

A NEW INFINITE GAME IN BANACH SPACES WITH APPLICATIONS

E. ODELL, TH. SCHLUMPRECHT AND A. ZSÁK

Dedicated to Nigel J. Kalton on the occasion of his 60th birthday.

1. INTRODUCTION

Let X be a separable infinite-dimensional Banach space, and let \mathcal{A} be a set of normalized sequences in X . We can consider a two-player game in X each move of which consists of player S (subspace chooser) selecting some element Y from the set $\text{cof}(X)$ of finite-codimensional subspaces of X , and P (point chooser) responding by selecting a vector y from the unit sphere S_Y of Y . The game, which we shall refer to as the \mathcal{A} -game, consists of an infinite sequence of such moves generating a sequence $X_1, x_1, X_2, x_2, \dots$, where $X_i \in \text{cof}(X)$ and $x_i \in S_{X_i}$ for all $i \in \mathbb{N}$. S wins the \mathcal{A} -game if $(x_i)_{i=1}^\infty \in \mathcal{A}$.

Of course this game, which has its roots in the game described by W. T. Gowers [5] and in the notion of asymptotic structure [9], has certain limitations. Unlike the theory of asymptotic structure (where, for each $n \in \mathbb{N}$, a game is considered that consists of n moves, where each move is the same as above), there is generally no unique smallest class \mathcal{A} (depending on X) for which S has a winning strategy. However, one can hypothesize certain specific classes \mathcal{A} for which S has a winning strategy for a given X and deduce certain structural consequences. For example, if for some $K > 0$ we let \mathcal{A} be the class of sequences K -equivalent to the unit vector basis of ℓ_p ($1 < p < \infty$), then any reflexive space X in which S has a winning strategy for the \mathcal{A} -game in X , embeds into an ℓ_p -sum of finite-dimensional spaces [10]. In fact, it was the problem of classifying subspaces of ℓ_p -sums of finite-dimensional spaces that motivated the study of this game.

The general theme here is to take a coordinate-free property of a space X , recast it in terms of S having a winning strategy in the \mathcal{A} -game for a suitable class \mathcal{A} , and then to show that X embeds into a space with an FDD (finite-dimensional decomposition) which has the “coordinatized” version of the property we started with. In addition to the ℓ_p result in [10] cited above this general theme was followed in [11] and [12]. In [11] reflexive spaces X were studied for which S has a winning strategy for both games corresponding to the classes \mathcal{A}_p of normalized basic sequences with an ℓ_p -lower estimate and \mathcal{A}^q of normalized basic sequences with an ℓ_q -upper estimate ($1 < q \leq p < \infty$). The end result was that X embeds into a reflexive space with an FDD such that every block sequence satisfies ℓ_p -lower and ℓ_q -upper estimates. A consequence of this is that one can construct a separable, reflexive space universal for the class of separable, uniformly convex spaces or, more generally, for the class $C_\omega = \{X : X \text{ is separable, reflexive, } \text{Sz}(X) \leq \omega, \text{Sz}(X^*) \leq \omega\}$, where $\text{Sz}(Y)$ denotes the Szlenk index of a separable Banach space Y . Recently an alternative proof of the universal result was given [3] using powerful set-theoretical notions (although, the FDD structural results cannot be obtained in this way). We should also note that a set-theoretical study of \mathcal{A} -games was given by C. Rosendal [14].

Date: December 21, 2006.

2000 Mathematics Subject Classification. Primary 46B20.

Research of the first two authors was supported by the National Science Foundation.

The motivation behind this paper arose from a problem posed to us by A. Pełczyński. Given $\alpha < \omega_1$, does there exist a separable, reflexive space universal for the class C_α (defined as above with ω replaced by α). Thus far the authors of [3] have been unable to extend their techniques to this problem. In researching Pełczyński's problem we discovered that it was necessary to consider a new game and solve the corresponding embedding problem in this context.

The game is played as follows. In each move of the game S (subspace chooser) selects $k \in \mathbb{N}$ and $Y \in \text{cof}(X)$, and then P (point chooser) responds by choosing $y \in S_Y$. The game then consists of an infinite sequence of moves generating a sequence $k_1, X_1, x_1, k_2, X_2, x_2, \dots$, where $k_i \in \mathbb{N}$, $X_i \in \text{cof}(X)$ and $x_i \in S_{X_i}$ for all $i \in \mathbb{N}$. Given a normalized, 1-unconditional sequence (v_i) , S is declared winner of this game if (v_{k_i}) is dominated by (x_i) . We also consider, for given normalized 1-unconditional sequence (u_i) , the version where S wins the game if (x_i) is dominated by (u_{k_i}) . In the case where (v_i) and (u_i) are the unit vector bases of ℓ_p and ℓ_q , respectively, this conforms to the games considered in [11], but Pełczyński's problem requires us to consider sequences (v_i) and (u_i) that are not subsymmetric. (In [12] in order to solve the problem of embedding an asymptotic ℓ_p space into one with an asymptotic ℓ_p FDD it was necessary to extend the results of [11] concerning ℓ_p -lower and ℓ_q -upper estimates to more general (v_i) -lower and (u_i) -upper estimates, but the game played did not change.)

