THE FULL CLARKSON-ERDŐS-SCHWARTZ THEOREM ON THE CLOSURE OF NON-DENSE MÜNTZ SPACES

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1. Introduction and Notation

Müntz's beautiful classical theorem characterizes sequences $(\lambda_j)_{j=0}^{\infty}$ with

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

for which the Müntz space

$$\text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is dense in $C[0,1]$. Here, and in what follows, the above span denotes the collection of finite linear combinations of the functions $x^{\lambda_0}, x^{\lambda_1}, \dots$ with real coefficients, and $C[a,b]$ is the space of all real-valued continuous functions on $[a,b] \subset \mathbb{R}$ equipped with the uniform norm. Müntz's Theorem [Bo-Er3, De-Lo, Go, Mü, Szá] states the following.

Theorem A (Müntz). *Suppose* $(\lambda_j)_{j=0}^{\infty}$ *is a sequence with*

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots .$$

Then

$$\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is dense in $C[0,1]$ *if and only if* $\sum_{j=1}^{\infty} 1/\lambda_j = \infty$.

The original Müntz Theorem proved by Müntz [Mü] in 1914, by Szász [Szá] in 1916, and anticipated by Bernstein [Be] was only for sequences of exponents tending to infinity. The point 0 is special in the study of Müntz spaces. Even replacing $[0,1]$ by an interval $[a,b] \subset [0,\infty)$ in Müntz's Theorem is a non-trivial issue. This is, in large measure, due to Clarkson and Erdős [Cl-Er] and Schwartz [Sch] whose works include the result that if $\sum_{j=1}^{\infty} 1/\lambda_j < \infty$ then every function belonging to the uniform closure of

$$\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

on $[a,b]$ can be extended analytically throughout the region $\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < b\}$.

There are many variations and generalizations of Müntz's Theorem [An, Be, Boa, Bo1, Bo2, Bo-Er1, Bo-Er2, Bo-Er3, Bo-Er4, Bo-Er5, Bo-Er6, Bo-Er7, B-E-Z, Ch, Cl-Er, De-Lo, Er-Jo, Go, Lu-Ko, Op, Sch, So]. There are also still many open problems. In [Bo-Er6] it is shown that the interval $[0,1]$ in Müntz's Theorem can be replaced by an arbitrary compact set $A \subset [0,\infty)$ of positive Lebesgue measure. That is, if $A \subset [0,\infty)$ is a compact set of positive Lebesgue measure, then $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ is dense in $C(A)$ if and only if $\sum_{j=1}^{\infty} 1/\lambda_j = \infty$. Here $C(A)$ denotes the space of all real-valued continous functions on A equipped with the uniform norm. If A contains an interval then this follows from the already mentioned results of Clarkson, Erdős, and Schwartz. However, their results and methods cannot handle the case when, for example, $A \subset [0,1]$ is a a Cantor type set of positive measure.

In the case that $\sum_{j=1}^{\infty} 1/\lambda_j < \infty$, analyticity properties of the functions belonging to the uniform closure of $\mathrm{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ on A are also established in [Bo-Er6].

In [Bo-Er3, Section 4.2] and in [Bo-Er4] the following result is proved.

Theorem B (Full Müntz Theorem in $C[0,1]$). *Suppose $(\lambda_j)_{j=1}^{\infty}$ is a sequence of distinct positive real numbers. Then* $\mathrm{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$ *is dense in $C[0,1]$ if and only if*

$$\sum_{j=1}^{\infty} \frac{\lambda_j}{\lambda_j^2 + 1} = \infty\,.$$

Moreover, if

$$\sum_{j=1}^{\infty} \frac{\lambda_j}{\lambda_j^2 + 1} < \infty\,,$$

then every function from the $C[0,1]$ closure of

$$\mathrm{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$$

is infinitely many times differentiable on $(0,1)$.

The new result of this paper is the following.

