Texas Geometry and Topology Conference

This is a report on the presentations at the 43nd meeting of the Texas Geometry and Topology Conference at Texas Christian University, February 26-28, 2010. This conference was partially supported by National Science Foundation Grant DMS-0904481 and Texas Christian University. Speakers reported on recent research. For this report, speakers have provided synopses of their talks together with broader discussions of the significance and context of their results.

Meeting 43. Texas Christian University, February 26-28, 2010

Jim Anderson, University of Southampton, *Small Filling Sets of Curves on a Surface*

This talk is joint work with Hugo Parlier and Alexandra Pettet.

We study the size of a collection of simple closed curves on a closed, orientable surface of genus $g \geq 2$ which fill the surface and pairwise intersect at most K times for some fixed $K \geq 1$. In particular, we give upper and lower bounds on the minimum numbers of curves in such a collection.

The original motivation for this work came from considering fillings sets of systoles on hyperbolic surfaces. Thurston suggested that the set X_g of surfaces in the Teichmüller space of closed, orientable surfaces of genus $g \geq 2$ which admit a filling set of systoles might form a mapping class group invariant spine; here, a systole is a closed geodesic of shortest length on a surface. However still very little is known about this set of surfaces, and whether for instance it is actually contractible remains to be shown.

A straightforward argument shows that the curves in a filling set of systoles pairwise intersect at most once. Hence the sets of curves that we consider are a topological generalization of the case of systoles. An implication of our work (Theorems 1 and 3) is that the topological condition that a set of curves pairwise intersect at most once is quite far from the geometric condition that a set of curves can arise as systoles.

Our main result is Theorem 1.

Theorem 1. *Let S be a closed, orientable surface of genus $g \geq 2$. If n is the number of elements in a filling set of simple closed curves which pairwise intersect at most $K \geq 1$ times, then n satisfies*

$$g \leq \frac{K(n^2 - n) + 2}{4}$$

If N is the smallest integer satisfying this inequality, then there exists a set of no more than $N + 1$ such curves on S.

Note that our theorem is only really interesting where g is much larger than K. We have the resulting corollary:

Corollary 2. *Let S be a closed, orientable surface of genus $g \geq 2$. Given $K \geq 1$, the number n of curves in a smallest filling set of simple closed curves which pairwise intersect at most K times satisfies $n \sim \frac{2\sqrt{g}}{\sqrt{K}}$. As a consequence,*

$$\lim_{g\to\infty} \frac{n}{\sqrt{g}} = \frac{2}{\sqrt{K}}$$

The next result says that the number of systoles in a smallest filling set grows with g at a rate of order strictly greater than $\sqrt{g}$.

Theorem 3. *Let S be a closed, orientable hyperbolic surface of genus $g \geq 2$ with a filling set of systoles $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$. Then*

$$n \geq \pi \frac{\sqrt{g(g-1)}}{\log(4g-2)}.$$

Furthermore there exist hyperbolic surfaces with filling sets of $n \leq 2g$ systoles.

The proof of Theorem 1 uses an Euler characteristic argument to obtain a lower bound on the number in a filling set of curves which pairwise intersect at most K times. To find an upper bound, we give a construction of such a set of curves whose cardinality is at most one larger than the lower bound. As it must work for all g and satisfy this small cardinality condition, this construction is rather cumbersome to describe, and will only be addressed if time allows.

To illustrate the difference between the topological condition of a filling set of curves that intersect pairwise at most once and a filling set of systoles, we produce a family of surfaces of such small sets of curves whose growth rate, though larger, still has order $\sqrt{g}$. We also give an explicit proof that these sets cannot be realized as systoles.

Scott Baldridge, Louisiana State University, *Cube Diagrams: The Topology and Geometry of Knots in* 3*- Dimensions*

In this talk we introduce the idea of a cube diagram of a knot. In 1926, Kurt Reidemeister described three essential "moves" on 2-dimensional knot projections and proved that two knots were equivalent if a knot projection of the first knot could be transformed into a knot projection of the second knot using applications of the three moves. Reidemeister's Theorem is useful because any computation that is invariant under these three moves is an invariant of the knot.

Reidemeister's result is inherently two dimensional: the full geometry of the knot as it sits in 3-dimensional space cannot be captured with 2-dimensional knot projections of the knot. Therefore it would be valuable to have a similar theorem for knots in 3-dimensional space that used a limited set of 3-dimensional moves. Last year I and my graduate student, Adam Lowrance, defined a new way to represent a knot in 3-dimensional space called a cube diagram and proved that any two cube diagrams of the same knot could be transformed into each other using only five cube diagram moves.

A *cube diagram* is a special piecewise-linear embedding of a knot into a 3-dimensional integral Cartesian grid. Intuitively, a cube diagram can be thought of as an embedding of a knot in an $n \times n \times n$ cube (using xyz-coordinates) such that the projections of the cube to each axis plane ($x = 0$, $y = 0$ and $z = 0$) are special 2-dimensional knot projections. A *cube diagram move* is a special ambient isotopy of a knot that takes one cube diagram to another cube diagram.