The main results of this paper are given in Section 4, Theorems 12 and 15, and Corollary 13. In brief these theorems say the following. Suppose we are given normalized, 1-unconditional bases (v_i) and (u_i) with certain properties, and a reflexive space X . Assume that S wins the subsequential (v_i) -lower and the subsequential (u_i) -upper games described above. Then X embeds into a space Z with an FDD such that every block sequence satisfies subsequential (v_i) -lower and (u_i) -upper estimates. (Precise definitions of these estimates will be given below.)

One application of these theorems is a new proof of the results of [11]. The application to the Pełczyński problem will appear in [13], where further machinery is necessary to exploit the results obtained here.

In Section 5 we derive some universal space consequences of our embedding theorems. Section 2 introduces our terminology, in particular we give precise definitions of various lower and upper norm estimates. Section 2 also contains some straightforward duality results concerning such norm estimates, and a combinatorial result (Proposition 5) that is key to embedding spaces satisfying the coordinate-free version of a certain property into a space with an FDD satisfying the "coordinatized" version of the same property.

In Section 3 we define the space $Z^V(E)$, where Z is a Banach space with an FDD $E = (E_i)$ and V is the closed linear span of a normalized, 1-unconditional sequence (v_i) . We develop the properties of $Z^V(E)$, in particular proving that, under appropriate hypotheses, $Z^V(E)$ is a reflexive space admitting subsequential V -lower estimates.

2. DEFINITIONS AND PRELIMINARY RESULTS

We begin with fixing some terminology. Let Z be a Banach space with an FDD $E = (E_n)$. For $n \in \mathbb{N}$ we denote by P_n^E the n -th *coordinate projection*, i.e., $P_n^E: Z \rightarrow E_n$ is the map defined by $\sum_i z_i \mapsto z_n$, where $z_i \in E_i$ for all $i \in \mathbb{N}$. For a finite set $A \subset \mathbb{N}$ we put $P_A^E = \sum_{n \in A} P_n^E$. The *projection constant* $K(E, Z)$ of (E_n) (in Z) is defined by

$$K = K(E, Z) = \sup_{m < n} \|P_{[m, n]}^E\|,$$

where $[m, n]$ denotes the interval $\{m, m+1, \dots, n\}$ in \mathbb{N} . Recall that K is always finite and, as in the case of bases, we say that (E_n) is *bimonotone (in Z)* if $K=1$. By passing to the equivalent norm

$$\|\cdot\|: Z \rightarrow \mathbb{R}, \quad z \mapsto \sup_{m < n} \|P_{[m, n]}^E(z)\|,$$

we can always renorm Z so that $K=1$.

For a sequence (E_i) of finite-dimensional spaces we define the vector space

$$c_{00}(\oplus_{i=1}^{\infty} E_i) = \{(z_i) : z_i \in E_i \text{ for all } i \in \mathbb{N}, \text{ and } \{i \in \mathbb{N} : z_i \neq 0\} \text{ is finite}\},$$

which is dense in each Banach space for which (E_i) is an FDD. For a set $A \subset \mathbb{N}$ we denote by $\bigoplus_{i \in A} E_i$ the linear subspace of $c_{00}(\oplus E_i)$ generated by the elements of $\bigcup_{i \in A} E_i$. As usual we denote the vector space of sequences in \mathbb{R} which are eventually zero by c_{00} . We sometimes will consider for the same sequence (E_i) of finite-dimensional spaces different norms on $c_{00}(\oplus E_i)$. In order to avoid confusion we will therefore often index the norm by the Banach space whose norm we are using, i.e., $\|\cdot\|_Z$ denotes the norm of the Banach space Z .

If Z has an FDD (E_i) , the vector space $c_{00}(\oplus_{i=1}^{\infty} E_i^*)$, where E_i^* is the dual space of E_i for each $i \in \mathbb{N}$, can be identified in a natural way with a w^* -dense subspace of Z^* . Note however that the embedding $E_i^* \hookrightarrow Z^*$ is, in general, not isometric unless $K=1$. We will always consider E_i^* with the norm it inherits from Z^* instead of the norm it has as the dual space of E_i . We denote the norm closure of $c_{00}(\oplus_{i=1}^{\infty} E_i^*)$ in Z^* by $Z^{(*)}$. Note that $Z^{(*)}$ is w^* -dense in Z^* , the unit ball $B_{Z^{(*)}}$ norms Z , and (E_i^*) is an FDD of $Z^{(*)}$ having a projection constant not exceeding $K(E, Z)$. If $K(E, Z)=1$, then $B_{Z^{(*)}}$ is 1-norming for Z and $Z^{(*)} = Z$.

