Texas Geometry and Topology Conference

This is a report on the presentations at the 35th meeting of the Texas Geometry and Topology Conference at the University of Houston, February 17-February 19, 2006. This meeting was a special themed conference in memory of Professor S. S. Chern. This conference was partially supported by National Science Foundation Grant DMS-0306628 and The University of Houston. Speakers reported on recent research. For this report, speakers have provided synopses of their talks together with broader discussions of the significance and context of their results.

Meeting 35. University of Houston, February 17-February 19, 2006

Robert L. Bryant, Duke University, *Finsler 2-spheres of constant curvature*

This talk concerns Finsler 2-spheres with constant Finsler-Gauss curvature and all geodesics closed.

First, I review Cartan's structure equations [2] for Finsler surfaces and discuss the meanings of his curvature invariants I, J, and K. It is the function K that generalizes the usual Gauss curvature for a surface to the Finsler-Gauss curvature for a Finsler surface.

Second, I describe some examples of Katok [3] that were recently shown by Shen [4] to have constant curvature.

Finally, I show that each Finsler 2-sphere with constant $K = 1$ lies in a 10-parameter family of such Finsler spheres, all of which have the same underlying path geometry. I.e., all of the Finsler structures in the family have the same (oriented) geodesics, up to reparametrization.

In fact, I show that the moduli space of such Finsler structures naturally carries the structure of a 5-dimensional complex manifold.

This generalizes the projectively flat case, which was already studied in [1].

References

[1] R. Bryant, *Projectively flat Finsler 2-spheres of constant curvature*, Selecta Math., New Series **3** (1997), 161–204. MR 98i:53101

[2] É. Cartan, *Sur un problème d'équivalence et la théorie des espaces métriques généralisés*, Mathematica **4** (1930), 114–136. (Reprinted in *Oeuvres Complètes*, partie III, vol. 2, Éditions du CNRS, 1984.)

[3] A. Katok, *Ergodic properties of degenerate integrable Hamiltonian systems*, Izs. Akad. Nauk SSSR Ser. Math. **37** (1973), 539–576. (English translation: Math. USSR Izv. **7** (1973), 535–572.) MR 48 #9758

[4] Z. Shen, *Two-dimensional Finsler metrics with constant flag curvature*, Manuscripta Math. **109** (2002), 349–366. MR 03k:53091

Brendan Hassett, Rice University, *Compact moduli spaces for surfaces of general type*

We will survey compactifications of moduli spaces of varieties of general type, as constructed using techniques from birational geometry. The stable curves of Deligne and Mumford generalize to arbitrary dimension, provided certain standard conjectures of birational geometry are valid. Moduli spaces of stable surfaces were constructed about ten years ago by Alexeev, Kollár, Shepherd-Barron, Viehweg and others. Functorial descriptions of the moduli space have only recently been completed by Kovács, Hacking, Abramovich, and the speaker. Analogs to moduli spaces of pointed stable curves are more elusive. However, Hacking, Keel, Tevelev, and others have made important recent contributions to our understanding of these spaces.

A good functorial compactification of the moduli space of stable surfaces would likely have significant applications to enumerative geometry, intersection theory, and higher-dimensional analogs of Gromov-Witten theory. It remains an open problem to construct a compactification admitting a virtual fundamental class.

Shanyu Ji, University of Houston, *Rigidity problems for proper holomorphic maps between balls*

In this talk, we should survey some problems in Complex Geometry and Complex Analysis in which Chern was interested and was working, and it includes equivalence problems of domains in the complex spaces and rigidity properties of proper holomorphic mappings between balls as well as some of their recent developments.

The rigidity problems for proper holomorphic mappings between domains (equidimensional or non equidimensional) are very fundamental ones, which were originated from Riemann and Poincaré, and were laid down the foundation by E. Cartan, Chern and Moser's work. It includes many basic problems such as classification, or characterization of two equivalence maps.

Dan Knopf, The University of Texas at Austin, *Ricci flow with surgery, as applied to Poincaré*

A natural question in Riemannian geometry is whether a smooth closed manifold $\mathcal{M}^n$ can be assigned a Riemannian metric that is in some sense optimal or canonical. The Ricci flow introduced by Richard Hamilton addresses this question by constructing a smooth one-parameter family of metrics $g(t)$ starting from an arbitrary initial metric g_0. Heuristically, one hopes that $g(t)$ improves the shape of $\mathcal{M}^n$ given by g_0. For example, the metric of a closed surface $\mathcal{M}^2$ evolving by the normalized Ricci flow

$$\frac{\partial}{\partial t} g(x,t) = \left(R_{\text{ave}}(t) - R(x,t)\right)\, g(x,t)$$

converges to a metric g_∞ of constant scalar curvature

$$r = \frac{4\pi\chi(\mathcal{M}^2)}{\text{Area}(\mathcal{M}^2, g_0)},$$

whose existence is guaranteed by the classical Uniformization Theorem. In this case, the canonical metric g_∞ yields a topological classification of the surface $\mathcal{M}^2$.

