REVERSE MARKOV- AND BERNSTEIN-TYPE INEQUALITIES FOR INCOMPLETE POLYNOMIALS

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January 28, 2019

Dedicated to Professor Richard Varga on the occasion of his 90th birthday

Abstract. Let $\mathcal{P}_k$ denote the set of all algebraic polynomials of degree at most k with real coefficients. Let $\mathcal{P}_{n,k}$ be the set of all algebraic polynomials of degree at most $n+k$ having at least $n+1$ zeros at 0. Let

$$\|f\|_A := \sup_{x\in A} |f(x)|$$

for real-valued functions f defined on a set $A \subset \mathbb{R}$. Let

$$V_a^b(f) := \int_a^b |f'(x)|\,dx$$

denote the total variation of a continuously differentiable function f on an interval $[a,b]$. We prove that there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \frac{n}{k} \le \min_{P\in\mathcal{P}_{n,k}} \frac{\|P'\|_{[0,1]}}{V_0^1(P)} \le \min_{P\in\mathcal{P}_{n,k}} \frac{\|P'\|_{[0,1]}}{|P(1)|} \le c_2 \left(\frac{n}{k}+1\right)$$

for all integers $n \ge 1$ and $k \ge 1$. We also prove that there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \left(\frac{n}{k}\right)^{1/2} \le \min_{P\in\mathcal{P}_{n,k}} \frac{\|P'(x)\sqrt{1-x^2}\|_{[0,1]}}{V_0^1(P)} \le \min_{P\in\mathcal{P}_{n,k}} \frac{\|P'(x)\sqrt{1-x^2}\|_{[0,1]}}{|P(1)|} \le c_2 \left(\frac{n}{k}+1\right)^{1/2}$$

for all integers $n \ge 1$ and $k \ge 1$.

Key words and phrases. reverse Markov- and Bernstein-type inequalities, polynomials with constraints, polynomials with restricted zeros, incomplete polynomials.

2010 Mathematics Subject Classifications. 41A17

1. Introduction and Notation

In April, 2018, A. Eskenazis and P. Ivanisvili [8] asked me if I knew polynomial inequalities of a certain type. The inequalities they were interested in looked to me immediately as reverse (or inverse) Markov- and Bernstein-type inequalities for incomplete polynomials on the interval $[0,1]$, but I have not been aware of any such inequalities in the literature. This short paper is a result of an effort to answer the questions raised by A. Eskenazis and P. Ivanisvili [8]. There is a conjecture on the Hamming cube $\{-1,1\}^n$ about a "reverse Bernstein–Markov inequality", see, for example, inequality (106) on page 37 of [16]. This conjecture is analogous to the L_p version of the Bohr-Favard inequality (see page 55 in [12], for example) stating that

$$\int_{-\pi}^{\pi} |Q'(t)|^p \, dt \geq (ck)^p \int_{-\pi}^{\pi} |Q(t)|^p \, dt$$

for every $Q \in \mathcal{T}_{n,k}$, $0 \leq k \leq n-1$, and $p \geq 1$, where

$$\mathcal{T}_{n,k} := \left\{ Q : Q(t) = \sum_{j=k+1}^{n} (a_j \cos(jt) + b_j \sin(jt)) : \quad a_j, b_j \in \mathbb{R} \right\} .$$

This conjecture is solved in [9] in an important particular case with the help of the main results of this paper. G.G. Lorentz, M. von Golitschek, and Y. Makovoz devotes Chapter 3 of their book [14] to incomplete polynomials. E.B. Saff and R.S. Varga were among the researches having contributed significantly to this topic. See [20] and [21], for instance. See also [1] written by I. Borosh, C.K. Chui, and P.W. Smith. Reverse Markov- and Bernstein type inequalities were first studied by P. Turán [22] and J. Erőd [7a] in 1939 (see also [7b]). The research on Turán and Erőd type reverses of Markov- and Bernstein-type inequalities got a new impulse suddenly in 2006 in large part by the work of Sz. Révész [18], and several results have been published on such inequalities in recent years, see [5], [6], [10], [11], [13], [15] [17], [19], [23], and [24], for example.