For $z \in c_{00}(\oplus E_i)$ we define the *support* $\text{supp}_E(z)$ of z with respect to (E_i) by

$$\text{supp}_E(z) = \{i \in \mathbb{N} : P_i^E(z) \neq 0\},$$

and we define the *range* $\text{ran}_E(z)$ of Z with respect to (E_i) to be the smallest interval in \mathbb{N} containing $\text{supp}_E(z)$. A sequence (z_i) (finite or infinite) of non-zero vectors in $c_{00}(\oplus E_i)$ is called a *block sequence of (E_i)* if

$$\max \text{supp}_E(z_n) < \min \text{supp}_E(z_{n+1}) \quad \text{whenever } n \in \mathbb{N} \text{ (or } n < \text{length}(z_i)).$$

A block sequence (z_i) of (E_i) is called *normalized (in Z)* if $\|z_i\|_Z = 1$ for all $i \in \mathbb{N}$.

Let $\bar{\delta} = (\delta_i) \subset (0, 1)$ with $\delta_i \downarrow 0$. A (finite or infinite) sequence (z_i) in S_Z is called a *$\bar{\delta}$ -skipped block sequence of (E_i)* if there exists a sequence $1 \leq k_0 < k_1 < k_2 < \dots$ in \mathbb{N} such that

$$\|z_n - P_{(k_{n-1}, k_n)}^E(z_n)\| < \delta_n \quad \text{for all } n \in \mathbb{N} \text{ (or } n \leq \text{length}(z_i)).$$

Remark. A sequence (F_i) of finite-dimensional spaces is called a *blocking of (E_i)* if for some sequence $m_1 < m_2 < \dots$ in \mathbb{N} we have $F_n = \bigoplus_{j=m_{n-1}+1}^{m_n} E_j$ for all $n \in \mathbb{N}$ ($m_0 = 0$). If (F_i) is a blocking of (E_i) , and if (x_i) is a $\bar{\delta}$ -skipped block sequence of (F_i) , then (x_i) is not necessarily a $\bar{\delta}$ -skipped block sequence of (E_i) (since in the definition of skipped block sequence we skip exactly one coordinate). Nevertheless it is clear that (x_i) is a $2K\bar{\delta}$ -skipped block sequence of (E_i) , where K is the projection constant of (E_i) in Z .

Definition. Given two sequences (e_i) and (f_i) in some Banach spaces, and given a constant $C > 0$, we say that (f_i) *C-dominates* (e_i) , or that (e_i) is *C-dominated by (f_i)* , if

$$\left\| \sum a_i e_i \right\| \leq C \left\| \sum a_i f_i \right\| \quad \text{for all } (a_i) \in c_{00}.$$

We say that (f_i) *dominates* (e_i) , or that (e_i) is *dominated by (f_i)* , if there exists a constant $C > 0$ such that (f_i) C-dominates (e_i) .

We shall now introduce certain lower and upper norm estimates for FDD's.

Definition. Let Z be a Banach space with an FDD (E_n) , let V be a Banach space with a normalized, 1-unconditional basis (v_i) and let $1 \leq C < \infty$.

We say that (E_n) *satisfies subsequential C - V -lower estimates (in Z)* if every normalized block sequence (z_i) of (E_n) in Z C -dominates (v_{m_i}) , where $m_i = \min \text{supp}_E(z_i)$ for all $i \in \mathbb{N}$, and (E_n) *satisfies subsequential C - V -upper estimates (in Z)* if every normalized block sequence (z_i) of (E_n) in Z is C -dominated by (v_{m_i}) , where $m_i = \min \text{supp}_E(z_i)$ for all $i \in \mathbb{N}$.

If U is another space with a normalized and 1-unconditional basis (u_i) , we say that (E_n) *satisfies subsequential C - (V, U) estimates (in Z)* if it satisfies subsequential C - V -lower and C - U -upper estimates in Z .

We say that (E_n) *satisfies subsequential V -lower, U -upper or (V, U) estimates (in Z)* if for some $C \geq 1$ it satisfies subsequential C - V -lower, C - U -upper or C - (V, U) estimates in Z , respectively.

Remark. Assume that (E_n) satisfies subsequential C - V -lower estimates in Z and that (z_i) is a normalized block sequence of (E_n) . If $\max \text{supp}_E(z_{i-1}) < m_i \leq \min \text{supp}_E(z_i)$ for all $i \in \mathbb{N}$ (where $\max \text{supp}_E(z_0) = 0$), then (z_i) C -dominates (v_{m_i}) .

Another easy fact is that if every normalized block sequence (z_i) of (E_n) in Z dominates (v_{m_i}) , where $m_i = \min \text{supp}_E(z_i)$ for all $i \in \mathbb{N}$, then (E_n) satisfies subsequential V -lower estimates in Z .

Analogous statements hold for upper estimates.

