Markov–Bernstein–Turán type inequalities for complex polynomials

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1. Introduction

We will be concerned with generalizations of the following classical polynomial inequalities

$$||p'||_I \leq n^2 ||p||_I \tag{1.1}$$

(Markov inequality), and

$$||p'||_D \leq n ||p||_D \tag{1.2}$$

(Bernstein inequality), where $I := [-1, 1]$, D is the closed unit disk, $p \in \mathcal{P}_n^c$ (=the space of polynomials of degree at most n with complex coefficients), and $|| \cdot ||$ means supremum norm over the set specified. There is a striking difference beteween n and n^2 (the so-called Markov-factors) in the above inequalities, due to the difference between the domains considered. Our purpose in this paper is to create a transition between these Markov-factors by considering ellipses whose limit cases are I and D. We shall also consider Turán type inequalities in these domains, i.e. we intend to give lower estimates for the Markov factors when the roots of the polynomials are constrained to these domains. Apart from constants, these inequalities turn out to be sharp. We shall also consider these problems for diamond shaped domains, but the results for such domains will be less complete.

2. Markov type inequalities

The most natural way of seeking connection between the inequalities (1.1) and (1.2) is to introduce the ellipse

$$E_\varepsilon := \{z = x + iy : \varepsilon^2 x^2 + y^2 \leq \varepsilon^2\}, \qquad 0 \leq \varepsilon \leq 1,$$

in the complex plane, and establish an inequality there. ($\varepsilon = 0$ and $\varepsilon = 1$ correspond to (1.1) and (1.2), respectively.)

Theorem 1. *There exists an absolute constant $C > 0$ such that*

$$\frac{1}{C} \min\left(\frac{n}{\varepsilon}, n^2\right) \leq \sup_{p \in \mathcal{P}_n^c} \frac{||p'||_{E_\varepsilon}}{||p||_{E_\varepsilon}} \leq \min C \left(\frac{n}{\varepsilon}, n^2\right), \qquad 0 \leq \varepsilon \leq 1. \tag{2.1}$$

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Proof. First we prove the upper estimate. Let $p \in \mathcal{P}_n^c$. Consider the trigonometric polynomial

$$q(t) := p(\cos t + i\varepsilon \sin t) \tag{2.2}$$

of order at most n. We have

$$q'(t) = (-\sin t + i\varepsilon \cos t) p'(\cos t + i\varepsilon \sin t),$$

whence

$$|p'(\cos t + i\varepsilon \sin t)| \le \frac{||q'||_{\mathbf{R}}}{|-\sin t + i\varepsilon \cos t|} = \frac{||q'||_{\mathbf{R}}}{\sqrt{\sin^2 t + \varepsilon^2 \cos^2 t}} \le \frac{||q'||_{\mathbf{R}}}{\varepsilon}.$$

According to Corollary 5.1.5 in Borwein–Erdélyi [1],

$$||q'||_{\mathbf{R}} \le n ||q||_{\mathbf{R}},$$

and since $\cos t + i\varepsilon \sin t \in E_\varepsilon,\ t \in \mathbf{R}$, implies $||q||_{\mathbf{R}} = ||p||_{E_\varepsilon}$ by (2.2), we get

$$||p'||_{E_\varepsilon} \le \frac{n}{\varepsilon} ||p||_{E_\varepsilon}.$$

This proves the first upper estimate in (2.1).

In order to prove the second upper estimate, let

$$\varrho = \varepsilon + \sqrt{\varepsilon^2 + 1}, \tag{2.3}$$

and define another ellipse

$$E(\varrho) := \left\{ z = x + iy : \frac{x^2}{\left(\frac{\varrho + \varrho^{-1}}{2}\right)^2} + \frac{y^2}{\left(\frac{\varrho - \varrho^{-1}}{2}\right)^2} \le 1 \right\} \supset E_\varepsilon.$$

(The inclusion follows from the fact that, because of (2.3), the minor axis $[-\varepsilon, \varepsilon]$ of E_ε coincides with that of $E(\varrho)$, while the major axis $[-1, 1]$ of E_ε is contained in that of $E(\varrho)$.) Thus by

$$||p||_{E(\varrho)} \le \varrho^n ||p||_I \qquad p \in \mathcal{P}_n^c, \quad \varrho > 1, \tag{2.4}$$

(cf. E.17d) of [1]) and (1.2),

$$||p'||_{E_\varepsilon} \le ||p'||_{E(\varrho)} \le \varrho^n ||p'||_I \le n^2 \left(\varepsilon + \sqrt{\varepsilon^2 + 1}\right)^n ||p||_I \le n^2 e^{2n\varepsilon} ||p||_I.$$

Now for $\varepsilon \le \frac{1}{2n}$ this yields the second upper estimate in (2.1), while in the opposite case the already proved first upper estimate gives the Markov factor $2n^2$, and the theorem is completely proved.