In creating cube diagrams, we were motivated by the search for a 3-dimensional data structure that was rigid enough to be able to easily define new invariants, yet robust enough to represent all knots and flexible enough that only a few types of moves were needed to transform one cube diagram of a knot to any other cube diagram of the same knot. That study lead us to put strong conditions and symmetries on how the knot was embedded in 3-dimensional space, so strong in fact that the types and number of possible ambient isotopy moves become severely limited. Thus, the conditions imposed in defining cube diagrams actually reduced the total number of isotopies to only five allowable moves. The content of our main theorem was that these moves are all that are necessary to get the full geometry of the knot as it sits in 3-dimensional space.

As an example of the usefulness of cube diagrams, we present a homology theory constructed from cube diagrams that is equivalent to knot Floer homology, one of the most powerful known knot invariants.

To learn more about cube diagrams, display them, or compute invariants of knots using them, please visit `http://code.google.com/p/cubeknots`.

Tim Cochran, Rice University, *The Fractal Nature of* 3*-Manifolds up to Homology Cobordism*

Using the unique torus decomposition, I will explain how the set of homeomorphism classes of 3-manifolds may, in a naive way, be viewed as a fractal set. Recent advances indicate that much of the complicated combinatorial and algebraic structure normally associated to graph manifolds survives even modulo homology cobordism (and exists even in hyperbolic 3-manifolds!). I will focus on examples that are knot and link exteriors and discuss an array of invariants starting from classical signatures and ending with noncommutative algebra and functional analysis, that have recently been useful in giving evidence for this extremely complicated fractal behavior. This is joint work with Shelly Harvey and Constance Leidy.

Emily Dryden, Bucknell University, *Upper Bounds on Eigenvalues of the Laplacian: Surfaces and Beyond*

The talk was a report on joint work with Bruno Colbois and Ahmad El Soufi.

The original isoperimetric inequality says that for a simple closed curve in $\mathbb{R}^2$ of length L, we have $L^2 \geq 4\pi A$, where A is the area enclosed by the curve. The circle can be characterized as the unique curve for which equality holds. If one considers the higher-dimensional analogue of the isoperimetric problem, namely to minimize surface area among domains with fixed volume in $\mathbb{R}^n$, the unique extremal is the domain bounded by a sphere. Not surprisingly, spheres also play an important role in more "physical" versions of the isoperimetric problem: one tries to extremize a physical quantity which is represented by the eigenvalues of a differential equation.

We will focus on the eigenvalues of the Laplace operator $\Delta = -\operatorname{div}\operatorname{grad}$ acting on smooth functions. In 1970, Joseph Hersch showed that when we consider all metrics on the sphere, the product of the lowest nonzero eigenvalue and the volume is at most 8π. Moreover, the bound is realized only by the usual constant curvature metric. In connection with Hersch's result, Marcel Berger asked whether there exists a constant $k(M)$ such that $\lambda_1(g)\mathrm{Vol}(g)^{2/n} \leq k(M)$ for any Riemannian metric g on an n-dimensional manifold M. Answers to Berger's question for surfaces were given by Paul Yang and Shing Tung Yau, Peter Li and Yau, and others using barycentric methods; using new techniques, Nick Korevaar showed that for a compact orientable surface of genus γ and for every integer $k \geq 1$,

$$\lambda_k(g)\mathrm{Vol}(g) \leq C(\gamma+1)k,$$

where C is some universal constant.

In higher dimensions, however, the story is completely different. A theorem of Colbois and Józef Dodziuk states that if (M^n, g) is a compact, closed, connected manifold of dimension at least three, then

$$\sup \lambda_1(g)\mathrm{Vol}(g)^{2/n} = \infty,$$

where the supremum is taken over all Riemannian metrics g on M. Thus to study extremal properties of the Laplace spectrum in dimensions greater than two, we must add more constraints. These constraints may be intrinsic or extrinsic, and we discuss an example of each type.

We first consider the setting in which we require our metrics and our eigenfunctions to be invariant with respect to a group action. This kind of condition was first considered by Miguel Abreu and Pedro Freitas and by Martin Engman in the setting of S^1-invariant metrics and eigenfunctions on S^2. We generalize their results in several ways. Specifically, let the Laplacian Δ act on $O(n)$-invariant functions and consider $O(n)$-invariant metrics g on S^n arising from *embeddings* of S^n in $\mathbb{R}^{n+1}$.

Theorem 1. *[CDE1] Let (S^n, g) be as above, with Vol$(g) = 1$. Then, for all $k \in \mathbb{Z}$,*

$$\lambda_k^{O(n)}(g) < \lambda_k^{O(n)}(D^n)\mathit{Vol}(D^n)^{2/n},$$

where D^n is the Euclidean n-ball of volume $1/2$.