Theorem 1.1 (Full Clarkson-Erdős-Schwartz Theorem). *Suppose* $(\lambda_j)_{j=1}^{\infty}$ *is a sequence of distinct positive numbers. Then*

$$\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$$

is dense in $C[0,1]$ *if and only if*

$$\sum_{j=1}^{\infty} \frac{\lambda_j}{\lambda_j^2 + 1} = \infty \,.$$

Moreover, if

$$\sum_{j=1}^{\infty} \frac{\lambda_j}{\lambda_j^2 + 1} < \infty \,,$$

then every function from the $C[0,1]$ *closure of*

$$\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$$

can be represented as an analytic function on

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < 1\}$$

restricted to $(0,1)$*.*

Theorem 1.3 (Full Müntz Theorem in $L_p(A)$ for $p \in (0, \infty)$ and for compact sets $A \subset [0, 1]$ with positive lower density at 0). *Let $A \subset [0, 1]$ be a compact set with positive lower density at 0. Let $p \in (0, \infty)$. Suppose $(\lambda_j)_{j=1}^{\infty}$ is a sequence of distinct real numbers greater than $-(1/p)$.*

(a) Then

$$\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$$

is dense in $L_p(A)$ if and only if

$$\sum_{j=1}^{\infty} \frac{\lambda_j + (1/p)}{(\lambda_j + (1/p))^2 + 1} = \infty\,.$$

(b) Moreover, if

$$\sum_{j=1}^{\infty} \frac{\lambda_j + (1/p)}{(\lambda_j + (1/p))^2 + 1} < \infty\,,$$

then every function from the $L_p(A)$ *closure of*

$$\mathrm{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$$

can be represented as an analytic function on

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < r_A\}$$

restricted to $A \cap (0, r_A)$*, where*

$$r_A := \sup\{y \in \mathbb{R} : m(A \cap [y, \infty)) > 0\}$$

($m(\cdot)$ *denotes the one-dimensional Lebesgue measure).*

This corrects, improves, and extends earlier results of Müntz [Mü], Szász [Szá], Clarkson and Erdős [Cl-Er], P. Borwein and Erdélyi [Bo-Er3, Bo-Er4], and Operstein [Op].

The notation

$$\|f\|_A := \sup_{x \in A} |f(x)|$$

is used throughout this paper for real-valued measurable functions f defined on a set $A \subset \mathbb{R}$ The space of all real-valued continuous functions on a set $A \subset \mathbb{R}$ equipped with the uniform norm is denoted by $C(A)$. Denote by $\operatorname{span}\{f_1, f_2, \dots\}$ the collection of all finite linear combinations of the functions $f_1, f_2, \dots$ over $\mathbb{R}$.

2. AUXILIARY RESULTS

The following result is the "bounded Remez-type inequality for non-dense Müntz spaces" due to P. Borwein and Erdélyi [Bo-Er6].

Theorem 2.1. *Suppose* $(\gamma_j)_{j=1}^\infty$ *is a sequence of distinct positive numbers satisfying*

$$\sum_{j=1}^\infty 1/\gamma_j < \infty\,.$$

Let $s > 0$. *Then there exists a constant* $c(\Gamma, s)$ *depending only on* $\Gamma := (\gamma_j)_{j=1}^\infty$ *and* s *(and not on* ϱ, A, *or the "length" of* f*) so that*

$$\|Q\|_{[0,\varrho]} \leq c(\Gamma, s)\,\|Q\|_A$$

for every $f \in \operatorname{span}\{1, x^{\gamma_1}, x^{\gamma_2}, \dots\}$ *and for every set* $A \subset [\varrho, 1]$ *of Lebesgue measure at least* s.