We now prove the lower estimates in (2.1). If $\varepsilon = 1$ then $p(z) = z^n$ proves the statement. Let now first $\frac{1}{n} \le \varepsilon < 1$, and consider

$$p(z) := T_m\left(\frac{z+\varepsilon\sqrt{z^2-1+\varepsilon^2}}{1-\varepsilon^2}\right)T_m\left(\frac{z-\varepsilon\sqrt{z^2-1+\varepsilon^2}}{1-\varepsilon^2}\right), \qquad m=[n/2], \tag{2.5}$$

where $T_m(x) := \cos(m \arccos x)$ is the Chebyshev polynomial of degree m, and $[\cdot]$ means integer part. First we show that $p \in \mathcal{P}_n$ (=the space of polynomials of degree at most n with real coefficients). This follows from the representation $T_m(x) = 2^{m-1}\prod_{k=1}^m (x-x_k)$, which implies

$$p(z) = 2^{2m-2}\prod_{k=1}^m\left(\frac{z}{1-\varepsilon^2}-x_k+\frac{\varepsilon\sqrt{z^2-1+\varepsilon^2}}{1-\varepsilon^2}\right)\left(\frac{z}{1-\varepsilon^2}-x_k-\frac{\varepsilon\sqrt{z^2-1+\varepsilon^2}}{1-\varepsilon^2}\right)=$$

$$=2^{2m-2}\prod_{k=1}^m\left[\left(\frac{z}{1-\varepsilon^2}-x_k\right)^2-\frac{\varepsilon^2(z^2-1+\varepsilon^2)}{(1-\varepsilon^2)^2}\right]\subset\mathcal{P}_{2m}\subset\mathcal{P}_n.$$

At the points $z=\cos t + i\varepsilon \sin t$ on the ellipse E_ε, we can give a simpler form of this polynomial. An easy calculation yields

$$\sqrt{z^2-1+\varepsilon^2} = \pm(\varepsilon\cos t + i\sin t),$$

whence by (2.5)

$$p(z) = T_m\left(\frac{(1+\varepsilon^2)\cos t + 2i\varepsilon\sin t}{1-\varepsilon^2}\right)T_m(\cos t), \qquad z=\cos t+i\varepsilon\sin t.$$

Hence, with the notation

$$\varrho := \frac{1+\varepsilon}{1-\varepsilon}, \tag{2.6}$$

we obtain

$$p(z) = T_m\left(\frac{\varrho^2+1}{2\varrho}\cos t + i\frac{\varrho^2-1}{2\varrho}\sin t\right)\cos mt\,, \qquad z=\cos t+i\varepsilon\sin t.$$

Here, with the notation $w := \varrho e^{it}$ we have

$$p(z) = T_m\left(\frac{w^2+1}{2w}\right)\cos mt = \frac{w^{2m}+1}{2w^m}\cos mt =$$

$$=\frac{1}{2}(\varrho^m e^{imt}+\varrho^{-m}e^{-imt})\cos mt\,, \qquad z=\cos t+i\varepsilon\sin t. \tag{2.7}$$

Hence it is clear that

$$||p||_{E(\varepsilon)} = \frac{\varrho^m+\varrho^{-m}}{2}. \tag{2.8}$$

Differentiating (2.7) with respect to t and putting $z = 1$ (i.e. $t = 0$) we obtain

$$p'(1)i\varepsilon = i\frac{m}{2}(\varrho^m - \varrho^{-m}),$$

whence by (2.8)

$$||p'||_{E_\varepsilon} \geq |p'(1)| = \frac{m}{\varepsilon}||p||_{E_\varepsilon}\frac{\varrho^m - \varrho^{-m}}{\varrho^m + \varrho^{-m}}.$$

Here by (2.6)

$$\frac{\varrho^m - \varrho^{-m}}{\varrho^m + \varrho^{-m}} = \frac{\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{2m} - 1}{\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{2m} + 1} \geq \frac{(1+\varepsilon)^m - 1}{(1+\varepsilon)^m + 1} \geq \frac{\varepsilon m}{\varepsilon m + 2} \geq \frac{1}{9},$$

provided $\varepsilon \geq \frac{1}{n} > \frac{1}{2m+1}$.