We shall need a coordinate-free version of subsequential lower and upper estimates. One way of defining this is reminiscent of the notion of asymptotic structure. Let V be a Banach space with a normalized and 1-unconditional basis (v_i) , and let $C \in [1, \infty)$. Assume that we are given an infinite-dimensional Banach space X . We say that X *satisfies subsequential C - V -lower estimates* (respectively, *subsequential C - V -upper estimates*) if

$$\begin{array}{lll} \exists k_1 \in \mathbb{N} & \exists X_1 \in \text{cof}(X) & \forall x_1 \in S_{X_1} \\ \exists k_2 \in \mathbb{N} & \exists X_2 \in \text{cof}(X) & \forall x_2 \in S_{X_2} \\ \exists k_3 \in \mathbb{N} & \exists X_3 \in \text{cof}(X) & \forall x_3 \in S_{X_3} \\ \vdots & \vdots & \vdots \end{array}$$

such that $k_1 < k_2 < \dots$ and (v_{k_i}) is C -dominated by (respectively, C -dominates) (x_i) . If U is another Banach space with a normalized, 1-unconditional basis (u_i) , then we say that X *satisfies subsequential C - (V, U) estimates* if it satisfies subsequential C - V -lower and C - U -upper estimates. Finally, we say that *the Banach space X satisfies subsequential V -lower, U -upper or (V, U) estimates* if for some constant C it satisfies subsequential C - V -lower, C - U -upper or C - (V, U) estimates, respectively.

The above definitions are given more formally in the language of games. Let us recall from the Introduction that in our games each move consists of S (subspace chooser) selecting $k \in \mathbb{N}$ and $Y \in \text{cof}(X)$, and then P (point chooser) responding by choosing $y \in S_Y$. The game then consists of an infinite sequence of moves generating a sequence $k_1, X_1, x_1, k_2, X_2, x_2, \dots$, where $k_i \in \mathbb{N}$, $X_i \in \text{cof}(X)$ and $x_i \in S_{X_i}$ for all $i \in \mathbb{N}$. Player S wins the game if (v_{k_i}) is C -dominated by (x_i) , otherwise P is declared the winner. The space X then satisfies subsequential C - V -lower estimates if and only if S has a winning strategy, i.e., there is a function ϕ such that given sequences (k_i) in \mathbb{N} , (X_i) in $\text{cof}(X)$, and (x_i) in X such that $x_i \in S_{X_i}$ and $(k_i, X_i) = \phi(x_1, x_2, \dots, x_{i-1})$ for all $i \in \mathbb{N}$, then $k_1 < k_2 < \dots$ and

(v_{k_i}) is C -dominated by (x_i) . The notions of subsequential C - V -upper estimates, subsequential C - (V, U) estimates, etc. can be formalized in a similar way.

Yet another way of expressing subsequential lower and upper estimates in a coordinate-free way uses infinite, countably branching trees (see Proposition 1 below). This is not surprising since a winning strategy in the game described above corresponds naturally to such a tree. We define for $\ell \in \mathbb{N}$

$$T_\ell = \{(n_1, n_2, \dots, n_\ell) : n_1 < n_2 < \dots < n_\ell \text{ are in } \mathbb{N}\}$$

and

$$T_\infty = \bigcup_{\ell=1}^{\infty} T_\ell, \quad T_\infty^{\text{even}} = \bigcup_{\ell=1}^{\infty} T_{2\ell}.$$

If $\alpha = (m_1, \dots, m_\ell) \in T_\ell$, we call ℓ the *length* of α and denote it by $|\alpha|$, and $\beta = (n_1, \dots, n_k) \in T_\infty$ is called an *extension* of α , or α is called a *restriction* of β , if $\ell < k$ and $m_i = n_i$ for $i = 1, \dots, \ell$. We then write $\alpha < \beta$ and with this order both T_∞ and T_∞^{even} are trees.

An *even tree* in a Banach space X is a family $(x_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ in X . Sequences of the form $(x_{(\alpha, n)})_{n > n_{2\ell-1}}$, where $\ell \in \mathbb{N}$ and $\alpha = (n_1, n_2, \dots, n_{2\ell-1}) \in T_\infty$, are called *nodes* of the tree. For a sequence $n_1 < n_2 < \dots$ of positive integers the sequence $(x_{(n_1, n_2, \dots, n_{2\ell})})_{\ell=1}^{\infty}$ is called a *branch* of the tree.

If $(x_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ is an even tree in a Banach space X and if $T' \subset T_\infty^{\text{even}}$ is closed under taking restrictions so that for each $\alpha \in T' \cup \{\emptyset\}$ and for each $m \in \mathbb{N}$ the set $\{n \in \mathbb{N} : (\alpha, m, n) \in T'\}$ is either empty or has infinite size, and moreover the latter occurs for infinitely many values of m , then we call $(x_\alpha)_{\alpha \in T'}$ a *full subtree* of $(x_\alpha)_{\alpha \in T_\infty^{\text{even}}}$. Note that $(x_\alpha)_{\alpha \in T'}$ could then be relabeled to a family indexed by T_∞^{even} , and note that the branches of $(x_\alpha)_{\alpha \in T'}$ are branches of $(x_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ and that the nodes of $(x_\alpha)_{\alpha \in T'}$ are subsequences of certain nodes of $(x_\alpha)_{\alpha \in T_\infty^{\text{even}}}$.

An even tree $(x_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ in a Banach space X is called *normalized* if $\|x_\alpha\| = 1$ for all $\alpha \in T_\infty^{\text{even}}$, and is called *weakly null* if every node is a weakly null sequence. If X has an FDD (E_n) , then $(x_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ is called a *block even tree* of (E_n) if every node is a block sequence of (E_n) .

