and Bernstein's inequalities*, C. R. Acad. Sci. Paris Sér. I Math. **321** (1995, 3), 313–318.

[QS2] H. Queffelec and B. Saffari, *On Bernstein's inequality and Kahane's ultraflat polynomials*, J. Fourier Anal. Appl. **2** (1996, 6), 519–582.

[Sa] B. Saffari, *The phase behavior of ultraflat unimodular polynomials*, in Probabilistic and Stochastic Methods in Analysis, with Applications (1992), Kluwer Academic Publishers, Printed in the Netherlands, 555–572.