Finally, if $0 \leq \varepsilon \leq 1/n$, then with the notation (2.3) we get $\varrho \leq 1 + \frac{2}{n}$, whence

$$||T_n||_{E_\varepsilon} \leq ||T_n||_{E(\varrho)} \leq \left(1 + \frac{2}{n}\right)^n ||T_n||_I < e^2,$$

i.e.

$$||T_n'||_{E_\varepsilon} \geq T_n'(1) = n^2 > \frac{n^2}{e^2}||T_n||_{E_\varepsilon},$$

which completes the proof of the lower estimate.

Remark. We can see from the first part of the proof that

$$|p'(\pm 1)| \leq \frac{n}{\varepsilon}||p||_{E_\varepsilon} \qquad \text{and} \qquad |p'(\pm i\varepsilon)| \leq n||p||_{E_\varepsilon},$$

i.e. the estimate is much better at the endpoints of the minor axis than at the endpoints of the major axis.

To close this section, we mention a Markov type inequality for the diamond shaped domain

$$S_\varepsilon := \{z = x + iy : \ \varepsilon|x| + |y| \leq \varepsilon\}, \qquad 0 \leq \varepsilon < \infty.$$

Proposition. *There are constants $c_1, c_2 > 0$ depending only on ε such that*

$$c_1 n^{2-\frac{2}{\pi}\arctan\varepsilon} \leq \sup_{p \in \mathcal{P}_n^c} \frac{||p'||_{S_\varepsilon}}{||p||_{S_\varepsilon}} \leq c_2 n^{2-\frac{2}{\pi}\arctan\varepsilon}, \qquad 0 \leq \varepsilon < \infty.$$

This is a special case of a classical result of Szegö [3]. In particular, for the square ($\varepsilon = 1$) we obtain $O(n^{3/2})$ for the order of magnitude of the Markov factor.

3. Lower estimates (Turán type results) for the ellipse

The estimates given by Theorem 1 are sharp. Of course, this still permits that for some polynomials the derivative estimates are much better. However, if we make some restrictions on the roots of the polynomial then the supremum norm of the derivative cannot be "too small" compared with that of the polynomial. In this respect we quote two classical results of P. Turán [4]. For a compact set M, denote by $\mathcal{P}_n^c(M)$ the set of all algebraic polynomials of exact degree n with complex coefficients having all their roots in M.

Theorem A. *We have*

$$||p'||_D \geq \frac{n}{2}||p||_D, \qquad p \in \mathcal{P}_n^c(D),$$

and

$$||p'||_I \geq \frac{\sqrt{n}}{6}||p||_I, \qquad p \in \mathcal{P}_n^c(I).$$

Our next result interpolates between n and $\sqrt{n}$ in the above estimates.

Theorem 2. *There exists an absolute constant $C > 0$ such that*

$$\frac{1}{C}(n\varepsilon + \sqrt{n}) \leq \inf_{p \in \mathcal{P}_n^c(E_\varepsilon)} \frac{||p'||_{E_\varepsilon}}{||p||_{E_\varepsilon}} \leq C(n\varepsilon + \sqrt{n}), \qquad 0 \leq \varepsilon \leq 1. \tag{3.1}$$

For the proof we need a slight generalization of the first inequality of Theorem A. In fact, in this section we need only the case $m = n$ of it, when we do not need to consider the half-plane H. The lemma below will be fully expoited only in the proof of Theorems 3 and 4 in Section 4. We remark that N. Levenberg and E. A. Poletsky [2, Proposition 2.1] have also used the case $m = n$ of the lemma below to prove their main result.

Lemma. *Let $\Gamma(a,r)$ be the circle of the complex plane centered at a with radius r. Let $z_0 \in \Gamma(a,r)$. Suppose $p \in \mathcal{P}_n^c$ has at least m zeros in the disk $D(a,r)$ bounded by $\Gamma(a,r)$ and it has all its zeros in the half-plane $H = H(a,r,z_0)$ containing a and bounded by the line tangent to $\Gamma(a,r)$ at z_0. Then*

$$\left|\frac{p'(z_0)}{p(z_0)}\right| \geq \frac{m}{2r}\,.$$

Proof. Let $p \in \mathcal{P}_n^c(H)$ be of the form

$$p(z) = c\prod_{k=1}^{n}(z - z_k)\,, \qquad c, z_k \in \mathbf{C}\,.$$

Then

$$r\left|\frac{p'(z_0)}{p(z_0)}\right| = \left|\frac{p'(z_0)(z_0 - a)}{p(z_0)}\right| = \left|\sum_{k=1}^{n}\frac{z_0 - a}{z_0 - z_k}\right| \geq \sum_{k=1}^{n}\operatorname{Re}\frac{z_0 - a}{z_0 - z_k} \geq \frac{m}{2},$$