Definition. Let V be a Banach space with a normalized and 1-unconditional basis (v_i) , and let $C \in [1, \infty)$. Assume that we are given an infinite-dimensional Banach space X . We say that X *satisfies subsequential C - V -lower tree estimates* if every normalized, weakly null even tree $(x_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ in X has a branch $(x_{(n_1, n_2, \dots, n_{2i})})$ which C -dominates $(v_{n_{2i-1}})$.

We say that X *satisfies subsequential C - V -upper tree estimates* if every normalized, weakly null even tree $(x_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ in X has a branch $(x_{(n_1, n_2, \dots, n_{2i})})$ which is C -dominated by $(v_{n_{2i-1}})$.

If U is a second space with a 1-unconditional and normalized basis (u_i) , we say that X *satisfies subsequential C - (V, U) tree estimates* if it satisfies subsequential C - V -lower and C - U -upper tree estimates.

We say that X *satisfies subsequential V -lower, U -upper or (V, U) tree estimates* if for some $1 \leq C < \infty$ X satisfies subsequential C - V -lower, C - U -upper or C - (V, U) tree estimates, respectively.

Remark. As in the FDD case, we do not need to fix a constant C in the above definitions: if every normalized, weakly null even tree $(x_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ in X has a branch $(x_{(n_1, n_2, \dots, n_{2i})})$ which dominates $(v_{n_{2i-1}})$, then there exists a constant $C \geq 1$ such that X satisfies subsequential C - V -lower tree estimates. The analogous statement for upper estimates also holds. (See [12, Proposition 1.2].)

Proposition 1 below shows that, under some mild hypotheses, the two coordinate-free versions of lower and upper estimates given above are essentially the same. Before stating this result we need a certain property of basic sequences defined in [2].

Definition. Let V be a Banach space with a normalized, 1-unconditional basis (v_i) and let $1 \leq C < \infty$.

We say that (v_i) is *C-right-dominant* (respectively, *C-left-dominant*) if for all sequences $m_1 < m_2 < \dots$ and $n_1 < n_2 < \dots$ of positive integers with $m_i \leq n_i$ for all $i \in \mathbb{N}$ we have that (v_{m_i}) is *C-dominated* by (respectively, *C-dominates*) (v_{n_i}) . We say that (v_i) is *right-dominant* or *left-dominant* if for some $C \geq 1$ it is *C-right-dominant* or *C-left-dominant*, respectively.

Remark. For (v_i) to be right-dominant (respectively, left-dominant) it is enough to have the property that (v_{m_i}) is dominated by (respectively, dominates) (v_{n_i}) for all sequences $m_1 < m_2 < \dots$ and $n_1 < n_2 < \dots$ of positive integers with $m_i \leq n_i$ for all $i \in \mathbb{N}$. Also, (v_i) is *C-right-dominant* (respectively, *C-left-dominant*) if and only if the sequence (v_i^*) of biorthogonal functionals in $V^{(*)}$ is *C-left-dominant* (respectively, *C-right-dominant*).

Proposition 1. *Let V be a Banach space with a normalized and 1-unconditional basis (v_i) and let $C, D \in [1, \infty)$. Let X be an infinite-dimensional Banach space.*

- (a) *Assume that (v_i) is D -left-dominant. If X satisfies subsequential C - V -lower estimates, then for all $\varepsilon > 0$ X satisfies subsequential $(CD + \varepsilon)$ - V -lower tree estimates.*
- (b) *Assume that X^* is separable. If X satisfies subsequential C - V -lower tree estimates, then it also satisfies subsequential C - V -lower estimates.*

Remark. Analogous results hold for upper estimates. For that in (a) we need to assume that (v_i) is D -right-dominant.

Proof. (a) Assume that for some $\varepsilon > 0$ there is a normalized, weakly null even tree $(x_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ in X such that for any sequence $n_1 < n_2 < \dots$ of positive integers the branch $(x_{(n_1, n_2, \dots, n_{2i})})$ does not $(CD + \varepsilon)$ -dominate $(v_{n_{2i-1}})$. We show that in this case X does not satisfy subsequential C - V -lower estimates by exhibiting a winning strategy for the point chooser P . Fix a sequence $(\delta_i) \subset (0, 1)$ with $\Delta = \sum_i \delta_i$ satisfying

$$C < \frac{CD + \varepsilon}{D} \cdot \left(1 + \Delta \frac{CD + \varepsilon}{D}\right)^{-1}.$$