Let $\mathcal{P}_k$ denote the set of all algebraic polynomials of degree at most k with real coefficients. Let $\mathcal{P}_{n,k}$ be the set of all algebraic polynomials of degree at most $n+k$ having at least $n+1$ zeros at 0. That is, every $P \in \mathcal{P}_{n,k}$ is of the form

$$P(x) = x^{n+1} R(x) \,, \qquad R \in \mathcal{P}_{k-1} \,.$$

Let

$$\|f\|_A := \sup_{x \in A} |f(x)|$$

for real-valued functions f defined on a set $A \subset \mathbb{R}$. Let

$$V_a^b(f) := \int_a^b |f'(x)| \, dx$$

denote the total variation of a continuously differentiable function f on an interval $[a,b]$.

2. New Results

Theorem 2.1. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1 \frac{n}{k} \le \min_{P \in \mathcal{P}_{n,k}} \frac{\|P'\|_{[0,1]}}{V_0^1(P)} \le \min_{P \in \mathcal{P}_{n,k}} \frac{\|P'\|_{[0,1]}}{|P(1)|} \le c_2 \left(\frac{n}{k} + 1\right)$$

for all integers $n \ge 1$ and $k \ge 1$. Here $c_1 = 1/12$ is a suitable choice.

Theorem 2.2. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1 \left(\frac{n}{k}\right)^{1/2} \le \min_{P \in \mathcal{P}_{n,k}} \frac{\|P'(x)\sqrt{1-x^2}\|_{[0,1]}}{V_0^1(P)} \le \min_{P \in \mathcal{P}_{n,k}} \frac{\|P'(x)\sqrt{1-x^2}\|_{[0,1]}}{|P(1)|} \le c_2 \left(\frac{n}{k} + 1\right)^{1/2}$$

for all integers $n \ge 1$ and $k \ge 1$. Here $c_1 = 1/7$ is a suitable choice.

3. Lemmas

Our first lemma is a simple consequence of the well known Chebyshev's inequality (see p. 235 of [4], for instance) on the growth of polynomials.

Lemma 3.1. *Let $a, b \in \mathbb{R}$ and $a < b$. We have*

$$|Q(x)| \le \left|\frac{4x - 2(a+b)}{b-a}\right|^k \|Q\|_{[a,b]}, \qquad x \in \mathbb{R} \setminus (a,b),$$

for every $Q \in \mathcal{P}_k$, $k \ge 0$.

We will use Lemmas 3.1 to prove Lemma 3.2.

Lemma 3.2. *Let $n \ge 1$ and $k \ge 1$ be integers, and let $S(x) := x^n R(x)$ with $R \in \mathcal{P}_k$. We have*

$$|S(x)| \le x^{n/2} \|S\|_{[0,1]}, \qquad x \in [0, 1 - 10k/n].$$

Lemma 3.3. *Let $n \ge 1$ and $k \ge 1$ be integers, and let $S(x) := x^n Q(x)\sqrt{1-x^2}$ with $Q \in \mathcal{P}_{k-1}$. We have*

$$|S(x)| \le x^{n/2} \|S\|_{[0,1]}, \qquad x \in [0, 1 - 10k/n].$$

To prove the upper bounds in Theorems 2.1 and 2.2 we need the following result proved in [2].

Lemma 3.4. *Let $\nu \ge 0$ and $\kappa \ge 1$ be nonnegative integers. There is an absolute constant $c_3 > 0$ such that*

$$|P'(x)| \le c_3 \left(\frac{(\nu+\kappa)\kappa}{x(1-x)}\right)^{1/2} \|P\|_{[0,1]}, \qquad x \in (0,1),$$

for every polynomial $P \in \mathcal{P}_{\nu+\kappa}$ having at most κ zeros in the open disk with diameter $(0,1)$.