since

$$\mathrm{Re}\,\frac{z_0-a}{z_0-z_k}=\mathrm{Re}\left(\left(1-\frac{z_k-a}{z_0-a}\right)^{-1}\right)\ge\frac{1}{2}\,,\qquad z_k\in D(a,r)\,,$$

and

$$\mathrm{Re}\,\frac{z_0-a}{z_0-z_k}\ge 0\,,\qquad z_k\in H(a,r,z_0)\,.$$

Proof of Theorem 2. In case $\varepsilon=0$ or $\varepsilon=1$ the lower estimate follows from Theorem A, so we may assume that $0<\varepsilon<1$. Let $z_0=x_0+iy_0$ be an arbitrary point on the boundary of E_ε, i.e.

$$\varepsilon^2x_0^2+y_0^2=\varepsilon^2,$$

and consider the circle

$$x^2+\left(y+\frac{1-\varepsilon^2}{\varepsilon^2}y_0\right)^2=\frac{1-(1-\varepsilon^2)x_0^2}{\varepsilon^2}.$$

An easy calculation shows that except the points $(\pm x_0,y_0)$, there are no other common points of this circle with the boundary of the ellipse E_ε. In fact, the disk $D(r,a)$ with

$$r=\frac{\sqrt{1-(1-\varepsilon^2)x_0^2}}{\varepsilon}\qquad\text{and}\qquad a=-\frac{1-\varepsilon^2}{\varepsilon^2}y_0$$

determined by this circle fully contains the ellipse E_ε. Therefore $p\in\mathcal{P}_n(D(r,a))$, whence Lemma with $m=n$ implies

$$|p'(z_0)|\ge\frac{n\varepsilon}{2\sqrt{1-(1-\varepsilon^2)x_0^2}}|p(z_0)|\ge\frac{n\varepsilon}{2}|p(z_0)|,$$

which proves the first part of the lower estimate in (3.1), since z_0 can be chosen as the point on the boundary ∂E_ε of E_ε where $||p||_{E_\varepsilon}$ is attained.

Now let $0<\varepsilon\le 1/\sqrt{n}$. Without loss of generality we may assume that

$$|p(z_0)|=||p||_{E_\varepsilon}=1$$

where $z_0=x_0+iy_0\in\partial E_\varepsilon$ is such that $0\le x_0\le 1$ and $0\le y_0\le\varepsilon$. Further assume that

$$||p'||_{E_\varepsilon}\le d\sqrt{n}||p||_{E_\varepsilon}=d\sqrt{n}\tag{3.2}$$

with some absolute constant $d>0$. We shall prove that the latter assumption leads to a contradiction if d is small enough, and this will prove the second lower estimate of the theorem.

By the assumptions made above, for $|z-z_0|\le\frac{1}{2d\sqrt{n}}$, $z\in E_\varepsilon$, we have

$$|p(z)-p(z_0)|=\left|\int_{z_0}^{z}p'(w)\,dw\right|\le|z-z_0|\cdot||p'||_{E_\varepsilon}\le\frac{1}{2},$$

whence

$$\frac{1}{2} \leq |p(z)| \leq 1, \qquad |z - z_0| \leq \frac{1}{2d\sqrt{n}}, \quad z \in E_\varepsilon. \tag{3.3}$$

Denote the intersection of the circle $(x-x_0)^2+(y-y_0)^2 = \frac{1}{4d^2n}$ with the ellipse $\varepsilon^2x^2+y^2 = \varepsilon^2$ in the quarter-plane

$$\{z \in \mathbf{C} : \operatorname{Re} z < x_0, \ \operatorname{Im} z < 0\}$$

by (x_1, y_1). In fact, if $d < 1/2$ and n is large enough then an intersection such that $-1 < x_1 < x_0$ certainly exists. This coupled with (3.3) yields that

$$p(z) \neq 0, \qquad x_1 \leq \operatorname{Re} z \leq x_0. \tag{3.4}$$

We also have

$$\frac{1}{4d^2n} \geq (x_1 - x_0)^2 = \frac{1}{4d^2n} - (y_1 - y_0)^2 \geq \frac{1}{4d^2n} - 4\varepsilon^2 \geq \frac{1}{4d^2n} - \frac{4}{n} \geq \frac{1}{9d^2n}$$