Suppose the game starts with S picking $k_1 \in \mathbb{N}$ and $X_1 \in \text{cof}(X)$. Since the nodes of $(x_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ are weakly null, there exist $n_1, n_2 \in \mathbb{N}$ such that $k_1 \leq n_1 < n_2$ and $d(x_{(n_1, n_2)}, X_1) < \delta_1$. P 's response will be a $y_1 \in S_{X_1}$ with $\|x_{(n_1, n_2)} - y_1\| < \delta_1$. In the second move S picks $k_2 \in \mathbb{N}$ and $X_2 \in \text{cof}(X)$. Then there exist $n_3, n_4 \in \mathbb{N}$ such that $n_2 < n_3 < n_4$, $k_2 \leq n_3$ and $d(x_{(n_1, n_2, n_3, n_4)}, X_2) < \delta_2$. P 's response will be some $y_2 \in S_{X_2}$ with $\|x_{(n_1, n_2, n_3, n_4)} - y_2\| < \delta_2$. In general, on the j^{th} move of the game ($j \geq 2$), S picks $k_j \in \mathbb{N}$, $X_j \in \text{cof}(X)$. Then one can find $n_{2j-1}, n_{2j} \in \mathbb{N}$ such that $n_{2j-2} < n_{2j-1} < n_{2j}$, $k_j \leq n_{2j-1}$ and $d(x_{(n_1, n_2, \dots, n_{2j})}, X_j) < \delta_j$. P 's j^{th} move will be some $y_j \in S_{X_j}$ such that $\|x_{(n_1, n_2, \dots, n_{2j})} - y_j\| < \delta_j$.

Since the branch $(x_{(n_1, n_2, \dots, n_{2i})})$ does not $(CD + \varepsilon)$ -dominate $(v_{n_{2i-1}})$, there exists $(a_i) \in c_{00}$ such that

$$\left\| \sum_i a_i v_{n_{2i-1}} \right\| > (CD + \varepsilon) \left\| \sum_i a_i x_{(n_1, n_2, \dots, n_{2i})} \right\|.$$

We may assume without loss of generality that $\|\sum a_i v_{k_i}\| = 1$. Using the D -left-dominant property of (v_i) and that $k_i \leq n_{2i-1}$ for all $i \in \mathbb{N}$, together with the choice of Δ , an easy computation now gives

$$\left\| \sum_i a_i v_{k_i} \right\| > C \left\| \sum_i a_i y_i \right\|.$$

Thus P wins the game.

(b) Assume that X does not satisfy subsequential C - V -lower estimates. This means that S does not have a winning strategy, which in turn implies that there is a winning strategy ϕ for the point chooser (this follows from the fact that closed games [4] or, more generally, Borel games [8] are determined). Thus given sequences (k_i) in \mathbb{N} , (X_i) in $\text{cof}(X)$, and (x_i) in X such that $k_1 < k_2 < \dots$ and $x_n = \phi(k_1, X_1, k_2, X_2, \dots, k_n, X_n)$ for all $n \in \mathbb{N}$, then $x_i \in S_{X_i}$ for all $i \in \mathbb{N}$ and (v_{k_i}) is not C -dominated by (x_i) . Fix a sequence (X_i) of finite-codimensional subspaces of X such that every bounded sequence (x_i) with $x_i \in X_i$ for all $i \in \mathbb{N}$ is weakly null. This exists by the assumption that X has separable dual.

We now construct a normalized, weakly null even tree in X by recursion to show that X does not satisfy subsequential C - V -lower tree estimates. For $\ell \in \mathbb{N}$ and $\alpha = (n_1, n_2, \dots, n_{2\ell}) \in T_{2\ell}$ we set $x_\alpha = \phi(n_1, X_{n_2}, n_3, X_{n_4}, \dots, n_{2\ell-1}, X_{n_{2\ell}})$. It is easy to verify that $(x_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ is a normalized, weakly null even tree in X , and that for any sequence $n_1 < n_2 < \dots$ of positive integers the branch $(x_{(n_1, n_2, \dots, n_{2i})})$ does not C -dominate $(v_{n_{2i-1}})$. \square

If V is a Banach space with a normalized, 1-unconditional basis, and if N is an infinite subset of \mathbb{N} , we write V_N for the closed linear span of $\{v_i : i \in N\}$. When we talk about subsequential V_N -lower estimates, etc., it will be with respect to the normalized, 1-unconditional basis $(v_i)_{i \in N}$ of V_N . We shall also write $\mathbb{N}^{(\omega)}$ for the set of all infinite subsets of \mathbb{N} .

Note that if V is a Banach space with a normalized, 1-unconditional, left-dominant basis (v_i) , then for any space Z with an FDD (E_i) satisfying subsequential V -lower estimates in Z , the FDD (E_i) will also satisfy subsequential V_N -lower estimates for any $N \in \mathbb{N}^{(\omega)}$. Later on we shall need a result that allows us to pass from subsequential V_N -lower estimates for some $N \in \mathbb{N}^{(\omega)}$ to subsequential V -lower estimates. Before stating this result we need a definition.

Definition. Let V be a Banach space with a normalized, 1-unconditional basis (v_i) and let $1 \leq C < \infty$.

We say that (v_i) is C -block-stable if any two normalized block bases (x_i) and (y_i) with

$$\max(\text{supp}(x_i) \cup \text{supp}(y_i)) < \min(\text{supp}(x_{i+1}) \cup \text{supp}(y_{i+1})) \quad \text{for all } i \in \mathbb{N}$$

are C -equivalent. We say that (v_i) is block-stable if it is C -block-stable for some constant C .

