MARKOV AND BERNSTEIN TYPE INEQUALITIES ON SUBSETS OF $[-1,1]$ AND $[-\pi,\pi]$

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Abstract. We extend Markov's, Bernsteins's, and Videnskii's inequalities to arbitrary subsets of $[-1,1]$ and $[-\pi,\pi]$, respectively.

The primary purpose of this note is to extend Markov's and Bernsteins's inequalities to arbitrary subsets of $[-1,1]$ and $[-\pi,\pi]$, respectively.

We denote by $\mathcal{P}_n$ the set of all real algebraic polynomials of degree at most n and let $m(\cdot)$ denote the Lebesque measure of a subset of $\mathbb{R}$. We were led to the results of this paper by the following problem. Can one give polynomials $p_n \in \mathcal{P}_n$ and numbers $a_n \in (0,1),\ n = 1,2,\cdots$, such that

(i) $\quad m(\{x \in [0,1] : |p_n(x)| \leq 1\}) \geq 1 - a_n,$

(ii) $\quad \max\limits_{0 \leq x \leq a_n} |p_n(x)| \leq 1$

and

(iii) $\quad \lim\limits_{n \to \infty} n^{-2} |p_n'(0)| = \infty$

are satisfied? This question was asked by Vilmos Totik, and a positive answer would have been used in proving a conjecture in the theory of orthogonal polynomials. However, Theorem 2 of this note shows that the answer to the above question is negative, in fact, it gives slightly more. In addition, our Theorem 1 answers the corresponding question for trigonometric polynomials. Though our results cannot be used for Totik's original purpose, our proofs illustrate well, how Remez-type inequalities can be used in proving various other polynomial inequalities.

In this note we prove the following pair of theorems.

Theorem 1. *Let $0 < a \leq 2\pi,\ 0 < L \leq 1$, let A be a closed subset of $[0,2\pi]$ with Lebesque measure $m(A) \geq 2\pi - a$. There is an absolute constant $c_1 > 0$ such that*

$$\max_{t \in I} |p'(t)| \leq c_1 L^{-1}(n + n^2 a) \max_{t \in A} |p(t)| \tag{1}$$

for every real trigonometric polynomial p of degree at most n and for every subinterval I of A with length at least La.

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Theorem 2. *Let $0 < a \le 1,\ 0 < M \le 1$, let A be a closed subset of $[0,1]$ with Lebesque measure $m(A) \ge 1-a$. There is an absolute constant $c_2 > 0$ such that*

$$\max_{x \in I} |p'(x)| \le c_2 M^{-1} n^2 \max_{x \in A} |p(x)| \tag{2}$$

for every real algebraic polynomial p of degree at most n and for every subinterval I of A with length at least Ma.

Up to the constant c_1, Theorem 1 is an extension of both Bernstein's [5, pp. 39-41] and Videnskii's [6] inequalities, while up to the constant c_2, Theorem 2 contains Markov's inequality [5, pp. 39-41] as a special case.

The key to the proof of Theorem 1 is a Remez-type inequality [2] proved recently for trigonometric polynomials, while the proof of Theorem 2 relies on Theorem 1.

Proof of Theorem 1. Denote by $\mathcal{T}_n$ the set of all trigonometric polynomials of degree at most n with real coefficients. If $\pi/2 \le a \le 2\pi$, then the theorem follows from an extension [1, Theorem 5] of an inequality of Videnskii [6]. Therefore, in the sequel we assume that $0 < a < \pi/2$. Let I be a subinterval of A such that $m(I) \ge La$ and $\pi \in I$. It is sufficient to prove that there is an absolute constant $c_1 > 0$ such that

$$|p'(\pi)| \le c_1 L^{-1}(n + n^2 a) \max_{t \in A} |p(t)| \tag{3}$$

for every $p \in \mathcal{T}_n$. Let T_n be the Chebyshev polynomial of degree n given by

$$T_n(x) = \cos(n \arccos x), \qquad -1 \le x \le 1, \tag{4}$$

and let

$$Q_{n,La}(t) := T_{2n}(\sin(t/2)(\cos(La/4))^{-1})(T_{2n}((\cos(La/4))^{-1}))^{-1}. \tag{5}$$

A simple calculation shows that

$$Q_{n,La}(\pi) = 1, \quad Q'_{n,La}(\pi) = 0, \quad \max_{t \in \mathbb{R}} |Q_{n,La}(t)| = 1, \tag{6}$$

and there is an absolute constant $c_3 > 0$ such that

$$|Q_{n,La}(t)| \le \exp(-c_3 n La), \qquad t \in [0, \pi - La/2] \cup [\pi + La/2, 2\pi]. \tag{7}$$

Let $p \in \mathcal{T}_n$ be such that

$$\max_{t \in A} |p(t)| = 1. \tag{8}$$

The Remez-type inequality for trigonometric polynomials [2, Theorem 2], $m(A) \ge 2\pi - a,\ 0 < a \le \pi/2$, and (8) yield that there is an absolute constant $c_4 > 0$ such that

$$\max |p(t)| \le \exp(c_4 na). \tag{9}$$

Denote the endpoints of the interval I by $\alpha < \beta$. Since $\beta - \alpha = m(I) \geq La$ and $\pi \in I$, we have either $\alpha \leq \pi - La/2$ or $\beta \geq \pi + La/2$. We may assume that

$$\beta \geq \pi + La/2, \tag{10}$$

otherwise we consider the trigonometric polynomial $\tilde{p} \in \mathcal{T}_n$ defined by $\tilde{p}(t) := p(\pi - t)$. Now let

$$m := [c_4 c_3^{-1} L^{-1} n] + 1 \quad \text{and} \quad Q := Q_{m,La}. \tag{11}$$