provided $d \leq \frac{\sqrt{5}}{12}$. Hence

$$\frac{1}{3d\sqrt{n}} \leq x_1 - x_0 \leq \frac{1}{2d\sqrt{n}}. \tag{3.5}$$

With the notation

$$t_1 = \frac{2x_1 + x_0}{3} \quad \text{and} \quad t_0 = \frac{x_1 + 2x_0}{3}$$

we obtain

$$4d\sqrt{n} \geq \left|\frac{p'(t_1)}{p(t_1)}\right| + \left|\frac{p'(t_0)}{p(t_0)}\right| \geq \left|\operatorname{Re}\left(\frac{p'(t_1)}{p(t_1)} - \frac{p'(t_0)}{p(t_0)}\right)\right| =$$

$$= \left|\operatorname{Re}\int_{t_1}^{t_0}\left(\frac{p'(x)}{p(x)}\right)' dx\right| \geq \left|\operatorname{Re}\int_{t_1}^{t_0}\frac{p''(x)}{p(x)}\, dx\right| - \int_{t_1}^{t_0}\left|\frac{p'(x)}{p(x)}\right|^2 dx \geq$$

$$\geq \left|\operatorname{Re}\int_{t_1}^{t_0}\frac{p''(x)}{p(x)}\, dx\right| - 4d^2n(t_1 - t_0) \geq \left|\int_{t_1}^{t_0}\operatorname{Re}\frac{p''(x)}{p(x)}\, dx\right| - \frac{2}{3}d\sqrt{n},$$

whence

$$\left|\int_{t_1}^{t_0}\operatorname{Re}\frac{p''(x)}{p(x)}\, dx\right| \leq \frac{14}{3}d\sqrt{n}.$$

The latter inequality together with (3.5) implies the existence of an $x \in [t_1, t_0]$ such that

$$\left|\operatorname{Re}\frac{p''(x)}{p(x)}\right| \leq 42d^2n. \tag{3.6}$$

Now denote the roots of $p(z)$ by $z_k = u_k + iv_k,\ k = 1, \dots, n$. By (3.4) and (3.5),

$$|x - u_k| \geq \frac{x_0 - x_1}{3} \geq \frac{1}{9d\sqrt{n}} \geq \frac{2}{\sqrt{n}} \geq 2|v_k|\,, \qquad k = 1, 2, \dots, n, \tag{3.7}$$

provided $d \leq \frac{1}{18}$. Thus using the identity

$$\frac{p'(x)^2}{p(x)^2} = \sum_{k=1}^{n} \frac{1}{(x - z_k)^2} + \frac{p''(x)}{p(x)}$$

as well as (3.6), we obtain

$$|p'(x)|^2 \geq \frac{1}{4}\left|\frac{p'(x)}{p(x)}\right|^2 \geq \frac{1}{4}\mathrm{Re}\,\frac{p'(x)^2}{p(x)^2} = \frac{1}{4}\mathrm{Re}\sum_{k=1}^{n} \frac{1}{(x - z_k)^2} + \frac{1}{4}\mathrm{Re}\frac{p''(x)}{p(x)} \geq$$

$$\geq \frac{1}{4}\frac{(x - u_k)^2 - v_k^2}{[(x - u_k)^2 + v_k^2]^2} - 42d^2n \geq \frac{3}{16}\sum_{k=1}^{n} \frac{(x - u_k)^2}{[(x - u_k)^2 + 1/n]^2} - 42d^2n.$$

Since the function $\frac{t}{(t+1/n)^2}$ is decreasing on $[1/n, \infty)$, and in our case $(x - u_k)^2 \geq 4/n$ by (3.7), each term in the above sum is greater than its value when $(x - u_k)^2 = 4$, whence

$$|p'(x)|^2 \geq \frac{3}{4}\sum_{k=1}^{n} \frac{1}{(4 + 1/n)^2} - 42d^2n \geq \frac{3n}{100} - 42d^2n \geq cn$$

provided d is small enough, which proves the second lower estimate of the theorem.

