

# THE NORM OF THE POLYNOMIAL TRUNCATION OPERATOR ON THE UNIT DISK AND ON $[-1, 1]$

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ABSTRACT. Let  $D$  and  $\partial D$  denote the open unit disk and the unit circle of the complex plane, respectively. We denote the set of all polynomials of degree at most  $n$  with real coefficients by  $\mathcal{P}_n$ . We denote the set of all polynomials of degree at most  $n$  with complex coefficients by  $\mathcal{P}_n^c$ . We define the truncation operator  $S_n$  for polynomials  $P_n \in \mathcal{P}_n^c$  of the form

$$P_n(z) := \sum_{j=0}^n a_j z^j, \quad a_j \in \mathbb{C},$$

by

$$(1.1) \quad S_n(P_n)(z) := \sum_{j=0}^n \tilde{a}_j z^j, \quad \tilde{a}_j := (a_j / |a_j|) \min\{|a_j|, 1\}$$

(here  $0/0$  is interpreted as 1). We define the norms of the truncation operators by

$$\|S_n\|_{\infty, \partial D}^{real} := \sup_{P_n \in \mathcal{P}_n} \frac{\max_{z \in \partial D} |S_n(P_n)(z)|}{\max_{z \in \partial D} |P_n(z)|}$$

and

$$\|S_n\|_{\infty, \partial D}^{comp} := \sup_{P_n \in \mathcal{P}_n^c} \frac{\max_{z \in \partial D} |S_n(P_n)(z)|}{\max_{z \in \partial D} |P_n(z)|}.$$

Our main theorem in this paper establishes the right order of magnitude of the norms of the operators  $S_n$ . This settles a question asked by S. Kwapien.

**Theorem.** *With the notation introduced above there is an absolute constant  $c_1 > 0$  such that*

$$c_1 \sqrt{2n+1} \leq \|S_n\|_{\infty, \partial D}^{real} \leq \|S_n\|_{\infty, \partial D}^{comp} \leq \sqrt{2n+1}.$$

Moreover, an analogous result in  $L_p(\partial D)$  for  $p \in [2, \infty]$  is also established and the case when the unit circle  $\partial D$  is replaced by the interval  $[-1, 1]$  is also studied.

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## 1. NEW RESULT

Let  $D$  and  $\partial D$  denote the open unit disk and the unit circle of the complex plane, respectively. We denote the set of all polynomials of degree at most  $n$  with real coefficients by  $\mathcal{P}_n$ . We denote the set of all polynomials of degree at most  $n$  with complex coefficients by  $\mathcal{P}_n^c$ . We define the truncation operator  $S_n$  for polynomials  $P_n \in \mathcal{P}_n^c$  of the form

$$P_n(z) := \sum_{j=0}^n a_j z^j, \quad a_j \in \mathbb{C},$$

by

$$(1.1) \quad S_n(P_n)(z) := \sum_{j=0}^n \tilde{a}_j z^j, \quad \tilde{a}_j := (a_j/|a_j|) \min\{|a_j|, 1\}$$

(here  $0/0$  is interpreted as 1). In other words, we take the coefficients  $a_j \in \mathbb{C}$  of a polynomial  $P_n$  of degree at most  $n$ , and we truncate them. That is, we leave a coefficient  $a_j$  unchanged if  $|a_j| < 1$ , while we replace it by  $a_j/|a_j|$  if  $|a_j| \geq 1$ . We form the new polynomial with the new coefficients  $\tilde{a}_j$  defined by (1.1), and we denote this new polynomial by  $S_n(P_n)$ . We define the norms of the truncation operators by

$$\|S_n\|_{\infty, \partial D}^{real} := \sup_{P_n \in \mathcal{P}_n} \frac{\max_{z \in \partial D} |S_n(P_n)(z)|}{\max_{z \in \partial D} |P_n(z)|}$$

and

$$\|S_n\|_{\infty, \partial D}^{comp} := \sup_{P_n \in \mathcal{P}_n^c} \frac{\max_{z \in \partial D} |S_n(P_n)(z)|}{\max_{z \in \partial D} |P_n(z)|}.$$

Our main theorem in this paper establishes the right order of magnitude of the norms of the operators  $S_n$ . This settles a question asked by S. Kwapien.