In general, if we replace S^n by a compact closed connected manifold M of dimension $n \geq 3$, replace $O(n)$ by a finite subgroup G of the group of diffeomorphisms acting on M, let Δ act on G-invariant functions, and consider G-invariant metrics on M, then $\lambda_1^G(g)\mathrm{Vol}(g)^{2/n}$ is unbounded [CDE1].

For an extrinsic constraint, one may consider compact connected submanifolds without boundary immersed in Euclidean space. For instance, if M is a compact convex hypersurface in $\mathbb{R}^{n+1}$ then

$$\lambda_1(M)\mathrm{Vol}(M)^{2/n} \leq A(n)\lambda_1(S^n)\mathrm{Vol}(S^n)^{2/n},$$

where $\lambda_1(S^n) = n$ and $A(n) = \frac{(n+2)\mathrm{Vol}(S^n)}{2\mathrm{Vol}(S^{n-1})}$ [CDE2]. To generalize this result to immersed submanifolds that are not necessarily convex, we introduce the notion of *intersection index*. For a submanifold M^n in $\mathbb{R}^{n+p}$, the intersection index of M is

$$i(M) = \sup_{\Pi} \#M \cap \Pi,$$

where Π runs over the set of p-planes transverse to M in $\mathbb{R}^{n+p}$.

Theorem 2. *[CDE2] For every compact n-dimensional immersed submanifold M of $\mathbb{R}^{n+p}$ and for every integer k,*

$$\lambda_k(M)\mathit{Vol}(M)^{2/n} \leq c(n)i(M)^{2/n}k^{2/n},$$

where $c(n)$ is an explicit constant depending only on the dimension n.

One may combine the result of Colbois-Dodziuk mentioned above with this and other related results in [CDE2] to get the following observation: Given a smooth manifold $\bar{M}$ of dimension $n \geq 3$, there exist Riemannian metrics g of volume 1 on $\bar{M}$ such that any immersion of $\bar{M}$ into a Euclidean space $\mathbb{R}^{n+p}$ which preserves g must have a very large intersection index and volume which concentrates into a small Euclidean ball. Thus imposing constraints in order to bound the eigenvalues of the Laplacian can lead to geometric insight about the metrics which are "bad" for finding bounds.

References

[CDE1] Bruno Colbois, Emily B. Dryden and Ahmad El Soufi, *Extremal G-invariant eigenvalues of the Laplacian of G-invariant metrics*, Math. Z. 258 (2008), 29–41.

[CDE2] Bruno Colbois, Emily B. Dryden and Ahmad El Soufi, *Bounding the eigenvalues of the Laplace-Beltrami operator on compact submanifolds,* Bull. Lond. Math. Soc. 42 (2010), 96-108.

Carolyn Gordon, Dartmouth College, *Spectral Geometry on Line Bundles over Flat Tori*

Let M be a closed Riemannian manifold and L a Hermitian line bundle over M. Each Hermitian connection on L gives rise to a Laplace operator acting on sections of L. We consider the question: How much information about the connection or the curvature of the connection is encoded in the spectrum of the Laplace operator? We also consider the analogous questions for Schrödinger operators on line bundles. We will focus primarily on the setting of line bundles over flat tori.

Mark Haskins, Imperial College, London, *Special Lagrangian Cones*

We give an introduction to special Lagrangian geometry explaining why singular special Lagrangian n-folds are important. The rest of the talk will concentrate on describing recent progress understanding the simplest singular special Lagrangian n-folds: cones with isolated singularities. We will describe recent work with Kapouleas constructing a plethora of special Lagrangian cones by gluing methods; i.e., a rigorous geometric version of singular perturbation theory.

Alexander Karabegov, Abilene Christian University, ***Formal Symplectic Groupoids***

A symplectic realization of a Poisson manifold M is a symplectic manifold Σ together with a Poisson map from Σ to M which is a surjective submersion. If this map has a cross section with the Lagrangian image Λ, then a neighborhood of Λ in Σ can be canonically equipped with the structure of a local symplectic groupoid for which Λ is the unit space and M is the object space. The space A of smooth functions on the formal neighborhood of Λ in Σ is a Poisson algebra. The groupoid operations induce dual mappings between the Poisson algebra A and the Poisson algebra of smooth functions on M. We give a self-contained algebraic description of the Poisson algebra A and use it to define a formal symplectic groupoid over the Poisson manifold M. We show that to each (natural) formal deformation quantization on M, one can relate a formal symplectic groupoid over M. Finally, we construct a unique formal symplectic groupoid "with separation of variables" over an arbitrary Kähler-Poisson manifold.

Markus Pflaum, University of Colorado, ***Algebraic Index Theorems for Orbifolds***

Algebraic index theory goes back to the work of Fedosov, Nest and Tsygan and is based on the idea to prove index theorems a la Atiyah-Singer by studying the noncommutative geometry of deformation quantizations. This approach to index theory will be explained in the talk and applied to obtain index theorems for orbifolds. The talk is about joint work with Posthuma and Tang.