Texas Geometry and Topology Conference

This is a report on the presentations at the 38th meeting of the Texas Geometry and Topology Conference at the Texas A&M University, October 19-October 21, 2007. This conference was partially supported by National Science Foundation Grant DMS-0605082 and Texas A&M University. Speakers reported on recent research. For this report, speakers have provided synopses of their talks together with broader discussions of the significance and context of their results.

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Simon Brendle, Stanford University, $1/4$-*pinched manifolds are space forms*

In this lecture, I will present the recent proof, joint with Richard Schoen, of the differentiable sphere theorem for manifolds with $1/4$-pinched sectional curvatures. Our main result is:

Theorem 1. *Let (M, g_0) be a compact Riemannian manifold of dimension $n \geq 4$. Assume that (M, g_0) has $1/4$-pinched sectional curvatures in the sense that $0 < K(\pi_1) < 4\,K(\pi_2)$ for all two-planes $\pi_1, \pi_2 \subset T_pM$. Then the normalized Ricci flow with initial metric g_0 exists for all time, and converges to a constant curvature metric as $t \to \infty$.*

The strict inequality in the pinching condition can be replaced by a weak one if we assume, in addition, that (M, g_0) is not isometric to a rank-one symmetric space:

Theorem 2. *Let (M, g_0) be a compact, simply connected Riemannian manifold of dimension $n \geq 4$. Assume that (M, g_0) has weakly $1/4$-pinched sectional curvatures in the sense that $0 < K(\pi_1) \leq 4\,K(\pi_2)$ for all two-planes $\pi_1, \pi_2 \subset T_pM$. If (M, g_0) is non-symmetric, then the normalized Ricci flow with initial metric g_0 exists for all time, and converges to a constant curvature metric as $t \to \infty$. In particular, M is diffeomorphic to S^n.*

Theorem 1 is a subcase of a more general convergence result for the Ricci flow in higher dimensions. More precisely, suppose that $(M, g(t))$ is a family of metrics evolving under the normalized Ricci flow. If the product $(M, g(t)) \times \mathbb{R}$ has positive isotropic curvature for $t = 0$, then this remains so for all $t \geq 0$, and $(M, g(t))$ converges to a constant curvature metric as $t \to \infty$.

Jean-Pierre Demailly, Université de Grenoble, *On compact complex manifolds covered by a torus*

The purpose of the talk is to show that if a compact complex manifold is a surjective image of a torus, then it is Kähler, and up to a finite étale cover, it is a product of projective spaces by a torus (joint work with J. M. Hwang and Th. Peternell). The proof relies on earlier results of Hwang-Mok, and on techniques developed during the 90's in the study of compact Kähler manifolds with nef tangent bundles.

Michael Eastwood, University of Adelaide, *The X-ray transform on complex projective space*

The classical Radon transform takes a function on the plane and integrates it over the straight lines in the plane:–

$$f \overset{\mathcal{R}}{\longmapsto} \Big(L \mapsto \int_L f\Big) \quad \text{for } L \text{ a line in } \mathbb{R}^2.$$

Its invertibility provides the mathematical basis of modern medical imaging techniques. The X-ray transform performs a similar task in n-space

$$f \overset{\mathcal{X}}{\longmapsto} \Big(L \mapsto \int_L f\Big) \quad \text{for } L \text{ a line in } \mathbb{R}^n,$$

the terminology being motivated by medical imaging. In both cases, the function f should decay sufficiently at infinity in order that these integrals make sense. As one might expect, these transforms are best viewed on real projective space. Specifically, if we view $\mathbb{R}^n \hookrightarrow \mathbb{RP}_n$ as a standard affine patch, then (with some fudge factors) we can extend the X-ray transform to

$$C^\infty(\mathbb{RP}_n) \ni f \longmapsto \mathcal{X}f \in C^\infty(\mathrm{Gr}_2(\mathbb{R}^{n+1})),$$

the Grassmannian on the right hand side arising as the space of geodesics on $\mathbb{RP}_n$ with its usual round metric. This is a more congenial formulation, automatically taking care of the decay conditions at infinity. When $n = 2$, Funk showed that this transform is, in fact, an isomorphism

$$\mathcal{X} : C^\infty(\mathbb{RP}_2) \xrightarrow{\simeq} C^\infty(\mathrm{Gr}_2(\mathbb{R}^3)) = C^\infty(\mathbb{RP}_2^*).$$

More generally, the X-ray transform on real projective space is injective.

In this talk, I shall discuss what happens on complex projective space

$$C^\infty(\mathbb{CP}_n) \ni f \longmapsto \Big(\gamma \mapsto \int_\gamma f\Big) \quad \text{for } \gamma \text{ a Fubini-Study geodesic in } \mathbb{CP}_n.$$

This transform is easily seen to be injective. More interesting are the X-ray transforms on symmetric covariant tensors:–

$$\mathcal{X} : \Gamma(\mathbb{CP}_n, \textstyle\bigodot^k \Lambda^1) \ni \omega \longmapsto \Big(\gamma \mapsto \displaystyle\int_\gamma \omega\Big).$$

If $\omega_{ab\cdots c} = \nabla_{(a}\phi_{b\cdots c)}$, then the field $\phi_{b\cdots c}$ is said to be a 'potential' for $\omega_{ab\cdots c}$. If ω has a potential, then $\mathcal{X}\omega = 0$. Conversely, joint work with Hubert Goldschmidt shows that if $\mathcal{X}\omega = 0$, then ω has a potential.