Let $k \geq 1$ be an integer. We define

$$\alpha_j = 1 + \cos\left(\frac{(2j-1)\pi}{4k}\right), \qquad j = 1, 2, \ldots, k.$$

Let $n = 2k + m$, where $m \geq 1$ is an integer. Let $1 > \gamma_1 > \gamma_2 > \cdots > \gamma_k > 0$ be defined by

$$\gamma_j := \frac{\alpha_j - (1 - m/k)}{1 + m/k}, \qquad j = 1, 2, \ldots, k.$$

Let $q_n \in \mathcal{P}_n$ be the unique polynomial of the form

$$q_n(x) = a_{k,n}(x+1)^{n-k} \prod_{j=1}^{k} (x - \rho_j), \qquad a_{k,n} \in \mathbb{R},$$

equioscillating $k+1$ times on $[-1,1]$ between -1 and 1, that is, there are

$$1 = x_0 > x_1 > \cdots > x_k > -1$$

satisfying

$$q_n(x_j) = (-1)^j = (-1)^j \|q_n\|_{[-1,1]}, \qquad j = 0, 1, \ldots, k.$$

To prove the upper bounds in Theorems 2.1 and 2.2 we also need the following lemma stating a key observation from the proof of Lemma 4 in [2].

Lemma 3.5. *With notation introduced above we have $\rho_j \leq \gamma_j$ for each $j = 1, 2, \ldots, k$. As a consequence, there is an absolute constant $c_4 > 0$ such that*

$$\rho_j \leq 1 - \frac{c_4 j^2}{nk}, \qquad j = 1, 2, \ldots, k.$$

For our purpose to prove the upper bounds in Theorems 2.1 and 2.2 the following version of Lemma 3.6 with $n := \nu + \kappa$ and $k := \kappa$ will be convenient for us.

Lemma 3.6. *Let $1 \leq \kappa \leq \nu - 1$ be integers. Let $T := T_{\nu,\kappa}$ be the Chebyshev polynomial for the Müntz space*

$$\operatorname{span}\{x^\nu, x^{\nu+1}, \ldots, x^{\nu+\kappa}\}$$

on $[0,1]$ normalized so that $T(1) = 1$. Denote the zeros of T in $(0,1)$ by

$$\beta_1 > \beta_2 > \cdots > \beta_\kappa.$$

We have

$$\beta_j \leq 1 - \frac{c_4 j^2}{2(\nu+\kappa)\kappa} \leq 1 - \frac{c_4 j^2}{4\nu\kappa}, \qquad j = 1, 2, \ldots, \kappa,$$

where $c_4 > 0$ is the absolute constant appearing in Lemma 3.6.

In fact, what we need in the proofs of the upper bounds in Theorems 2.1 and 2.2 is the following easy consequence of Lemma 3.6.

Lemma 3.7. *Let $\kappa \geq 2$ and $20\kappa \leq \nu$ be integers. Let $T := T_{\nu,\kappa}$ be the Chebyshev polynomial for the Müntz space*

$$\operatorname{span}\{x^\nu, x^{\nu+1}, \dots, x^{\nu+\kappa}\}$$

on $[0,1]$ normalized so that $T(1) = 1$. There is an absolute constant $c_5 > 0$ such that

$$\int_0^1 T(u)^2 \, du \geq \frac{c_5 \kappa}{\nu} \, .$$

4. Proofs of the Lemmas

Proof of Lemma 3.1. Without loss of generality we may assume that $[a,b] = [-1,1]$, the lemma in general follows from this case by a simple linear transformation. Let T_k be the k-th Chebyshev polynomial defined by

$$T_k(\cos t) = \cos(kt) \, , \qquad t \in \mathbb{R} \, .$$

It is well known that

$$T_k(x) = 2^{k-1} \prod_{j=1}^{k} (x - x_j) \, ,$$

where

$$x_j = \cos\left(\frac{(2j-1)\pi}{2k}\right) \, , \qquad j = 1, 2, \dots, k \, ,$$

