Upper Bounds for the Derivative of Exponential Sums

Peter Borwein and Tamás Erdélyi

Department of Mathematics
Statistics and Computing Science
Dalhousie University
Halifax, Nova Scotia
Canada B3H 3J5

November 5, 2013

ABSTRACT. The equality
\[ \sup_p \frac{|p'(a)|}{\|p\|_{[a,b]}} = \frac{2n^2}{b-a} \]
is shown, where the supremum is taken for all exponential sums \( p \) of the form
\[ p(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}, \quad a_j \in \mathbb{R}, \]
with nonnegative exponents \( \lambda_j \). The inequalities
\[ \|p'\|_{[a+\delta,b-\delta]} \leq 4(n + 2)^{3}\delta^{-1}\|p\|_{[a,b]} \]
and
\[ \|p'\|_{[a+\delta,b-\delta]} \leq 4\sqrt{2}(n + 2)^{3}\delta^{-3/2}\|p\|_{L_2[a,b]} \]
are also proved for all exponential sums of the above form with arbitrary real exponents. These results improve inequalities of Lorentz and Schmidt and partially answer a question of Lorentz.

Classification Number: 41A17

Key Words: Exponential sums, Müntz Polynomials, Markov Inequality, Bernstein Inequality.

1. Introduction and Notation
Let $\Lambda_n := \{\lambda_1 < \lambda_2 < \cdots < \lambda_n\}, \quad \lambda_j \neq 0, \quad j = 1, 2, \ldots, n,$

$$E(\Lambda_n) = \{f : f(t) = a_0 + \sum_{j=1}^n a_j e^{\lambda_j t}, \quad a_j \in \mathbb{R}\}$$

and

$$E_n := \bigcup_{\Lambda_n} E(\Lambda_n) = \{f : f(t) = a_0 + \sum_{i=1}^n a_j e^{\lambda_j t}, \quad a_j, \lambda_j \in \mathbb{R}\}.$$ 

We will use the norms

$$\|f\|_{[a,b]} := \max_{x \in [a,b]} |f(x)|$$

and

$$\|f\|_{L^2[a,b]} := \left( \int_a^b |f(x)|^2 \, dx \right)^{1/2}$$

for functions $f \in C[a,b]$.

Schmidt [3] proved that there is a constant $c(n)$ depending only on $n$ so that

$$\|p'\|_{[a+\delta,b-\delta]} \leq c(n)\delta^{-1}\|p\|_{[a,b]}$$

for every $p \in E_n$ and $\delta \in (0,(b-a)/2)$. Lorentz [2] improved Schmidt’s result by showing that for every $\alpha > \frac{1}{2}$ there is a constant $c(\alpha)$ depending only on $\alpha$ so that $c(n)$ in the above inequality can be replaced by $c(\alpha)n^{\alpha \log n}$, and he speculated that there may be an absolute constant $c$ so that Schmidt’s inequality holds with $c(n) = cn$. Theorem 2 of this paper shows that Schmidt’s inequality holds with $c(n) = 4(n+2)^3$. Our first theorem establishes the sharp inequality

$$|p'(a)| \leq \frac{2n^2}{b-a}\|p\|_{[a,b]}$$

for every $p \in E_n$ with nonnegative exponents $\lambda_j$.

2. New Results

Theorem 1. We have

$$\sup_{p} \frac{|p'(a)|}{\|p\|_{[a,b]}} = \frac{2n^2}{b-a}$$

for every $a < b$, where the supremum is taken for all exponential sums $p \in E_n$ with nonnegative exponents. The equality

$$\sup_{p} \frac{|p'(a)|}{\|p\|_{[a,b]}} = \frac{2n^2}{a(\log b - \log a)}$$
also holds for every $0 < a < b$, where the supremum is taken for all Müntz polynomials of the form

$$p(x) = a_0 + \sum_{j=1}^{n} a_j x^{\lambda_j}, \quad a_j \in \mathbb{R}, \quad \lambda_j \geq 0.$$ 

**Theorem 2.** The inequalities

$$\|p'\|_{[a+\delta,b-\delta]} \leq 4(n+2)^3 \delta^{-1} \|p\|_{[a,b]}$$

and

$$\|p'\|_{[a+\delta,b-\delta]} \leq 4\sqrt{2}(n+2)^3 \delta^{-3/2} \|p\|_{L_2[a,b]}$$

hold for every $p \in E_n$ and $\delta \in (0,(b-a)/2)$.