**Theorem 1.1.** *With the notation introduced above there is an absolute constant  $c_1 > 0$  such that*

$$c_1 \sqrt{2n+1} \leq \|S_n\|_{\infty, \partial D}^{real} \leq \|S_n\|_{\infty, \partial D}^{comp} \leq \sqrt{2n+1}.$$

In fact we are able to establish an  $L_p(\partial D)$  analogue of this as follows. For  $p \in (0, \infty)$ , let

$$\|S_n\|_{p, \partial D}^{real} := \sup_{P_n \in \mathcal{P}_n} \frac{\|S_n(P_n)\|_{L_p(\partial D)}}{\|P_n\|_{L_p(\partial D)}}$$

and

$$\|S_n\|_{p, \partial D}^{comp} := \sup_{P_n \in \mathcal{P}_n^c} \frac{\|S_n(P_n)\|_{L_p(\partial D)}}{\|P_n\|_{L_p(\partial D)}}.$$

**Theorem 1.2.** *With the notation introduced above there is an absolute constant  $c_1 > 0$  such that*

$$c_1(2n+1)^{1/2-1/p} \leq \|S_n\|_{p,\partial D}^{real} \leq \|S_n\|_{p,\partial D}^{comp} \leq (2n+1)^{1/2-1/p}$$

for every  $p \in [2, \infty)$ .

Note that it remains open what is the right order of magnitude of  $\|S_n\|_{p,\partial D}^{real}$  and  $\|S_n\|_{p,\partial D}^{comp}$ , respectively when  $0 < p < 2$ . In particular it would be interesting to see if  $\|S_n\|_{p,\partial D}^{comp} \leq c$  is possible for any  $0 < p < 2$  with an absolute constant  $c$ . We record the following observation, due to S. Kwapien, in this direction.

**Theorem 1.3.** *There is an absolute constant  $c > 0$  such that*

$$\|S_n\|_{1,\partial D}^{real} \geq c\sqrt{\log n}.$$

If the unit circle  $\partial D$  is replaced by the interval  $[-1, 1]$ , we get a completely different order of magnitude of the polynomial truncation projector. In this case the norms of the truncation operators  $S_n$  are defined in the usual way. That is, let

$$\|S_n\|_{\infty,[-1,1]}^{real} := \sup_{P_n \in \mathcal{P}_n} \frac{\max_{x \in [-1,1]} |S_n(P_n)(x)|}{\max_{x \in [-1,1]} |P_n(x)|}$$

and

$$\|S_n\|_{\infty,[-1,1]}^{comp} := \sup_{P_n \in \mathcal{P}_n^c} \frac{\max_{x \in [-1,1]} |S_n(P_n)(x)|}{\max_{x \in [-1,1]} |P_n(x)|}.$$

**Theorem 1.4.** *With the notation introduced above we have*

$$2^{n/2-1} \leq \|S_n\|_{\infty,[-1,1]}^{real} \leq \|S_n\|_{\infty,[-1,1]}^{comp} \leq \sqrt{2n+1} \cdot 8^{n/2}.$$

## 2. LEMMAS

To prove the lower bound of the theorem we need two lemmas. Our first one is from [LSV].

**Lemma 2.1 (Lovász, Spencer, Vesztergombi).** *Let  $a_{j,k}, j = 1, 2, \dots, n_1, k = 1, 2, \dots, n_2$  be such that  $|a_{j,k}| \leq 1$ . Let also  $p_1, p_2, \dots, p_{n_2} \in [0, 1]$ . Then there are choices*

$$\varepsilon_k \in \{-p_k, 1 - p_k\}, \quad k = 1, 2, \dots, n_2,$$

such that for all  $j$ ,

$$\left| \sum_{k=1}^{n_2} \varepsilon_k a_{j,k} \right| \leq C\sqrt{n_1}$$

with an absolute constant  $C$ .

Our second lemma is a direct consequence of the well-known Bernstein inequality (see Theorem 1.1 on page 97 of [DL]) and the Mean Value Theorem.

**Lemma 2.2.** *Suppose  $Q_n$  is a polynomial of degree  $n$  (with complex coefficients),*

$$\theta_n := \exp\left(\frac{2\pi}{14n}\right),$$

$$z_j := \exp(ij\theta_n), \quad j = 1, 2, \dots, 14n,$$

and

$$|Q_n(z_j)| \leq M, \quad j = 1, 2, \dots, 14n.$$

Then

$$\max_{z \in \partial D} |Q_n(z)| \leq 2M.$$

The inequalities below (see Theorem 2.6 on page 102 of [DL]) will be needed to prove the upper bound of the Theorem.

**Lemma 2.3 (Nikolskii Inequality).** *Let  $0 < q \leq p \leq \infty$ . If  $P_n$  is a polynomial of degree at most  $n$  with complex coefficients then*

$$\|P_n\|_{L_p(\partial D)} \leq \left(\frac{2nr+1}{2\pi}\right)^{1/q-1/p} \|P_n\|_{L_q(\partial D)},$$

where  $r = r(q)$  is the smallest integer not less than  $q/4$ .

The next lemma may be found in [Ri].