and hence

$$1 > x_1 > x_2 > \dots > x_k > -1 \, .$$

Using Chebyshev's inequality (see E.2 on page 235 of [4], for instance) we have

$$\begin{aligned} |Q(x)| \leq & |T_k(x)| \dot{\|} Q\|_{[-1,1]} = \left(2^{k-1} \prod_{j=1}^{k} |x - x_j|\right) \|Q\|_{[-1,1]} \\ = & \left(2^{k-1} \prod_{j=1}^{k} |x^2 - x_j^2|^{1/2}\right) \|Q\|_{[-1,1]} \leq |2x|^k \|Q\|_{[-1,1]} \end{aligned}$$

for every $Q \in \mathcal{P}_k$ and $x \in \mathbb{R} \setminus (-1,1)$. $\square$

Proof of Lemma 3.2. If $1 - 10k/n < 0$ there is nothing to prove, so we may assume that $1 - 10k/n \geq 0$. Without loss of generality we may also assume that $\|S\|_{[0,1]} = 1$. Let $\delta := k/n \in (0,1)$. As $S(x) := x^n R(x)$, we have

$$\|R\|_{[1-\delta,1]} \leq (1-\delta)^{-n} \, .$$

Combining this with Lemma 3.1 we obtain that if $x \in [0, 1-\delta]$, then

$$\begin{aligned}|S(x)| \leq x^n |R(x)| \leq\ & x^n \left| \frac{4x - (4-2\delta)}{\delta} \right|^k \|R\|_{[1-\delta,1]} \\ \leq & x^{n/2} \cdot x^{n/2} \left(\frac{4-4x}{\delta} \right)^k (1-\delta)^{-n} = x^{n/2} f(x)\,,\end{aligned}$$

where

$$f(x) := x^{n/2} \left(\frac{4-4x}{\delta} \right)^k (1-\delta)^{-n}\,. \tag{4.1}$$

To finish the proof we need to show that $|f(x)| \leq 1$ for every $x \in [0, 1-10k/n]$. The function is clearly nonnegative on $[0,1]$, and by examining the sign of $f'(x)$ it is easy to see that f is increasing on the interval $[0, n/(n+2k)]$, and hence on $[0, 1-2k/n] \subset [0, n/(n+2k)]$ as well. Using (4.1) to estimate the value of f at $x_0 := 1 - 10k/n \geq 0$, we obtain

$$\begin{aligned}f(x_0) &= \left(1 - \frac{10k}{n}\right)^{n/2} 40^k \left(1 - \frac{k}{n}\right)^{-n} \leq \left(1 - \frac{5k}{n}\right)^{n} 40^k \left(1 - \frac{k}{n}\right)^{-n} \\ &\leq \left(1 - \frac{4k}{n}\right)^{n} 40^k \leq e^{-4k} 40^k = \left(\frac{40}{e^4}\right)^k \leq 1\,,\end{aligned}$$

hence $0 \leq f(x) \leq 1$ for every $x \in [0, 1-10k/n]$, indeed. $\square$

Proof of Lemma 3.3. Applying Lemma 3.2 with S defined by $S(x)^2 = x^{2n} R(x)$, where $R \in \mathcal{P}_{2k}$ is defined by $R(x) = Q(x)^2(1-x^2)$, we obtain the lemma. $\square$