**3. Proofs**

To prove Theorem 1 we need some notation. If $\Lambda_n := \{\lambda_1 < \lambda_2 < \cdots < \lambda_n\}$ is a set of positive real numbers then the real span of

$$\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots, x^{\lambda_n}\}, \quad x \geq 0,$

will be denoted by $M(\Lambda_n)$. It is well-known that these are Chebyshev spaces (see [1] for instance), so $M(\Lambda_n)$ possesses a unique Chebyshev “polynomial” $T_{\Lambda_n}$ on $[a,b]$, $0 < a < b$, with the properties

(i) $T_{\Lambda_n} \in M(\Lambda_n)$,

(ii) $\|T_{\Lambda_n}\|_{[a,b]} = 1$

and

(iii) there are $a = x_0 < x_1 < \cdots < x_n = b$ so that

$$T_{\Lambda_n}(x_j) = (-1)^j, \quad j = 0, 1, \ldots, n.$$ 

It is routine to prove (see [1] again) that $T_{\Lambda_n}$ has exactly $n$ distinct zeros on $(a,b)$,

$$\max_{0 \neq p \in M(\Lambda_n)} \frac{|p'(a)|}{\|p\|_{[a,b]}} = \frac{|T'_{\Lambda_n}(a)|}{\|T_{\Lambda_n}\|_{[a,b]}} = |T'_{\Lambda_n}(a)| \quad (1)$$

and

$$\max_{0 \neq p \in M(\Lambda_n)} \frac{|p(0)|}{\|p\|_{[a,b]}} = \frac{|T_{\Lambda_n}(0)|}{\|T_{\Lambda_n}\|_{[a,b]}} = |T_{\Lambda_n}(0)| \quad . (2)$$

**Lemma 3.** Let

$\Lambda_n := \{\lambda_1 < \lambda_2 < \cdots < \lambda_n\}$ and $\Gamma_n := \{\gamma_1 < \gamma_2 < \cdots < \gamma_n\}$
be so that $0 < \lambda_j \leq \gamma_j$ for each $j = 1, 2, \ldots, n$. Then
\[ |T'_{\Gamma_n}(a)| \leq |T'_{\Lambda_n}(a)|. \] (3)

**Proof.** Without loss of generality we may assume that there is an index $m$, $1 \leq m \leq n$, so that $\lambda_m < \gamma_m$ and $\lambda_j = \gamma_j$ if $j \neq m$, since repeated applications of the result in this situation give the lemma in the general case. First we show that
\[ |T_{\Gamma_n}(0)| < |T_{\Lambda_n}(0)|. \] (4)

Indeed, let $R_{\Gamma_n} \in M(\Gamma_n)$ interpolate $T_{\Lambda_n}$ at the zeros of $T_{\Lambda_n}$, and be normalized so that $R_{\Gamma_n}(0) = T_{\Lambda_n}(0)$. Then the Improvement Theorem of Pinkus and Smith [4, Theorem 2] yields
\[ |R_{\Gamma_n}(x)| \leq |T_{\Lambda_n}(x)| \leq 1, \quad x \in [a, b]. \]

Hence, using (2) with $\Lambda_n$ replaced by $\Gamma_n$, we obtain
\[ |T_{\Lambda_n}(0)| = |R_{\Gamma_n}(0)| \leq |T_{\Gamma_n}(0)|, \]
which proves (4). Using the defining properties of $T_{\Lambda_n}$ and $T_{\Gamma_n}$, we deduce that $T_{\Lambda_n} - T_{\Gamma_n}$ has at least $n + 1$ zeros in $[a, b]$ (we count every zero without sign change twice). Now assume that (3) does not hold, then
\[ |T'_{\Lambda_n}(a)| > |T'_{\Gamma_n}(a)|. \]

This, together with (4), implies that $T_{\Lambda_n} - T_{\Gamma_n}$ has at least one zero in $(0, a)$. Hence $T_{\Lambda_n} - T_{\Gamma_n}$ has at least $n + 2$ zeros in $(0, b]$. This is a contradiction, since
\[ T_{\Lambda_n} - T_{\Gamma_n} \in \text{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots, x^{\lambda_n}, x^{\gamma_m}\}, \]
and every function from the above span can have only at most $n + 1$ zeros in $(0, \infty)$ (see [3]). \qed

**Proof of Theorem 1.** It is sufficient to prove only the second statement of the theorem, the first one can be obtained by the change of variable $x = e^t$. We obtain from (1) and Lemma 3 that
\[ \frac{|p'(a)|}{\|p\|_{[a,b]}} \leq \lim_{\delta \to 0^+} \frac{|T'_{\Lambda_n,\delta}(a)|}{\|T_{\Lambda_n,\delta}\|_{[a,b]}} = \lim_{\delta \to 0^+} \frac{|T'_{\Lambda_n,\delta}(a)|}{\|T_{\Lambda_n,\delta}\|_{[a,b]}}. \]
for every $p$ of the form
\[ p(x) = a_0 + \sum_{j=1}^{n} a_j x^{\lambda_j}, \quad a_j \in \mathbb{R}, \quad \lambda_j > 0, \]
where
\[ \Lambda_{n,\delta} := \{ \delta, 2\delta, 3\delta, \ldots, n\delta \} \]
and $T_{n,\delta}$ is the Chebyshev “polynomial” of $\alpha(\Lambda_{n,\delta})$ on $[a,b]$. From the definition and uniqueness of $T_{\Lambda_{n,\delta}}$ it follows that
\[ T_{\Lambda_{n,\delta}}(x) = T_n\left( \frac{2}{b^\delta - a^\delta} x^\delta - \frac{b^\delta + a^\delta}{b^\delta - a^\delta} \right), \]
where $T_n(y) := \cos(n \arccos y)$. Therefore
\[ |T'_{\Lambda_{n,\delta}}(a)| = \frac{2 n^2}{\delta^{-1}(b^\delta - 1) - \delta^{-1}(a^\delta - 1)} a^{-1} \delta \rightarrow b^+ \frac{2 n^2}{a(\log b - \log a)} \]
and the theorem is proved. \(\square\)

To prove Theorem 2 we need two lemmas.