Combining a result of Clarkson and Erdős [Cl-Er] and its extension given by Schwartz [Sch] we can state the following

Theorem 2.2. *Suppose* $(\gamma_j)_{j=1}^{\infty}$ *is a sequence of distinct positive numbers satisfying*

$$\sum_{j=1}^{\infty} 1/\gamma_j < \infty\,.$$

Then $\operatorname{span}\{1, x^{\gamma_1}, x^{\gamma_2}, \dots\}$ *is not dense in* $C[0,1]$*. In addition, if the gap condition*

$$\inf\{\gamma_{j+1} - \gamma_j : \quad j = 1, 2, \dots\} > 0$$

holds, then every function $f \in C[0,1]$ *belonging to the* $C[0,1]$ *closure of* $\operatorname{span}\{1, x^{\gamma_1}, x^{\gamma_2}, \dots\}$ *can be represented as*

$$f(x) = \sum_{j=1}^{\infty} a_j x^{\gamma_j}\,, \qquad x \in [0,1)\,,$$

If the gap condition (2.1) does not hold, then every function $f \in C[0,1]$ *belonging to the* $C[0,1]$ *closure of* $\operatorname{span}\{1, x^{\gamma_1}, x^{\gamma_2}, \dots\}$ *can still be represented as an analytic function on*

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < 1\}$$

restricted to $(0,1)$*.*

Now we offer a sufficient condition for a sequence $(\beta_j)_{j=1}^{\infty}$ of distinct positive numbers converging to 0 to guarantee the non-denseness of $\operatorname{span}\{x^{\beta_1}, x^{\beta_2}, \dots\}$ in $C[0,1]$.

Theorem 2.3. *Suppose that* $(\beta_j)_{j=1}^{\infty}$ *is a sequence of distinct real numbers greater than* 0 *satisfying*

$$\sum_{j=1}^{\infty} \beta_j =: \eta < \infty \,.$$

Then $\operatorname{span}\{x^{\beta_1}, x^{\beta_2}, \dots\}$ *is not dense in* $C[0,1]$. *In addition, every function in the* $C[0,1]$ *closure of* $\operatorname{span}\{x^{\beta_1}, x^{\beta_2}, \dots\}$ *can be represented as an analytic function on* $\mathbb{C} \setminus (-\infty, 0]$ *restricted to* $(0,1)$.

Proof of Theorem 2.3. The theorem is a consequence of D. J. Newman's Markov-type inequality [Bo-Er3, Theorem 6.1.1 on page 276] (see also [Ne]). We state this as Theorems 2.4. Repeated applications of Theorem 2.4 with the substitution $x = e^{-t}$ imply that

$$\|Q(e^{-t}))^{(m)}\|_{[0,\infty)} \leq (9\eta)^m \|Q(e^{-t})\|_{[0,\infty)}\,,$$

$$m = 1, 2, \dots\,,$$

in particular

$$|(Q(e^{-t}))^{(m)}(0)| \leq (9\eta)^m \|Q(e^{-t})\|_{[0,\infty)}\,,$$

$$m = 1, 2, \dots\,,$$

for every $Q \in \operatorname{span}\{x^{\beta_1}, x^{\beta_2}, \dots\}$. By using the Taylor series expansion of $Q(e^{-t})$ around 0, we obtain that

$$|Q(z)| \leq c_1(K,\eta)\|Q\|_{[0,1]}\,, \qquad z \in K\,, \tag{2.1}$$

for every $Q \in \operatorname{span}\{x^{\beta_1}, x^{\beta_2}, \dots\}$ and for every compact $K \subset \mathbb{C} \setminus \{0\}$, where

$$c_1(K,\eta) := \sum_{m=0}^{\infty} \frac{(9\eta)^m \Big(\max_{z\in K} |\log z|\Big)^m}{m!}$$

$$= \exp\left(9\eta \max_{z\in K} |\log z|\right)$$

is a constant depending only on K and η.

Now (2.1) shows that if

$$Q_n \in \operatorname{span}\{x^{\beta_1}, x^{\beta_2}, \dots\}$$

converges in $C[0,1]$, then it converges uniformly on every compact $K \subset \mathbb{C} \setminus \{0\}$, and the theorem is proved. $\square$

The next result is a Markov-type inequality for Müntz polynomials due to Newman [Bo-Er3, Theorem 6.1.1 on page 276] (see also [Ne]).

