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By a celebrated work of Calabi's every germ of holomorphic isometry of a simply-connected Kähler manifold into the projective space $\mathbb{P}^n$ extends to a global isometry. Furthermore, any local holomorphic isometry of a Hermitian symmetric manifold X of the compact type into the projective space $\mathbb{P}^n$ is equivariant with respect to the isometry group of X. For instance, any local holomorphic isometry between projective spaces equipped with Fubini-Study metrics must be congruent to a Veronese embedding.

For germs of holomorphic isometeric immersions between bounded symmetric domains it is commonly believed that the situation is even more rigid. For instance, using the notion of the *diastasis* introduced by Calabi, Umehara proved that any local holomorphic isometry between complex unit balls equipped with the Bergman metric must be totally geodesic. However, since a higher-rank bounded symmetric domain *cannot* be realized as a Kähler submanifold of the complex unit ball, the general problem remained unresolved.

More generally, let $f : D \to D'$ be a germ of holomorphic isometry up to a normalizing constant between two bounded domains equipped with the Bergman metric. We posed the question of characterizing such maps and of finding conditions which force the map to be totally geodesic. The special case where D is the unit disk and D' is a polydisk was studied by Clozel-Ullmo in connection to a problem in Arithmetic Geometry. There first of all they proved that f extends to an algebraic map by making use of functional identities arising from equating potential functions of Kähler metrics.

We have now developed a general method for the analytic continuation of germs of holomorphic isometries. Starting with the same functional identities and polarizing we obtain in the general situation an infinite number of holomorphic identities, and the first question is to determine whether the functional identities are sufficiently nondengenerate to force analytic continuation. We solve this problem by studying deformations of solutions of the holomorphic functional identities, and force analytic continuation by showing that, in the event that there are nontrivial deformations, the germ of holomorphic isometry must take values on linear sections of the embedding of the domain into the infinite-dimensional projective space $\mathbb{P}^\infty$.

The linear sections that we obtain correspond to extremal functions with respect to the Bergman metric. In the event that the Bergman kernel function $K(z,\overline{w})$ is rational in $(z,\overline{w})$, as is the case for bounded symmmetric domains, they yield algebraic equations satisfied by the germ of map, forcing analytic continuation to a proper algebraic map.

Between certain bounded symmetric domains we have now produced examples of Kähler embeddings (i.e., holomorphic isometric embeddings) which are *not* totally geodesic. The simplest examples, which disprove a Conjecture of Clozel-Ullmo's, are Kähler embeddings of the unit disk into polydisks whose algebraic extensions develop branch points on the unit circle. However, any Kähler embedding of the unit disk into a bounded symmetric domain must be aymtotically totally geodesic at a general point of the unit circle. As a consequence, there are no exotic Kähler embeddings between bounded symmetric domains equivariant with respect to a lattice Γ, and any Kähler embedding between two bounded symmetric domains equivariant with respect to the automorphism group of the domain must be totally geodesic.