**Lemma 4.** For every set $\Lambda_n := \{ \lambda_1 < \lambda_2 < \ldots < \lambda_n \}$ of nonzero real numbers there is a point $y \in [-1,1]$ depending only on $\Lambda_n$ so that
\[ |p'(y)| \leq 2(n+2)^3\|p\|_{L_2[-1,1]} \]
for every $p \in E(\Lambda_n)$.

**Proof.** Take the orthonormal set $\{p_k\}_{k=0}^{n}$ on $[-1,1]$ defined by
(i) $p_k \in \text{span}\{1, e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_k t}\}, \quad k = 0, 1, \ldots, n,$
(ii) $\int_{-1}^{1} p_i p_j = \delta_{i,j}, \quad 0 \leq i \leq j \leq n.$

Writing $p \in E(\Lambda_n)$ as a linear combination of the functions $p_k, \ k = 0, 1, \ldots, n$, and using the Cauchy-Schwartz inequality and the orthonormality of $\{p_k\}_{k=0}^{n}$ on $[-1,1]$, we obtain in a standard fashion that
\[ \max_{p \in E(\Lambda_n)} \frac{|p'(t_0)|}{\|p\|_{L_2[-1,1]}} = \left( \sum_{k=0}^{n} p_k(t_0)^2 \right)^{1/2}, \quad t_0 \in \mathbb{R}. \]
Let
\[ A_k := \{ t \in [-1,1] : |p_k(t)| \geq (n+1)^{1/2} \}, \quad k = 0, 1, \ldots, n \]
and

\[ B_k := \{ t \in [-1, 1] \setminus A_k : | p'_k(t) | \geq 2(n + 2)^{5/2} \}, \quad k = 0, 1, \ldots, n. \]

Since \( \int_{-1}^{1} p_k^2 = 1 \), we have

\[ m(A_k) \leq (n + 1)^{-1}, \quad k = 0, 1, \ldots, n. \]

Since \( \text{span}\{1, e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_k t}\} \) is a Chebyshev system, each \( \tilde{A}_k := [-1, 1] \setminus A_k \) comprises of at most \( k+1 \) intervals, and each \( B_k \) comprises of at most \( 2(k+1) \) intervals. Therefore

\[ 2(n + 2)^{5/2} m(B_k) \leq \int_{B_k} | p'_k(t) | \, dt \leq 4(k + 1)\sqrt{n + 1}, \]

whence

\[ \sum_{k=0}^{n} m(B_k) \leq \frac{2\sqrt{n + 1}}{(n + 2)^{5/2}} \frac{(n + 1)(n + 2)}{2} < 1. \]

Now let

\[ A := [-1, 1] \setminus \bigcup_{k=0}^{n} (A_k \cup B_k). \]

Then

\[ m(A) \geq 2 - \sum_{k=0}^{n} m(A_k) - \sum_{k=0}^{n} m(B_k) \]
\[ > 2 - (n + 1)(n + 1)^{-1} - 1 > 0, \]

so there is a point \( y \in A \subset [-1, 1] \), where

\[ | p'(y) | \leq 2(n + 1)^{5/2}, \quad k = 0, 1, \ldots, n, \]

hence

\[ \left( \sum_{k=0}^{n} p'_k(y)^2 \right)^{1/2} \leq 2(n + 2)^3, \]

and the lemma is proved. \( \square \)

**Lemma 5.** We have

\[ | p'(0) | \leq 2(n + 2)^3 || p ||_{L_2[-2,2]} \leq 2(n + 2)^3 || p ||_{[-2,2]} \]

for every \( p \in E_n \).
**Proof.** Let $\Lambda_n := \{\lambda_1 < \lambda_2 < \cdots, \lambda_n\}$ be a fixed set of nonzero real numbers, and let $y \in [-1, 1]$ be chosen by Lemma 4. Let $0 \neq p \in E(\Lambda_n)$. Then
\[
q(t) := p(t - y) \in E(\Lambda_n),
\]
therefore, applying Lemma 4 to $q$, we obtain
\[
\frac{|p'(0)|}{\|p\|_{L^2[-2,2]}} \leq \frac{|p'(0)|}{\|p\|_{L^2[-1-y,1-y]}} = \frac{|q'(y)|}{\|q\|_{L^2[-1,1]}} \leq 2(n + 2)^3,
\]
and the lemma is proved.

**Proof of Theorem 2.** Let $t_0 \in [a + \delta, b - \delta]$. Applying Lemma 5 to $q(t) := p(\delta t/2 + t_0)$, we get the theorem.

**References**