Theorem 2.4. *Let $\beta_1, \beta_2, \dots, \beta_n$ be distinct non-negative numbers. Then*

$$\|xQ'(x)\|_{[0,1]} \leq 9 \left(\sum_{j=1}^{n} \beta_j \right) \|Q\|_{[0,1]}$$

for every $Q \in \operatorname{span}\{x^{\beta_1}, x^{\beta_2}, \dots, x^{\beta_n}\}$.

We will also need the bounded Bernstein-type inequality below (see [Bo-Er3, page 178]).

Theorem 2.5. *Suppose* $\Gamma := (\gamma_j)_{j=1}^{\infty}$ *is a sequence of distinct nonnegative numbers satisfying* $\sum_{j=1}^{\infty} 1/\gamma_j < \infty$ *. Then*

$$\|Q'\|_{[0,x]} \leq c(x,\Gamma)\|Q\|_{[0,1]}$$

for every $Q \in \operatorname{span}\{1, x^{\gamma_1}, x^{\gamma_2}, \dots\}$ *and for every* $x \in [0,1)$*, where* $c(x,\Gamma)$ *depends only on* x *and* Γ*).*

The following simple fact will also be needed.

Lemma 2.6. *Let* $U \subset C[0,1]$ *be a cosed linear subspace of* $C[0,1]$ *and let* $V \subset C[0,1]$ *be a finite dimensional (hence closed) linear subspace of* $C[0,1]$*. Then* $U + V$ *is closed.*

4. PROOF OF THEOREMS 1.1

Proof of Theorem 1.1. The first part of the theorem is contained in Theorem B, so we need to prove only the second part. Suppose $(\lambda_j)_{j=1}^{\infty}$ is a sequence of distinct positive numbers satisfying

$$\sum_{j=1}^{\infty} \frac{\lambda_j}{\lambda_j^2+1} < \infty\,.$$

Then there are positive numbers η, β_j, γ_j, and δ_j such that

$$\{\lambda_j : j=1,2,\dots\} =$$
$$\{\beta_j : j=1,2,\dots\} \cup \{\gamma_j : j=1,2,\dots\} \cup$$
$$\cup \{\delta_j : j=1,2,\dots,k\}\,,$$

where

$$\sum_{j=1}^{\infty} \beta_j \le \eta\,, \qquad \sum_{j=1}^{\infty} 1/\gamma_j < \infty\,,$$

and with $\Gamma := (\gamma_j)_{j=1}^{\infty}$ we have

$$c(\Gamma, 1/2) < \frac{36}{\eta}$$

($c(\Gamma,1/2)$ is defined in Theorem 2.1).

Let

$$H_\beta := \operatorname{span}\{x^{\beta_1}, x^{\beta_2}, \dots\}\,,$$

$$H_\gamma := \operatorname{span}\{1, x^{\gamma_1}, x^{\gamma_2}, \dots\}\,,$$

and

$$H_\delta := \operatorname{span}\{x^{\delta_1}, x^{\delta_2}, \dots, x^{\delta_k}\}\,.$$

Every $Q \in H_\beta + H_\gamma$ can be written as $Q = Q_\beta + Q_\gamma$ with some $Q_\beta \in H_\beta$ and $Q_\gamma \in H_\gamma$. First we show that there are constans C_β and C_γ depending only on H_β and H_γ, respectively, so that

$$\|Q_\beta\|_{[0,1]} \leq C_\beta \|Q\|_{[0,1]} \tag{3.1}$$

and

$$\|Q_\gamma\|_{[0,1]} \leq C_\gamma \|Q\|_{[0,1]} \tag{3.2}$$

for every $Q \in H_\beta + H_\gamma$. Suppose to the conrary that, say the first inequality fails. Then there are Müntz polynomials $Q_{\beta,n} \in H_\beta$ and $Q_{\gamma,n} \in H_\gamma$ so that

$$\|Q_{\beta,n}\|_{[0,1]} = 1\,, \qquad \|Q_{\gamma,n}\|_{[0,1]} = 1\,, \tag{3.3}$$