Proof of Lemma 3.7. As in Lemma 3.6, denote the zeros of T in $(0,1)$ by

$$\beta_1 > \beta_2 > \cdots > \beta_\kappa\,.$$

We introduce the points of equioscillation $1 = x_0 > x_1 > \cdots > x_\kappa (> 0)$, that is, $T(x_j) = (-1)^j$ and $\beta_j \in (x_j, x_{j-1})$ for $j = 1, 2, \ldots, \kappa$, where $x_\kappa \geq 1 - 10\kappa/\nu \geq 1/2$ follows from Lemma 3.2 and the assumption $20\kappa \leq \nu$. We define $y_j \in (\beta_{j+1}, x_j)$ by

$$T(y_j) = (-1)^j (1/2)\,, \qquad j = 1, 2, \ldots, \kappa - 1\,.$$

The Mean Value Theorem and Lemma 3.5 imply that there are a $\xi_j \in (y_j, x_j)$ such that

$$\begin{aligned}1/2 =& |T(x_j) - T(y_j)| = (x_j - y_j)|T'(\xi_j)| \leq (x_j - y_j) c_3 \left(\frac{(\nu+\kappa)\kappa}{\xi_j(1-\xi_j)} \right)^{1/2} \\ \leq& c_3 (x_j - y_j) \left(\frac{(\nu+\kappa)\kappa}{(1/2)(1-\beta_j)} \right)^{1/2}, \qquad j = 1, 2, \ldots, \kappa - 1\,,\end{aligned}$$

and hence

$$x_j - y_j \geq c_6 \frac{(1-\beta_j)^{1/2}}{(\nu\kappa)^{1/2}}\,, \qquad j = 1, 2, \dots, \kappa - 1\,,$$

with an absolute constant $c_6 > 0$. Observe that $|T(x)| \geq 1/2$ on each of the intervals $[y_j, x_j]$, $j = 1, 2, \dots, \kappa - 1$, so

$$m(\{x \in [0,1] : |T(x)| \geq 1/2\}) \geq \sum_{j=1}^{\kappa-1} (x_j - y_j) \geq \sum_{j=1}^{\kappa-1} c_6 \frac{(1-\beta_j)^{1/2}}{(\nu\kappa)^{1/2}}\,,$$

where $m(A)$ denotes the Lebesgue measure of a set $A \subset \mathbb{R}$. Combining this with Lemma 3.6 and recalling the assumption $\kappa \geq 2$, we obtain

$$m(\{x \in [0,1] : |T(x)| \geq 1/2\}) \geq \sum_{j=1}^{\kappa-1} c_6 \frac{(c_4 j^2/(4\nu\kappa))^{1/2}}{(\nu\kappa)^{1/2}} \geq c_7 \sum_{j=1}^{\kappa-1} \frac{j}{\nu\kappa} \geq \frac{c_5\kappa}{\nu}$$

with some absolute constants $c_7 > 0$ and $c_5 > 0$, and the lemma follows. $\square$

5. Proofs of Theorems 2.1 and 2.2.

Proof of the lower bound in Theorem 2.1. Let $P \in \mathcal{P}_{n,k}$ be of the form

$$P(x) = x^{n+1} R(x)\,, \qquad R \in \mathcal{P}_{k-1}\,.$$

We define

$$S(x) := P'(x) = x^n Q(x)\,,$$

where $Q \in \mathcal{P}_{k-1}$ is defined by

$$Q(x) = (n+1)R(x) + xR'(x)\,.$$

We also define $y := \max\{1 - 10k/n, 0\} \geq 0$. We have

$$V_0^1(P) = \int_0^1 |P'(x)|\,dx = \int_0^y |P'(x)|\,dx + \int_y^1 |P'(x)|\,dx\,. \tag{5.1}$$

The first term at the right-hand side of (5.1) can be estimated by Lemma 3.2 as

$$\int_0^y |P'(x)|\,dx = \int_0^y |S(x)|\,dx \leq \int_0^y (x^{n/2}\|S\|_{[0,1]})\,dx \leq \frac{2}{n}\,\|S\|_{0,1]}\,, \tag{5.2}$$

while the second term at the right-hand side of (5.1) can be estimated as

$$\int_y^1 |P'(x)|\,dx = \int_y^1 |S(x)|\,dx \leq (1-y)\|S\|_{[0,1]} \leq \frac{10k}{n}\,\|S\|_{[0,1]}\,. \tag{5.3}$$