**Lemma 2.4 (Erdős).** *Suppose that  $z_0 \in \mathbb{C}$  and  $|z_0| \geq 1$ . Then*

$$|P_n(z_0)| \leq |T_{2n}(z_0)|^{1/2} \cdot \max_{x \in [-1, 1]} |P_n(x)|, \quad P_n \in \mathcal{P}_n^c,$$

where  $T_{2n} \in \mathcal{P}_{2n}$  defined by

$$T_n(x) := \cos(n \arccos x), \quad x \in [-1, 1],$$

is the Chebyshev polynomial of degree  $2n$ . As a consequence, writing

$$T_{2n}(z) = 2^{2n-1} \prod_{j=1}^n (z^2 - x_j^2), \quad x_j \in (0, 1),$$

we have

$$\max_{z \in \partial D} |P_n(z)| \leq 8^{n/2} \max_{x \in [-1, 1]} |P_n(x)|.$$

## 3. PROOF OF THE THEOREM

*Proof of Theorem 1.1.* We apply Lemma 2.1 with  $n_1 = 14n, n_2 = n$ ,

$$\theta_n := \exp(2\pi/(14n)), \quad a_{j,k} := \exp(ijk\theta_n),$$

and  $p_1 = p_2 = \dots = p_n = 1/3$ , and with the choices

$$\varepsilon_k \in \left\{ -\frac{1}{3}, \frac{2}{3} \right\}, \quad k = 1, 2, \dots, n,$$

coming from Lemma 2.1, we define

$$Q_n(z) = 3 \sum_{j=1}^n \varepsilon_j z^j.$$

Then  $Q_n$  is a polynomial of degree  $n$  with each coefficient in  $\{-1, 2\}$ , and with the notation

$$z_j := \exp(ij\theta_n), \quad j = 1, 2, \dots, 14n,$$

we have

$$|Q_n(z_j)| \leq 3C\sqrt{14n}, \quad j = 1, 2, \dots, 14n.$$

Hence Lemma 2.2 yields

$$(3.1) \quad \max_{z \in \partial D} |Q_n(z)| \leq 24C\sqrt{n}$$

In particular, if we denote by  $m$  the number of indices  $k$  for which  $\varepsilon_k = 2/3$ , then

$$|3m - n| = |2m - (n - m)| = |Q_n(1)| \leq 24C\sqrt{n},$$

hence

$$(3.2) \quad |S_n(Q_n)(1)| = |m - (n - m)| = |2m - n| \geq \frac{n}{3} - 32C\sqrt{n}.$$

Now (3.1) and (3.2) give the lower bound of the theorem.

To see the upper bound of the theorem, observe that Lemma 2.3 implies

$$\begin{aligned} \max_{z \in \partial D} |S_n(P_n)(z)| &\leq \frac{\sqrt{2n+1}}{\sqrt{2\pi}} \|S_n(P_n)\|_{L_2(\partial D)} \leq \frac{\sqrt{2n+1}}{\sqrt{2\pi}} \|P_n\|_{L_2(\partial D)} \\ &\leq \sqrt{2n+1} \max_{z \in \partial D} |P_n(z)| \end{aligned}$$

for all polynomials  $P_n$  of degree at most  $n$  with complex coefficients. This proves the upper bound of the theorem.  $\square$

*Proof of Theorem 1.2.* Let  $p \in [2, \infty)$ . Using (3.2) and the Nikolskii-type inequality of Lemma 3.4, we obtain that

$$(3.3) \quad \|S_n(Q_n)\|_{L_p(\partial D)} \geq c_1 n^{1-1/p}$$

with an absolute constant  $c_1 > 0$ . On the other hand (3.1) implies

$$(3.4) \quad \|Q_n\|_{L_p(\partial D)} \leq c_2 n^{1/2}$$

with an absolute constant  $c_2 > 0$ , and the lower bound of the theorem follows.

To see the upper bound of the theorem, observe that Lemma 2.3 implies

$$\begin{aligned} \|S_n(P_n)\|_{L_p(\partial D)} &\leq \left(\frac{2n+1}{2\pi}\right)^{1/2-1/p} \|S_n(P_n)\|_{L_2(\partial D)} \\ &\leq \left(\frac{2n+1}{2\pi}\right)^{1/2-1/p} \|P_n\|_{L_2(\partial D)} \\ &\leq (2n+1)^{1/2-1/p} \|P_n\|_{L_p(\partial D)} \end{aligned}$$

for all polynomials  $P_n$  of degree at most  $n$  with complex coefficients. This proves the upper bound of the theorem.  $\square$

*Proof of Theorem 1.3.* Let  $n = 2^{m+2} - 2$ . Consider the polynomial

$$P_n(z) = 4z^{2^{m+1}-1} \prod_{k=0}^m \left(1 + \frac{z^{2^k} + z^{-2^k}}{2}\right).$$