Yakov Eliashberg, Stanford University, ***Orderability of groups of contact transformations and related topics***

There is a natural candidate for a partial order on the universal cover $G(M, \xi)$ of the group of contact diffeomorphisms of a contact manifold (M, ξ). Namely, we say that $f > g$ if fg^{-1} can be generated by a positive Hamiltonian function. However, it is not clear whether this order is non-trivial. Examples of *orderable* contact manifolds, e.g. manifolds for which this order is non-trivial were given by A. Givental in [3] (odd-dimensional real projective spaces) and by L. Polterovich and the author in [2] (the unit cotangent bundle of a torus). In a more recent paper [1] of S.-S. Kim, L. Polterovich and the author there were discovered a class of non-orderable contact manifolds which included standard contact spheres. It was also shown that the orderability problem is tightly related with many other interesting contact-geometric problems, and in particular, with a contact analog of Gromov's famous non-squeezing theorem (1985) which states that the standard symplectic ball cannot be symplectically squeezed into the cylinder of smaller radius. It was shown in this paper that the contact non-squeezing phenomenon exists on large scales, but it disappears on small scales. In the talk there will be discussed these results as well as more recent progress in this direction, in particular by I. Milin and G. Ben-Simon. We will also discuss connections with other problems, such as existence of quasi-morphisms on $G(M, \xi)$.

References

[1] Y. Eliashberg and L. Polterovich, Partially ordered groups and geometry of contact transformations, *GAFA*, 1 (2000), 1448–1476.

[2] Y. Eliashberg, S.-S. Kim, L. Polterovich, Geometry of contact transformations: orderability versus squeezing, *Geom. and Topol.*, 10 (2006), 1635–1748.

[3] A. Givental, Nonlinear generalization of the Maslov index, in *Theory of singularities and its applications*, pp. 71-103, Adv. Soviet Math., 1, Amer. Math. Soc., Providence, RI, 1990.

Dan Freed, University of Texas at Austin, *Remarks on topological field theory*

Topological field theories were introduced in the late 1980s. It was quickly realized that some invariants in low dimensional topology fit into this structure, and new ones were introduced. Recently there have been advances in understanding the structure of such theories in low dimensions. I will report on some of these results organized broadly into two categories: generators/relations and a priori constructions. The main example to be discussed is Chern-Simons theory.

Bruce Kleiner, Columbia University, *BiLipschitz embedding in Banach spaces*

A mapping between metric spaces is L-*biLipschitz* if it stretches distances by a factor of at most L, and compresses them by a factor no worse than $1/L$. A basic problem in geometric analysis is to determine when one metric space can be bi-Lipschitz embedded in another, and if so, to estimate the optimal bi-Lipschitz constant. In recent years this question has generated great interest in computer science, since many data sets can be represented as metric spaces, and associated algorithms can be simplified, improved, or estimated, provided one knows that the metric space space in question can be biLipschitz embedded (with controlled bi-Lipschitz constant) in a nice space, such as L^2 or L^1.

The lecture will discuss several new existence and non-existence results for bi-Lipschitz embeddings in Banach spaces. One approach to non-existence theorems is based on generalized differentiation theorems in the spirit of Rademacher's theorem on the almost everywhere differentiability of Lipschitz functions on $\mathbb{R}^n$. We first show that earlier differentiation based results of Pansu and Cheeger, which proved non-existence of embeddings into $\mathbb{R}^k$, generalize to many Banach space targets, such as L^p for $1 < p < \infty$. We then focus on the case when the target is L^1, where differentiation theory is known to fail, and the embedding questions are of particular interest in computer science. When the domain is the Heisenberg group with its Carnot-Caratheodory metric, we show that a modified form of differentiation still holds for Lipschitz maps into L^1, by exploiting a new connection with functions of bounded variation, and some very recent advances in geometric measure theory. This leads to a proof of a conjecture of Assaf Naor.

This is joint work with Jeff Cheeger.

Ngaiming Mok, University of Hong Kong, *Analytic continuation of local holomorphic maps isometric with respect to the Bergman metric*

By a celebrated work of Calabi's every germ of holomorphic isometry of a simply-connected Kähler manifold into the projective space $\mathbb{P}^n$ extends to a global isometry. Furthermore, any local holomorphic isometry of a Hermitian symmetric manifold X of the compact type into the projective space $\mathbb{P}^n$ is equivariant with respect to the isometry group of X. For instance, any local holomorphic isometry between projective spaces equipped with Fubini-Study metrics must be congruent to a Veronese embedding.

For germs of holomorphic isometeric immersions between bounded symmetric domains it is commonly believed that the situation is even more rigid. For instance, using the notion of the *diastasis* introduced by Calabi, Umehara proved that any local holomorphic isometry between complex unit balls equipped with the Bergman metric must be totally geodesic. However, since a higher-rank bounded symmetric domain *cannot* be realized as a Kähler submanifold of the complex unit ball, the general problem remained unresolved.