Combining (5.1), (5.2), and (5.3) we obtain

$$V_0^1(P) = \int_0^1 |P'(x)|\,dx \le \frac{2}{n}\|S\|_{[0,1]} + \frac{10k}{n}\,\|S\|_{[0,1]} \le \frac{10k+2}{n}\,\|P'\|_{[0,1]}\,,$$

and the lower bound of Theorem 2.1 follows. $\square$

Proof of the lower bound in Theorem 2.2. Let $P \in \mathcal{P}_{n,k}$ be of the form

$$P(x) = x^{n+1}R(x)\,, \qquad R \in \mathcal{P}_{k-1}\,.$$

We define

$$S(x) := P'(x)\sqrt{1-x^2} = x^n Q(x)\sqrt{1-x^2}\,,$$

where $Q \in \mathcal{P}_{k-1}$ is defined by

$$Q(x) = (n+1)R(x) + xR'(x)\,.$$

We also define $y := \max\{1 - 10k/n, 0\} \ge 0$, as in the proof of Theorem 2.1. We have

$$V_0^1(P) = \int_0^1 |P'(x)|\,dx = \int_0^y |P'(x)|\,dx + \int_y^1 |P'(x)|\,dx\,. \tag{5.4}$$

The first term at the right-hand side of (5.4) can be estimated by Lemma 3.3 as

$$\begin{aligned}\int_0^y |P'(x)|\,dx &= \int_0^y |S(x)|(1-x^2)^{-1/2}\,dx \\ &\le \int_0^y (x^{n/2}\|S\|_{[0,1]})(1-y^2)^{-1/2}\,dx \\ &\le \frac{2}{n}\,\|S\|_{[0,1]}(1-y)^{-1/2} \le \frac{2}{n}\,\|S\|_{[0,1]}(10k/n)^{-1/2} \\ &\le (kn)^{-1/2}\|S\|_{0,1]}\,,\end{aligned} \tag{5.5}$$

while the second term at the right-hand side of (5.4) can be estimated as

$$\begin{aligned}\int_y^1 |P'(x)|\,dx &= \int_y^1 |S(x)|(1-x^2)^{-1/2}\,dxi \le \|S\|_{[0,1]}\int_y^1 (1-x^2)^{-1/2}\,dx \\ &\le \|S\|_{[0,1]}(\arccos y - \arccos 1) = \|S\|_{[0,1]}\arccos y \\ &\le \|S\|_{[0,1]}(\pi(1-y))^{1/2} \le \left(\frac{10\pi k}{n}\right)^{1/2}\|S\|_{[0,1]}\,,\end{aligned} \tag{5.6}$$

where we used the inequality $\arccos y \le (\pi/2)\left(1-y^2\right)^{1/2} \le (\pi(1-y))^{1/2}$ which follows from the inequality $\sin\tau \ge (2/\pi)\tau$ with $\tau = \arccos y$. (Note also that (5.5) and (5.6) show

that in the sum on the right-hand side of (5.4) the second term is the dominating one.) Combining (5.4), (5.5), and (5.6) we obtain

$$V_0^1(P) = \int_0^1 |P'(x)|\,dx \le (kn)^{-1/2}\|S\|_{[0,1]} + \left(\frac{10\pi k}{n}\right)^{1/2} \|S\|_{[0,1]} \le 7(k/n)^{1/2}\|S\|_{[0,1]}\,,$$

and the lower bound of Theorem 2.2 follows. □

Proof of the upper bound in Theorem 2.1. If $1 \le k \le 5$, then the upper bound of the theorem follows by considering $P \in \mathcal{P}_{n,k}$ defined by $P(x) = x^{n+1}$. So we can assume that $k \ge 6$. Observe that for a fixed nonnegative integer n the function f_n defined by