The proof of the upper estimate in (3.1) works with the same example as in [4]. Let

$$p(z) = (1 - z^2)^{[n/2]} \in \mathcal{P}_n\,. \tag{3.8}$$

Then

$$||p||_{E(\varepsilon)} = |p(\pm i\varepsilon)| = (1 + \varepsilon^2)^{[n/2]}\,. \tag{3.9}$$

With the notation $z = \cos t + i\varepsilon \sin t \in E_\varepsilon$ we obtain

$$|p'(z)| \leq n|z| \cdot |1 - z^2|^{[n/2]-1} =$$

$$= n\sqrt{\cos^2 t + \varepsilon^2 \sin^2 t}\,\left\{(1 + \varepsilon^2)^2 \sin^4 t + \varepsilon^2 \sin^2 2t\right\}^{\frac{[n/2]-1}{2}} \leq$$

$$\leq n||p||_{E(\varepsilon)}(|\cos t| + \varepsilon \sin t)\left\{\sin^4 t + \frac{\varepsilon^2}{(1 + \varepsilon^2)^2}\sin^2 2t\right\}^{\frac{[n/2]-1}{2}} \leq$$

$$\leq n||p||_{E_\varepsilon}(|\cos t| \sin^{[n/2]-1} t + \varepsilon).$$

It is easily seen that the function on the right hand side attains its absolute maximum at $\cos t = \frac{1}{\sqrt{[n/2]}}$ whence

$$\|p'\|_{E_\varepsilon} \le n\|p\|_{E_\varepsilon} \left(\frac{1}{\sqrt{[n/2]}} + \varepsilon \right), \tag{3.10}$$

and this completely proves Theorem 2.

4. Turán-type inequalities on diamonds

For a compact set M, denote by $\mathcal{P}_n(M)$ the set of all algebraic polynomials of exact degree n with real coefficients having all their roots in M. Recall that S_ε ($\varepsilon \ge 0$) denotes the diamond of the complex plane with diagonals $[-1,1]$ and $[-i\varepsilon, i\varepsilon]$.

Theorem 3. *There exists an absolute constant $C > 0$ such that*

$$\frac{1}{C}(n\varepsilon + \sqrt{n}) \le \inf_p \frac{\|p'\|_{S_\varepsilon}}{\|p\|_{S_\varepsilon}} \le C(n\varepsilon + \sqrt{n}), \qquad 0 \le \varepsilon \le 1,$$

where the infimum is taken for all $p \in \mathcal{P}_n^c(S_\varepsilon)$ with the property

$$|p(z)| = |p(-z)|, \qquad z \in \mathbf{C}, \tag{4.1}$$

or where the infimum is taken for all $p \in \mathcal{P}_n(S_\varepsilon)$.

Proof. For $n \ge 2$, the upper bound can be obtained by considering again the polynomial (3.8). Namely, the absolute maximum of this polynomial in the ellipse E_ε or in the diamond S_ε is attained at $\pm i\varepsilon$, hence $\|p\|_{S_\varepsilon} = \|p\|_{E(\varepsilon)}$. On the other hand, because of $S_\varepsilon \subset E(\varepsilon)$, inequality (3.10) proves the statement. For $n = 1$ we can take $p(z) := z - 1$.

To prove the lower bound we consider three cases.

Case 1: Property (4.1) holds and $\varepsilon \in [n^{-1/2}, 1]$. Choose a point z_0 on the boundary of S_ε such that

$$|p(\pm z_0)| = \|p\|_{S_\varepsilon} \tag{4.2}$$

(cf. Property (4.1)). Without loss of generality we may assume that $z_0 \in [i\varepsilon, 1]$. A simple calculation shows that there are disks D_1 and D_2 of the complex plane such that D_1 and D_2 have radii $r = c\varepsilon^{-1}$, and they are tangent to $[i\varepsilon, 1]$ at z_0 and to $[-1, -i\varepsilon]$ at $-z_0$, respectively, and $S_\varepsilon \subset D_1 \cup D_2$ for every sufficiently large absolute constant $c > 0$. (In fact, geometrical considerations show that the 'worst" situation is when $z_0 = 1$, and then $c = 5\sqrt{2}/4$ suffices.) Since $p \in \mathcal{P}_n^c(S_\varepsilon)$, p has at least $n/2$ zeros either in D_1 or in D_2. Thus Lemma and (4.2) imply (with an appropriate choice of $\pm$)

$$\frac{\|p'\|_{S_\varepsilon}}{\|p\|_{S_\varepsilon}} \ge \frac{|p'(\pm z_0)|}{\|p\|_{S_\varepsilon}} = \left| \frac{p'(\pm z_0)}{p(\pm z_0)} \right| \ge \frac{n}{4r} = \frac{1}{4c} n\varepsilon.$$

