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Simon Brendle, Stanford University, $1/4$-*pinched manifolds are space forms*

In this lecture, I will present the recent proof, joint with Richard Schoen, of the differentiable sphere theorem for manifolds with $1/4$-pinched sectional curvatures. Our main result is:

Theorem 1. *Let (M, g_0) be a compact Riemannian manifold of dimension $n \geq 4$. Assume that (M, g_0) has $1/4$-pinched sectional curvatures in the sense that $0 < K(\pi_1) < 4\,K(\pi_2)$ for all two-planes $\pi_1, \pi_2 \subset T_pM$. Then the normalized Ricci flow with initial metric g_0 exists for all time, and converges to a constant curvature metric as $t \to \infty$.*

The strict inequality in the pinching condition can be replaced by a weak one if we assume, in addition, that (M, g_0) is not isometric to a rank-one symmetric space:

Theorem 2. *Let (M, g_0) be a compact, simply connected Riemannian manifold of dimension $n \geq 4$. Assume that (M, g_0) has weakly $1/4$-pinched sectional curvatures in the sense that $0 < K(\pi_1) \leq 4\,K(\pi_2)$ for all two-planes $\pi_1, \pi_2 \subset T_pM$. If (M, g_0) is non-symmetric, then the normalized Ricci flow with initial metric g_0 exists for all time, and converges to a constant curvature metric as $t \to \infty$. In particular, M is diffeomorphic to S^n.*

Theorem 1 is a subcase of a more general convergence result for the Ricci flow in higher dimensions. More precisely, suppose that $(M, g(t))$ is a family of metrics evolving under the normalized Ricci flow. If the product $(M, g(t)) \times \mathbb{R}$ has positive isotropic curvature for $t = 0$, then this remains so for all $t \geq 0$, and $(M, g(t))$ converges to a constant curvature metric as $t \to \infty$.