and

$$\lim_{n \to \infty} \|Q_{\beta,n} + Q_{\gamma,n}\|_{[0,1]} = 0\,. \tag{3.4}$$

Then by Theorem 2.4 $\{Q_{\beta,n} : n = 1, 2, \dots\}$ is a family of bounded, equi-continuous functions on $[1/3, 1]$, while $\{Q_{\gamma,n} : n = 1, 2, \dots\}$ is a family of bounded, equi-continuous functions on $[0, 2/3]$. So by the Arzela-Ascoli Theorem there are a subsequence of $(Q_{\beta,n})$ (without loss of generality we may assume that this is $(Q_{\beta,n})$ itself) and a subsequence of $(Q_{\gamma,n})$ (without loss of generality we may assume that this is $(Q_{\gamma,n})$ itself) so that

$$(3.5) \qquad \lim_{n\to\infty} \|Q_{\beta,n} - f\|_{[1/3,1]} = 0$$

and

$$(3.6) \qquad \lim_{n\to\infty} \|Q_{\gamma,n} - g\|_{[0,2/3]} = 0$$

with some continuous functions f and g on $[1, 3, 1]$ and $[0, 2/3]$, respectively. By (3.4), (3.5), and (3.6) we have $f = -g$ on $[1/3, 2/3]$, so the function

$$(3.7) \qquad h(x) := \begin{cases} f(x) & x \in [1/3, 1] \\ -g(x) & x \in [0, 2/3] \end{cases}$$

is well-defined. By (3.4) – (3.7) we can deduce that

$$(3.8) \qquad \lim_{n\to\infty} \|Q_{\beta,n} - h\|_{[0,1]} = 0$$

and

$$(3.9) \qquad \lim_{n\to\infty} \|Q_{\gamma,n} - h\|_{[0,1]} = 0$$

Using (3.3), (3.8), Theorem 2.4, and

$$\sum_{j=1}^{\infty} \beta_j \le \eta$$

we can deduce that

$$h(x) - h(1) \le 18\eta\,, \qquad x \in [1/2, 1]\,.$$

Note that (3.1) implies $\|h\|_{[0,1]} = 1$, and obviously $h(0) = 0$. Now observe that the function $h - h(1)$ is in the uniform closure of

$$H_\gamma = \operatorname{span}\{1, x^{\gamma_1}, x^{\gamma_2}, \dots\}\,,$$

hence Theorem 2.1 implies

$$\|h - h(1)\|_{[0,1]} \le c(\Gamma, 1/2)\, \|h - h(1)\|_{[1/2,1]} \le$$
$$\le c(\Gamma, 1/2)\, 18\eta < 1/2\,.$$

This contradicts the facts that

$$h(0) = 0 \qquad \text{and} \qquad \|h\|_{[0,1]} = 1\,.$$

Hence the proof of (3.1) is finished. The proof of (3.2) goes in the same way, so we omit it.

Let $\overline{H}$ denote the uniform closure of a subspace $H \subset C[0,1]$. We want to prove that

$$\overline{H_\beta + H_\gamma + H_\delta} \subset \mathcal{A}\,,$$

where $\mathcal{A} \subset C[0,1]$ denotes the collection of functions $f \in C[0,1]$, which can be represented as an analytic function on

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < 1\}$$

restricted to $(0,1)$. Since H_δ is finite dimensional, Theorem 2.6 implies that

$$\overline{H_\beta + H_\gamma + H_\delta} \subset \overline{H_\beta + H_\gamma} + H_\delta\,.$$

so it is sufficient to prove that

$$\overline{H_\beta + H_\gamma} \subset \mathcal{A} \tag{3.10}$$

However, (3.1) and (3.2) imply that

$$\overline{H_\beta + H_\gamma} \subset \overline{H_\beta} + \overline{H_\gamma}\,,$$

where $\overline{H_\beta} \subset \mathcal{A}$ by Theorem 2.3 and $\overline{H_\gamma} \subset \mathcal{A}$ by Theorem 2.2. Hence (3.10) holds, indeed, and the proof of the theorem is finished. $\square$