Case 2: $p \in \mathcal{P}_n(S_\varepsilon)$ and $\varepsilon \in [n^{-1/2}, 1]$. Then we can choose a point z_0 on the boundary of S_ε such that

$$|p(z_0)| = |p(\overline{z}_0)| = \|p\|_{S_\varepsilon}. \tag{4.3}$$

Without loss of generality we may assume that $z_0 \in [i\varepsilon, 1]$. Let D_1 and D_2 be disks of the complex plane with radius $r = c\varepsilon^{-1}$ such that they are tangent to $[i\varepsilon, 1]$ at z_0 from below, and to $[-1, -i\varepsilon]$ at $\overline{z}_0$ from above, respectively. Denote the boundaries of D_1 and D_2 by Γ_1 and Γ_2, respectively. A simple calculation shows that if the absolute constant $c > 0$ is sufficiently large, then Γ_1 intersects the boundary of S_ε only at $a_1 \in [-1, i\varepsilon]$ and $b_1 \in [-i\varepsilon, 1]$, while Γ_2 intersects the boundary of S_ε only at $\overline{a}_1 \in [-1, -i\varepsilon]$ and $\overline{b}_1 \in [i\varepsilon, 1]$. Also, if the absolute constant $c > 0$ is sufficiently large, then

$$|a_1 - i\varepsilon| = |\overline{a}_1 + i\varepsilon| \le \frac{1}{64}\,, \quad |b_1 - 1| = |\overline{b}_1 - 1| \le \frac{1}{64}\,. \tag{4.4}$$

In the sequel let the absolute constant $c > 0$ be so large that inequalities (4.4) hold. If $p \in \mathcal{P}_n(S_\varepsilon)$ has at least αn zeros in D_1 or in D_2, then by using Lemma and (4.3), we deduce

$$\frac{\|p'\|_{S_\varepsilon}}{\|p\|_{S_\varepsilon}} \ge \frac{|p'(\zeta)|}{\|p\|_{S_\varepsilon}} = \left|\frac{p'(\zeta)}{p(\zeta)}\right| \ge \frac{\alpha n}{2r} = \frac{\alpha}{2c} n\varepsilon\,,$$

where ζ is z_0 or $\overline{z}_0$, respectively. Hence we may assume that $p \in \mathcal{P}_n(S_\varepsilon)$ has at least $(1-\alpha)n$ zeros both in $S_\varepsilon \setminus D_1$ and in $S_\varepsilon \setminus D_2$. In the light of (4.4) this yields that $p \in \mathcal{P}_n(E_\varepsilon)$ has at least $(1-2\alpha)n$ zeros in the disk centered at 1 with radius $1/32$. However, we show that this situation cannot occur if the absolute constant $\alpha > 0$ is sufficiently small. Indeed, let $p \in \mathcal{P}_n(E_\varepsilon)$ be of the form $p = fg$ with

$$f(z) = \prod_{j=1}^{n_1} (z - u_j) \qquad \text{and} \qquad g(z) = \prod_{j=1}^{n_2} (z - v_j)\,, \tag{4.5}$$

where

$$u_j \in \mathbf{C}\,, \qquad j = 1, 2, \ldots, n_1\,, \quad n_1 \le 2\alpha n\,, \tag{4.6}$$

and

$$|v_j - 1| \le \frac{1}{32}\,, \qquad j = 1, 2, \ldots, n_2\,, \quad n_2 \ge (1 - 2\alpha)n\,. \tag{4.7}$$

Let I be the subinterval of $[-1, i\varepsilon]$ with endpoint -1 and length $1/32$. Let $y_0 \in I$ be chosen so that $|f(y_0)| = \|f\|_I$. We show that $|p(z_0)| < |p(y_0)|$, a contradiction. Indeed, by Chebyshev's inequality and (4.6) we have

$$|f(y_0)| \ge \left(\frac{1}{128}\right)^{n_1} \ge \left(\frac{1}{128}\right)^{2\alpha n}\,,$$

hence

$$\left|\frac{f(y_0)}{f(z_0)}\right| \ge \left(\frac{1}{256}\right)^{2\alpha n}\,. \tag{4.8}$$

Also, (4.7) implies

$$\left|\frac{g(y_0)}{g(z_0)}\right| \ge \frac{\left(\frac{31}{16}\right)^{n_2}}{\left(\sqrt{2} + \frac{1}{32}\right)^{n_2}} \ge \left(\frac{31}{24}\right)^{(1-2\alpha)n}\,. \tag{4.9}$$

By (4.8) and (4.9)

$$\left|\frac{p(y_0)}{p(z_0)}\right| = \left|\frac{f(y_0)}{f(z_0)}\right| \cdot \left|\frac{g(y_0)}{g(z_0)}\right| \geq \left(\left(\frac{1}{256}\right)^{2\alpha}\left(\frac{31}{24}\right)^{(1-2\alpha)}\right)^n,$$

if $\alpha > 0$ is a sufficiently small absolute constant. This finishes the proof in this case.