Recently, Grisha Perelman has made dramatic progress in Hamilton's program to use Ricci flow to address two of the most important questions in 3-manifold topology, the Poincaré and Geometrization Conjectures for closed 3-manifolds.

Abstract. In the Hamilton–Perelman program to resolve the Poincaré and Geometrization Conjectures by Ricci flow, it is necessary to consider Ricci flow with surgery. A central goal of this program is to construct an algorithm that assigns to the initial data $(\mathcal{M}_0^3, g_0)$ a discrete sequence of surgery times

$$0 = T_0^- < T_0^+ = T_1^- < T_1^+ < \cdots < T \leq \infty$$

and smooth solutions

$$(\mathcal{M}_k^3, g(t) : T_k^- \leq t < T_k^+)$$

of Ricci flow with certain properties that make it possible to track the geometric and topological changes made to $\mathcal{M}_0^3$.

In this talk, I will introduce the Hamilton–Perelman program and describe its geometric-topological surgery algorithm.

References

[1] **Colding, Tobias H.; Minicozzi, William P., II.** Estimates for the extinction time for the Ricci flow on certain 3-manifolds and a question of Perelman. *J. Amer. Math. Soc.* **18** (2005), no. 3, 561–569.

[2] **Hamilton, Richard S.** The formation of singularities in the Ricci flow. *Surveys in differential geometry, Vol. II* (Cambridge, MA, 1993), 7–136, Internat. Press, Cambridge, MA, 1995.

[3] **Perelman, Grisha.** The entropy formula for the Ricci flow and its geometric applications. `arXiv:math.DG/0211159`.

[4] **Perelman, Grisha.** Ricci flow with surgery on three-manifolds. `arXiv:math.DG/0303109`.

[5] **Perelman, Grisha.** Finite extinction time for the solutions to the Ricci flow on certain three-manifolds. `arXiv:math.DG/0307245`.

Zhongmin Shen, NSF and IUPUI, *Randers metrics of scalar flag curvature*

Randers metrics are special Finsler metrics including Riemannian metrics. This is a rich class of Finsler metrics with many non-Riemannian properties. I will introduce Randers metrics from two different approaches: Hamilton-Jacobi equations on a compact domain and the navigation problem in a windy Riemannian space.

The positive viscosity solution of a Hamilton-Jacobi equation on a domain is the distance function from the boundary. Thus it is natural to study the relationship between the distance function and geometric quantities on a compact Finsler manifold with boundary.

The navigation problem is to find the shortest time path under the influence of an external force. This study leads to the discovery of some Randers metrics of constant flag curvature and eventually the classification of these metrics. The examples and classification enrich Finsler geometry.

Yum-Tong Siu, Harvard University, *Multiplier ideals – a new technique linking analysis and algebraic geometry*

A multiplier is a function such that local a priori estimates for partial differential equations hold only after the test function is multiplied by it. The ideal sheaf consisting of multipliers identifies the location and the jet orders where local a priori estimates fail to hold. Solvability of a partial differential equation is reduced to algebraic conditions which force the multiplier ideal sheaf to be the structure sheaf. On the side of analysis, the method of multiplier ideal sheaves has been applied to problems such as the global regularity problem of the complex Neumann equation on pseudoconvex domains and the existence of Kaehler-Einstein metrics of Fano manifolds. On the side of algebraic geometry, the method of multiplier ideal sheaves has been successfully applied to solving, or making substantial progress towards solving, a number of long outstanding problems in algebraic geometry such as the Fujita conjecture, the effective Matsusaka big theorem, the deformational invariance of plurigenera, and the finite generation of canonical rings.

Shoucheng Zhang, Stanford University, *Chern numbers and Chern–Simons terms in physics*

In this talk I shall review the Chern-Simons theory of the quanum Hall effect and the importance of the Chern number in characterizing topological properties of physical systems. The quantum Hall effect is physically realized in a strongly interacting quantum many-body system, and it has many interesting emergent properties such as fractional charge and fractional statistics. These emergent properties are best described by an effective field theory, which uses the Chern-Simons term in mathematics in a crucial way. In physics, one is also interested in the question of how to classify physical states according to their topological properties, some of which can be directly measured in experiments. Here the Chern numbers play a crucial role in the classification.

Through this talk, I hope to initiate an intimate communication between physics and mathematics. The timeless achievements of the late Prof. S. S Chern inspire us to search for deeper understanding of the physical world in terms of beautiful mathematical concepts and principles.