More generally, let $f : D \to D'$ be a germ of holomorphic isometry up to a normalizing constant between two bounded domains equipped with the Bergman metric. We posed the question of characterizing such maps and of finding conditions which force the map to be totally geodesic. The special case where D is the unit disk and D' is a polydisk was studied by Clozel-Ullmo in connection to a problem in Arithmetic Geometry. There first of all they proved that f extends to an algebraic map by making use of functional identities arising from equating potential functions of Kähler metrics.

We have now developed a general method for the analytic continuation of germs of holomorphic isometries. Starting with the same functional identities and polarizing we obtain in the general situation an infinite number of holomorphic identities, and the first question is to determine whether the functional identities are sufficiently nondengenerate to force analytic continuation. We solve this problem by studying deformations of solutions of the holomorphic functional identities, and force analytic continuation by showing that, in the event that there are nontrivial deformations, the germ of holomorphic isometry must take values on linear sections of the embedding of the domain into the infinite-dimensional projective space $\mathbb{P}^\infty$.

The linear sections that we obtain correspond to extremal functions with respect to the Bergman metric. In the event that the Bergman kernel function $K(z, \overline{w})$ is rational in $(z, \overline{w})$, as is the case for bounded symmmetric domains, they yield algebraic equations satisfied by the germ of map, forcing analytic continuation to a proper algebraic map.

Between certain bounded symmetric domains we have now produced examples of Kähler embeddings (i.e., holomorphic isometric embeddings) which are *not* totally geodesic. The simplest examples, which disprove a Conjecture of Clozel-Ullmo's, are Kähler embeddings of the unit disk into polydisks whose algebraic extensions develop branch points on the unit circle. However, any Kähler embedding of the unit disk into a bounded symmetric domain must be aymtotically totally geodesic at a general point of the unit circle. As a consequence, there are no exotic Kähler embeddings between bounded symmetric domains equivariant with respect to a lattice Γ, and any Kähler embedding between two bounded symmetric domains equivariant with respect to the automorphism group of the domain must be totally geodesic.

Natasa Sesum, Columbia University, *Compactness results for the Kähler-Ricci flow*

We consider the Kähler-Ricci flow $\frac{\partial}{\partial t} g_{i\bar{j}} = g_{i\bar{j}} - R_{i\bar{j}}$ on a compact Kähler manifold M with $c_1(M) > 0$, of complex dimension k. We prove the ϵ-regularity lemma for the Kähler-Ricci flow, based on Moser's iteration. Assume that $|\mathrm{Ric}|$ and $\int_M |\mathrm{Rm}|^k \, dV_t$ are uniformly bounded along the flow. Using the ϵ-regularity lemma we derive the compactness result for the Kähler-Ricci flow. Under our assumptions, if $k \geq 3$ in addition, using the compactness result we show that $|\mathrm{Rm}| \leq C$ holds uniformly along the flow. This means the flow does not develop any singularities at infinity. We use some ideas of Tian from to prove the smoothing property in that case.

To be more precise we first prove the following result.

Theorem 1. *Let $g(t)$ be the Kähler-Ricci flow on a compact, Kähler manifold M, with $c_1(M) > 0$, with $\int_M |\mathrm{Rm}(g(t))|^{n/2} dV_{g(t)} \leq C$ and $|\mathrm{Ric}| \leq C$ along the flow. Then for every sequence $t_i \to \infty$ there is a subsequence so that $(M, g(t_i + t))$ converges to $(M_\infty, g_\infty(t))$, where*

(a) M_∞ is an orbifold with finitely many isolated singular points, $\{p_1, \ldots, p_N\}$, and the convergence is smooth outside those singular points.

(b) The limit metric g_∞ is a Kähler-Ricci soliton in an orbifold sense, that is, satisfies the Kähler-Ricci soliton equation,

$$\begin{aligned} (g_\infty)_{i\bar{j}} - R_{i\bar{j}}(g_\infty) &= \partial_i \partial_{\bar{j}} f_\infty, \qquad (1) \\ \partial_{\bar{i}} \partial_{\bar{j}} f_\infty &= 0, \end{aligned}$$

off the singular points. Moreover, for every singular point p_j, there is a neighbourhood in M_∞ which lifts to an open set $D_j \subset \mathrm{C}^{n/2}$, and the lifting of an orbifold metric g_∞ satisfies equivalent equations to (1) in D_j.

By using the previous compactness theorem for the Kähler-Ricci flow we prove the following.

Theorem 2. *Let $g(t)$ be a Kähler-Ricci flow on a compact, Kähler manifold M of complex dimension k ($n = 2k$), with $c_1(M) > 0$ and with $\int_M |\mathrm{Rm}(g(t))|^{n/2} dV_{g(t)} \leq C$ along the flow. Then if $k \geq 3$, the curvature operator is uniformly bounded along the flow.*

Theorem 2 tells us the blow up of curvature does not happen at infinity which means we could potentially talk about smooth limits of the flow.