Observe that (6) - (11) imply

$$|(pQ)(t)| \leq 1, \qquad t \in E, \tag{12}$$

where

$$E := [0, \pi - La/2] \cup [\pi, 2\pi]. \tag{13}$$

Note that E is an interval of the period with length $2\pi - La/2$, and $\pi \in E$. Therefore an extension [1, Theorem 5] of an inequality of Videnskii [6], $0 < L \leq 1$ and (8) yield that there are absolute constants $c_5 > 0$ and $c_1 > 0$ such that

$$\begin{aligned} |(pQ)'(\pi)| &\leq ((n+m) + c_5 (n+m)^2 La/2) \\ &\leq c_1 L^{-1} (n + n^2 a) \max_{t \in A} |p(t)|. \end{aligned} \tag{14}$$

Recalling (6), we have

$$p'(\pi) = (pQ)'(\pi), \tag{15}$$

which, together with (14) gives the theorem. ■

Proof of Theorem 2. If $1/4 \leq a \leq 1$, then the theorem follows from the Markov inequality [5, pp. 39-41]. Therefore, in what follows we may assume that $0 < a \leq 1/4$. Without loss of generality we may assume that $I = [0, b]$, where $Ma \leq b \leq 1$, the general case can be deduced from this easily by a linear transformation. Let $p \in \mathcal{P}_n$,

$$y(t) := 1/2 + (1/2 + a)\cos t, \tag{16}$$

$$\tilde{p}(t) := p(y(t)) \in \mathcal{T}_n \tag{17}$$

$$\tilde{A} := \{t \in [0, 2\pi] : \quad y(t) \in A\}, \tag{18}$$

$$\tilde{I} := \{t \in [0, \pi] : \ y(t) \in I\} \tag{19}$$

and

$$\tilde{a} := 2\pi - m(\tilde{A}), \ \text{i.e.} \ m(\tilde{A}) = 2\pi - \tilde{a}. \tag{20}$$

It is easy to see that $0 < a \leq 1/4$, $A \subset [0, 1]$, $m(A) \geq 1 - a$, $m(I) \geq Ma$, (16), (18), (19), and (20) imply that

$$\tilde{a} \leq c \sqrt{a} \tag{21}$$

and

$$m(\tilde{I}) \ge c_7 M\sqrt{a} \ge c_7 c_6^{-1} M\tilde{a} \tag{22}$$

with suitable absolute constants $c_6 > 0$ and $c_7 > 0$. If $L := c_7 c_6^{-1} M \le 1$ and $a \ge n^{-2}$, then Theorem 1, (20), (21) and (22) yield

$$\begin{aligned} \max_{t\in\tilde{I}} |\tilde{p}'(t)| &\le c_1 c_7^{-1} c_6 M^{-1}(n + n^2\tilde{a}) \max_{t\in\tilde{A}} |\tilde{p}(t)| \\ &\le c_8 M^{-1} n^2 \sqrt{a} \max_{x\in A} |p(x)| \end{aligned} \tag{23}$$

with a suitable absolute constant $c_8 > 0$. Also, (16) - (19) and $I \subset [0,1]$ imply that

$$|\tilde{p}'(t)| = |p'(y(t))y'(t)| = |p'(y(t))|(1/2 + a)\sin t \ge c_9 |p'(y(t))|\sqrt{a}$$

for every $t \in \tilde{I}$ with a suitable absolute constant $c_9 > 0$. Since every $x \in I$ is of the form $x = y(t)$ with some $t \in \tilde{I}$, (23) and (24) imply that

$$\max_{x\in I} |p'(x)| \le c_8 c_9^{-1} M^{-1} n^2 \max_{x\in A} |p(x)|, \tag{25}$$

whenever $c_7 c_6^{-1} M \le 1$ and $a \ge n^{-2}$. If $c_7 c_6^{-1} M \ge 1$, i.e. $M \ge c_6 c_7^{-1}$, and $a \ge n^{-2}$, then I can be divided into subintervals of length $k^{-1}m(I)$, where $k := [c_6 c_7^{-1}] + 1$, and the already proved part gives the theorem. If $0 < a < n^{-2}$, $A \subset [0,1]$ and $m(A) \ge 1 - a$, then the Remez inequality [4, p. 119-121] or [3] yields that

$$\max_{0\le x\le 1} |p(x)| \le c_{10} \max_{x\in A} |p(x)| \tag{26}$$

for every $p \in \mathcal{P}_n$, where $c_{10} > 0$ is a suitable absolute constant. Combining this with the Markov inequality [5, p. 39-41], we obtain

$$\begin{aligned} \max_{x\in I} |p'(x)| &\le \max_{0\le x\le 1} |p'(x)| \le 2n^2 \max_{0\le x\le 1} |p(x)| \\ &\le 2c_{10} n^2 \max_{x\in A} |p(x)|, \end{aligned} \tag{27}$$

and the theorem is completely proved. ■

It may be interesting to compare Theorem 2 with the following

Example 3. Let $0 < a \le 1/2$, $A = [0, 1-a] \cup \{1\}$ and

$$P_n(x) = (x-1)T_n(2(1-a)^{-1}x - 1), \qquad n = 1, 2, \cdots,$$

where T_n is the Chebyshev polynomial of degree n defined by $T_n(x) = \cos(n \arccos x)$, $-1 \le x \le 1$. Then

$$\begin{aligned} \max_{x\in A} |P_n'(x)| \ge |P_n'(1)| &= T_n(2(1-a)^{-1} - 1) \ge T_n(1+2a) \\ &\ge 2^{-1}(1+2\sqrt{a})^n \ge 2^{-1}(1+2\sqrt{a})^n \max_{x\in A} |P_n(x)|. \end{aligned}$$

A similar example can be given in the trigonometric case.

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