$$f_n(k) := \min_{P \in \mathcal{P}_{n,k}} \frac{\|P'\|_{[0,1]}}{|P(1)|}$$

is a decreasing on the set of natural numbers k. Note also that in the case $n \le 20k-20$ the upper bound of the theorem follows simply from the case $n = 20k-20$. So without loss of generality, in addition to $k \ge 6$ we may also assume that $n = 2\nu \ge 2$, $k = 2\kappa + 2 \ge 6$ are even, and $20\kappa \le \nu$. Let $T := T_{\nu,\kappa}$ be the Chebyshev polynomial for the Müntz space

$$\operatorname{span}\{x^\nu, x^{\nu+1}, \dots, x^{\nu+\kappa}\}$$

on $[0,1]$ normalized so that $T(1) = 1$. We define $P \in \mathcal{P}_{n,k-1} \subset \mathcal{P}_{n,k}$ of the form

$$P(x) = x^{n+1}Q(x)\,, \qquad Q \in \mathcal{P}_{k-2}\,,$$

by

$$P(x) = \int_0^x T(u)^2\,du\,.$$

Lemma 3.7 gives that

$$P(1) = |P(1)| \ge \frac{c_5\kappa}{\nu} = \frac{c_5(k-2)}{n}\,. \tag{5.7}$$

Observe that

$$|P'(y)| = T(y)^2 \le 1\,, \qquad y \in [0,1]\,,$$

and hence

$$\|P'\|_{[0,1]} \le 1\,. \tag{5.8}$$

Combining (5.7) and (5.8) we have

$$\frac{\|P'\|_{[0,1]}}{|P(1)|} \le \frac{1}{c_5(k-2)/n} \le \frac{c_8 n}{k}$$

with an absolute constant $c_8 > 0$, and the upper bound of Theorem 2.1 follows. □

Proof of the upper bound in Theorem 2.2. If $1 \le k \le 5$, then the upper bound of the theorem follows by considering $P \in \mathcal{P}_{n,k}$ defined by $P(x) = x^{n+1}$. So we can assume that $k \ge 6$. Observe that for a fixed nonnegative integer n the function f_n defined by

$$f_n(k) := \min_{P \in \mathcal{P}_{n,k}} \frac{\|P'(x)\sqrt{1-x^2}\|_{[0,1]}}{|P(1)|}$$

is decreasing on the set of natural numbers k. Note also that in the case $20k - 20 \ge n$ the upper bound of the theorem follows simply from the case $n = 20k - 20$. So without loss of generality, in addition to $k \ge 6$ we may also assume that $n = 2\nu \ge 2$, $k = 2\kappa + 2 \ge 6$ are even, and $20\kappa \le \nu$. We define $P \in \mathcal{P}_{n,k-1} \subset \mathcal{P}_{n,k}$ of the form

$$P(x) = x^{n+1} Q(x)\,, \qquad Q \in \mathcal{P}_{k-2}\,,$$

by

$$P(x) = \int_0^x T(u)^2\, du\,,$$

as in the proof of the upper bound in Theorem 2.1. Let y be a number such that

$$|P'(y)\sqrt{1-y^2}| = \|P'(x)\sqrt{1-x^2}\|_{[0,1]}\,.$$

Lemma 3.3 gives that $y \ge 1 - 10k/n$, and hence

$$\|P'(x)\sqrt{1-x^2}\|_{[0,1]} = |P'(y)\sqrt{1-y^2}| = T(y)^2\sqrt{1-y^2} \le \sqrt{1-y^2} \le \left(\frac{20k}{n}\right)^{1/2}. \tag{5.9}$$

Combining (5.7) and (5.9) we have

$$\frac{\|P'(x)\sqrt{1-x^2}\|_{[0,1]}}{|P(1)|} \le \frac{(20k/n)^{1/2}}{c_5(k-2)/n} \le c_9 \left(\frac{n}{k}\right)^{1/2}$$

with an absolute constant $c_9 > 0$, and the upper bound of Theorem 2.2 follows. □

6. Acknowledgement

The author thanks Szilárd Révész for checking the details of the proofs in this paper and for his suggestions to make the paper more readable.

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