Case 3: $\varepsilon \in [0, n^{-1/2}]$. The proof of Theorem 2 in this case can be copied almost word for word.

It is an interesting question whether or not the inequality of Theorem 3 holds for all $p \in \mathcal{P}_n^c(S_\varepsilon)$. As our next result shows this is the case at least when $\varepsilon = 1$.

Theorem 4. *There exists an absolute constants $C > 0$ such that*

$$\frac{1}{C}n \leq \inf_{p \in \mathcal{P}_n^c(S_1)} \frac{\|p'\|_{S_1}}{\|p\|_{S_1}} \leq Cn\,.$$

Proof. Choose a point $z_0 \in S_1$ such that $|p(z_0)| = \|p\|_{S_1}$. Without loss of generality we may assume that $z_0 \in \left[1, \frac{1}{2}(1+i)\right]$. A simple calculation shows that there is an absolute constant $r > 0$ such that the circle Γ with radius r that is tangent to $[1, i]$ at z_0 and intersects the boundary of S_1 only at $a \in [-1, i]$ and $b \in [-i, 1]$. Moreover, if the $r > 0$ is sufficiently large, then

$$|a - i| \leq \frac{\sqrt{2}}{64} \qquad \text{and} \qquad |b - 1| \leq \frac{\sqrt{2}}{64}\,. \tag{4.10}$$

If $p \in \mathcal{P}_n^c(S_1)$ has at least αn zeros in the disk D with boundary Γ, then by Lemma we deduce

$$\frac{\|p'\|_{S_1}}{\|p\|_{S_1}} \geq \frac{|p'(z_0)|}{\|p\|_{S_1}} = \left|\frac{p'(z_0)}{p(z_0)}\right| \geq \frac{\alpha n}{2r}\,.$$

Hence we may assume that $p \in \mathcal{P}_n^c(S_1)$ has at most αn zeros in D, therefore that $p \in \mathcal{P}_n^c(1)$ has at least $(1-\alpha)n$ zeros in $S_1 \setminus D$. However, we show that this situation cannot occur if the absolute constant $\alpha > 0$ is sufficiently small. Indeed, let $p \in \mathcal{P}_n^c(S_1)$ be of the form $p = fg$ with (4.5), where

$$u_j \in \mathbf{C}\,, \qquad j = 1, 2, \ldots, n_1\,, \quad n_1 \leq \alpha n\,, \tag{4.11}$$

and

$$v_j \in S_1 \setminus D\,, \qquad j = 1, 2, \ldots, n_2\,, \quad n_2 \geq (1-\alpha)n\,. \tag{4.12}$$

Let I be the subinterval of $[-1, -i]$ with endpoint -1 and length $\sqrt{2}/4$. Let $y_0 \in I$ be chosen so that $|f(y_0)| = \|f\|_I$. We show that $|p(z_0)| < |p(y_0)|$, a contradiction. Indeed, by Chebyshev's inequality and (4.11) we have

$$|f(y_0)| \geq \left(\frac{\sqrt{2}}{16}\right)^{n_1} \geq \left(\frac{\sqrt{2}}{16}\right)^{\alpha n},$$

hence

$$\left|\frac{f(y_0)}{f(z_0)}\right| \geq \left(\frac{\sqrt{2}}{32}\right)^{\alpha n}. \tag{4.13}$$

Also, (4.10) and (4.12) imply

$$\left|\frac{g(y_0)}{g(z_0)}\right| \geq \left(\sqrt{2}\left(\left(1-\frac{1}{64}\right)^2 + \left(\frac{1}{4}\right)^2\right)^{1/2}\right)^{n_2} \left(\sqrt{2}\left(1+\left(\frac{1}{64}\right)^2\right)^{1/2}\right)^{n_2} \tag{4.14}$$

$$\geq \left(\frac{66}{65}\right)^{n_2/2} \geq \left(\frac{66}{65}\right)^{(1/2-\alpha)n}.$$

By (4.13) and (4.14)

$$\left|\frac{p(y_0)}{p(z_0)}\right| = \left|\frac{f(y_0)}{f(z_0)}\right| \left|\frac{g(y_0)}{g(z_0)}\right| \geq \left(\left(\frac{\sqrt{2}}{32}\right)^{\alpha} \left(\frac{66}{65}\right)^{(1/2-\alpha)}\right)^{n} > 1$$

if $\alpha > 0$ is a sufficiently small absolute constant.

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