Remark. It is routine to check that (v_i) is C -block-stable if and only if the sequence (v_i^*) of biorthogonal functionals in $V^{(*)}$ is C -block-stable.

A block-stable basis is a special case of a *block-norm-determined FDD* introduced by H. P. Rosenthal, who has initiated an exhaustive study of such FDDs [15].

Lemma 2. *Let V and U be Banach spaces with normalized, 1-unconditional, block-stable bases (v_i) and (u_i) , respectively, and assume that (v_i) is dominated by (u_i) . Let $M \in \mathbb{N}^{(\omega)}$ and let Z be a Banach space with an FDD $E = (E_i)$ satisfying subsequential (V_M, U_M) estimates in Z . Then $W = Z \oplus_{\ell_\infty} V_{\mathbb{N} \setminus M}$ has an FDD $F = (F_i)$ satisfying subsequential (V, U) estimates in W .*

Proof. Choose constants $B, C, D \in [1, \infty)$ such that (v_i) and (u_i) are B -block-stable, (v_i) is D -dominated by (u_i) , and (E_i) satisfies subsequential C -(V_M, U_M) estimates in Z . For each $n \in \mathbb{N}$ define

$$F_n = \begin{cases} E_i & \text{if } n = m_i \text{ for some } i \in \mathbb{N}, \\ \mathbb{R} \cdot v_n & \text{if } n \notin M. \end{cases}$$

Then $F = (F_n)$ is an FDD for W with projection constant $K(F, W) \leq K(E, Z)$. We now show that (F_i) satisfies subsequential \bar{C} -(V, U) estimates in W , where $\bar{C} = B \cdot \max\{2C, D\}$.

Let (z_i) be a normalized block sequence of (F_n) in W . For each $i \in \mathbb{N}$ let $k_i = \min \text{supp}_F(z_i)$ and write

$$z_i = z_i^{(1)} + z_i^{(2)}, \quad \text{where } z_i^{(1)} \in Z, \quad z_i^{(2)} \in V_{\mathbb{N} \setminus M}.$$

Fix $(a_i) \in c_{00}$. We have

$$\begin{aligned} \left\| \sum_i a_i z_i^{(1)} \right\|_Z &\geq \frac{1}{C} \left\| \sum_i a_i \cdot \|z_i^{(1)}\|_{Z \cdot v_{\min \text{supp}_F(z_i^{(1)})}} \right\|_V \\ &\geq \frac{1}{BC} \left\| \sum_i a_i \cdot \|z_i^{(1)}\|_{Z \cdot v_{k_i}} \right\|_V \end{aligned}$$

and

$$\left\| \sum_i a_i z_i^{(2)} \right\|_V \geq \frac{1}{B} \left\| \sum_i a_i \cdot \|z_i^{(2)}\|_{Z \cdot v_{k_i}} \right\|_V.$$

It follows that

$$\begin{aligned} \left\| \sum_i a_i z_i \right\|_W &\geq \frac{1}{BC} \max \left\{ \left\| \sum_i a_i \cdot \|z_i^{(1)}\|_{Z \cdot v_{k_i}} \right\|_V, \left\| \sum_i a_i \cdot \|z_i^{(2)}\|_{Z \cdot v_{k_i}} \right\|_V \right\} \\ &\geq \frac{1}{2BC} \left\| \sum_i a_i v_{k_i} \right\|_V. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \left\| \sum_i a_i z_i^{(1)} \right\|_Z &\leq C \left\| \sum_i a_i \cdot \|z_i^{(1)}\|_{Z \cdot u_{\min \text{supp}_F(z_i^{(1)})}} \right\|_U \\ &\leq BC \left\| \sum_i a_i \cdot \|z_i^{(1)}\|_{Z \cdot u_{k_i}} \right\|_U \end{aligned}$$

and

$$\begin{aligned} \left\| \sum_i a_i z_i^{(2)} \right\|_V &\leq B \left\| \sum_i a_i \cdot \|z_i^{(2)}\|_{Z \cdot v_{k_i}} \right\|_V \\ &\leq BD \left\| \sum_i a_i \cdot \|z_i^{(2)}\|_{Z \cdot u_{k_i}} \right\|_U. \end{aligned}$$

It follows that

$$\begin{aligned} \left\| \sum_i a_i z_i \right\|_W &\leq \max \left\{ BC \left\| \sum_i a_i \cdot \|z_i^{(1)}\|_{Z \cdot u_{k_i}} \right\|_U, BD \left\| \sum_i a_i \cdot \|z_i^{(2)}\|_{Z \cdot u_{k_i}} \right\|_U \right\} \\ &\leq B \cdot \max\{C, D\} \cdot \left\| \sum_i a_i u_{k_i} \right\|_U. \end{aligned}$$

□

The next two results show how norm estimates in a space and in its dual are related.

Proposition 3. *Assume that Z has an FDD (E_i) , and let V be a space with a normalized and 1-unconditional basis (v_i) . The following statements are equivalent:*

- (a) (E_i) satisfies subsequential V -lower estimates in Z .
- (b) (E_i^*) satisfies subsequential $V^{(*)}$ -upper estimates in $Z^{(*)}$.