Then

$$|P_n(z)| = 4 \prod_{k=0}^m \left(1 + \frac{z^{2^k} + z^{-2^k}}{2}\right),$$

and hence  $\|P_n\|_{L_1(\partial D)} = 4$ . Also

$$P_n(z) - S_n(P_n)(z) = z^{2^{m+1}-1} \left(3 + \sum_{k=0}^m (z^{2^k} + z^{-2^k})\right).$$

Let

$$R_n(z) := 3 + \sum_{k=0}^m (z^{2^k} + z^{-2^k}).$$

Then

$$\|S_n(P_n)\|_{L_1(\partial D)} \geq \|S_n(P_n) - P_n\|_{L_1(\partial D)} - \|P_n\|_{L_1(\partial D)} = \|R_n\|_{L_1(\partial D)} - 4.$$

We will prove that  $\|R_n\|_{L_1(\partial D)} \geq c\sqrt{m}$  for some absolute constant  $c > 0$ . It is easy to see that if  $b, a_0, a_1, \dots, a_m$  are complex numbers and

$$F(z) = b + \sum_{k=0}^m a_k (z^{2^k} + z^{-2^k}),$$

then we have

$$\|F\|_{L_4(\partial D)} \leq \sqrt[4]{3} \left( |b|^2 + \sum_{k=0}^m |2a_k|^2 \right)^{1/2}.$$

Therefore

$$\|R_n\|_{L_4(\partial D)} \leq \sqrt[4]{3} \sqrt{9 + 4(m+1)}.$$

Moreover

$$\|R_n\|_{L_2(\partial D)} = \sqrt{9 + 2(m+1)}.$$

By Hölder's Inequality

$$\|R_n\|_{L_4(\partial D)}^{2/3} \|R_n\|_{L_1(\partial D)}^{1/3} \geq \|R_n\|_{L_2(\partial D)}.$$

Hence we obtain

$$\left(\sqrt[4]{3} \sqrt{9 + 4(m+1)}\right)^{2/3} \left(\|R_n\|_{L_1(\partial D)}\right)^{1/3} \geq \sqrt{9 + 2(m+1)},$$

and thus  $\|R_n\|_{L_1(\partial D)} \geq c\sqrt{m}$ . This gives

$$\frac{\|S_n(P_n)\|_{L_1(\partial D)}}{\|P_n\|_{L_1(\partial D)}} \geq c'\sqrt{m} \geq c''\sqrt{\log n}$$

with absolute constants  $c' > 0$  and  $c'' > 0$ .  $\square$

*Proof of Theorem 1.4.* Let

$$P_n(z) := \sum_{j=0}^n a_j z^j, \quad a_j \in \mathbb{C},$$

and

$$S_n(P_n)(z) := \sum_{j=0}^n \tilde{a}_j z^j, \quad \tilde{a}_j := (a_j/|a_j|) \min\{|a_j|, 1\}.$$

First we prove the upper bound. Using Lemma 1.4 we obtain

$$\begin{aligned} \max_{x \in [-1, 1]} |S_n(P_n)(x)| &\leq \max_{z \in \partial D} |S_n(P_n)(z)| \\ &\leq \left(\frac{2n+1}{2\pi}\right)^{1/2} \|S_n(P_n)\|_{L_2(\partial D)} \\ &\leq \left(\frac{2n+1}{2\pi}\right)^{1/2} \|P_n\|_{L_2(\partial D)} \\ &\leq \left(\frac{2n+1}{2\pi}\right)^{1/2} \cdot 8^{n/2} \sqrt{2\pi} \max_{x \in [-1, 1]} |P_n(x)|, \end{aligned}$$

which proves the upper bound of the theorem.

Now we turn to the lower bound. We define  $Q_n \in \mathcal{P}_{4n}$  by

$$Q_n(z) := z^{2n}(1-z^2)^n = z^{2n} \sum_{j=0}^n (-1)^j \binom{n}{j} z^{2j}.$$

Then

$$(3.5) \quad \max_{x \in [-1, 1]} |Q_n(x)| = \left(\frac{1}{4}\right)^n .$$

Also

$$S_n(Q_n)(z) = z^{2n} \sum_{j=0}^n (-1)^j z^{2j} ,$$

hence for every positive even  $n$

$$(3.6) \quad |S_n(Q_n)(1)| = 1 .$$

Now we conclude the the lower bound of the theorem by combining (3.5) and (3.6).  $\square$

#### 4. Acknowledgment.

I thank Stanislaw Kwapien for raising some of the questions we settled here, and for several discussions about the topic. His method based on the Salem-Zygmund Theorem gave  $c(n/\log n)^{1/2}$  lower bound rather than the right  $cn^{1/2}$  one in Theorem 1.1. In addition, Theorem 1.3 is due to Kwapien.

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