

Texas Geometry and Topology Conference
Meeting 38. Texas A&M University, October 19-21, 2007

Michael Eastwood, University of Adelaide, *The X-ray transform on complex projective space*

The classical Radon transform takes a function on the plane and integrates it over the straight lines in the plane:–

$$f \xrightarrow{\mathcal{R}} \left(L \mapsto \int_L f \right) \quad \text{for } L \text{ a line in } \mathbb{R}^2.$$

Its invertibility provides the mathematical basis of modern medical imaging techniques. The X-ray transform performs a similar task in n -space

$$f \xrightarrow{\mathcal{X}} \left(L \mapsto \int_L f \right) \quad \text{for } L \text{ a line in } \mathbb{R}^n,$$

the terminology being motivated by medical imaging. In both cases, the function f should decay sufficiently at infinity in order that these integrals make sense. As one might expect, these transforms are best viewed on real projective space. Specifically, if we view $\mathbb{R}^n \hookrightarrow \mathbb{RP}_n$ as a standard affine patch, then (with some fudge factors) we can extend the X-ray transform to

$$C^\infty(\mathbb{RP}_n) \ni f \longmapsto \mathcal{X}f \in C^\infty(\text{Gr}_2(\mathbb{R}^{n+1})),$$

the Grassmannian on the right hand side arising as the space of geodesics on \mathbb{RP}_n with its usual round metric. This is a more congenial formulation, automatically taking care of the decay conditions at infinity. When $n = 2$, Funk showed that this transform is, in fact, an isomorphism

$$\mathcal{X} : C^\infty(\mathbb{RP}_2) \xrightarrow{\cong} C^\infty(\text{Gr}_2(\mathbb{R}^3)) = C^\infty(\mathbb{RP}_2^*).$$

More generally, the X-ray transform on real projective space is injective.

In this talk, I shall discuss what happens on complex projective space

$$C^\infty(\mathbb{CP}_n) \ni f \longmapsto \left(\gamma \mapsto \int_\gamma f \right) \quad \text{for } \gamma \text{ a Fubini-Study geodesic in } \mathbb{CP}_n.$$

This transform is easily seen to be injective. More interesting are the X-ray transforms on symmetric covariant tensors:–

$$\mathcal{X} : \Gamma(\mathbb{CP}_n, \odot^k \Lambda^1) \ni \omega \longmapsto \left(\gamma \mapsto \int_\gamma \omega \right).$$

If $\omega_{ab\dots c} = \nabla_{(a} \phi_{b\dots c)}$, then the field $\phi_{b\dots c}$ is said to be a ‘potential’ for $\omega_{ab\dots c}$. If ω has a potential, then $\mathcal{X}\omega = 0$. Conversely, joint work with Hubert Goldschmidt shows that if $\mathcal{X}\omega = 0$, then ω has a potential.