(Here subsequential $V^{(*)}$ -upper estimates are with respect to (v_i^*) , the sequence of biorthogonal functionals to (v_i)).

Moreover, if (E_i) is bimonotone in Z , then the equivalence holds true if one replaces, for some $C \geq 1$, V -lower estimates by C - V -lower estimates in (a) and $V^{(*)}$ -upper estimates by C - $V^{(*)}$ -upper estimates in (b).

Remark. By duality, Proposition 3 holds if we interchange the words *lower* and *upper* in (a) and (b).

Proof. Without loss of generality we may assume that (E_i) is bimonotone in Z . “(a) \Rightarrow (b)” Let (z_i^*) be a normalized block sequence of $E^* = (E_n^*)$ in $Z^{(*)}$, and for each $i \in \mathbb{N}$ let $m_i = \min \text{supp}_{E^*}(z_i^*)$. Given $(a_i) \in c_{00}$, choose $z \in S_Z$ with finite support with respect to (E_n) such that $\|\sum a_i z_i^*\| = \sum a_i z_i^*(z)$. For each $i \in \mathbb{N}$ write

$$P_{[\min \text{supp}_{E^*}(z_i^*), \min \text{supp}_{E^*}(z_{i+1}^*)]}^E(z) = b_i z_i,$$

where $z_i \in S_Z$ and $|b_i| \leq 1$. Since (E_n) satisfies subsequential C - V -lower estimates in Z , we have $\|\sum b_i v_{m_i}\| \leq C \|\sum b_i z_i\| \leq C$. Hence

$$\begin{aligned} \left\| \sum a_i z_i^* \right\| &= \sum a_i b_i z_i^*(z_i) \leq \sum |a_i| |b_i| \\ &\leq \left\| \sum a_i v_{m_i}^* \right\| \cdot \left\| \sum b_i v_{m_i} \right\| \leq C \left\| \sum a_i v_{m_i}^* \right\|, \end{aligned}$$

as required.

“(b) \Rightarrow (a)” Let (z_i) be a normalized block sequence of (E_n) in Z , and for each $i \in \mathbb{N}$ let $m_i = \min \text{supp}_E(z_i)$. Given $(a_i) \in c_{00}$, choose $(b_i) \in c_{00}$ such that $\|\sum b_i v_{m_i}^*\| = 1$ and $\|\sum a_i v_{m_i}\| = \sum a_i b_i$. For each $i \in \mathbb{N}$ there exists $z_i^* \in S_{Z^{(*)}}$ such that $z_i^*(z_i) = 1$ and $\text{ran}_{E^*}(z_i^*) \subset \text{ran}_E(z_i)$. Since (E_n^*) satisfies subsequential C - $V^{(*)}$ -upper estimates in $Z^{(*)}$, we have $\|\sum b_i z_i^*\| \leq C$, and hence

$$\left\| \sum a_i z_i \right\| \geq \frac{1}{C} \sum a_i b_i = \frac{1}{C} \left\| \sum a_i v_{m_i} \right\|.$$

This completes the proof. \square

Proposition 4. *Assume that U is a space with a normalized, 1-unconditional basis (u_i) which is D -right-dominant for some $D \geq 1$, and that X is a reflexive space which satisfies subsequential C - U -upper tree estimates for some $C \geq 1$.*

Then, for any $\varepsilon > 0$, X^ satisfies subsequential $(2CD + \varepsilon)$ - $U^{(*)}$ -lower tree estimates.*

Remark. One might ask whether or not the converse of Proposition 4 is true, i.e., similar to the FDD case, whether X satisfies subsequential U -upper tree estimates if X^* satisfies subsequential $U^{(*)}$ -lower tree estimates.

The answer is affirmative under certain conditions on U , but we do not give a direct proof for that fact. Instead, we shall deduce it from one of our main embedding theorems (see Corollary 14 in Section 4).

Proof. We start with a simple observation. Let (x_i^*) be a normalized, weakly null sequence in X^* . For each $n \in \mathbb{N}$ pick $x_n \in S_X$ with $x_n^*(x_n) = 1$. There exist $y \in X$

and $k_1 < k_2 < \dots$ in \mathbb{N} such that $x_{k_n} \xrightarrow{w} y$. Given $\eta \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $|x_{k_n}^*(y)| < \eta$ for all $n \geq n_0$. Set

$$y_n^* = x_{k_{n_0+n}}^* \quad \text{and} \quad y_n = \frac{x_{k_{n_0+n}} - y}{\|x_{k_{n_0+n}} - y\|}, \quad n \in \mathbb{N}.$$

We have found, for given $\eta \in (0, 1)$, a subsequence (y_i^*) of (x_i^*) and a normalized, weakly null sequence (y_i) in X satisfying $y_n^*(y_n) > (1-\eta)/2$ for all $n \in \mathbb{N}$.

Now let $(x_\alpha^*)_{\alpha \in T_\infty^{\text{even}}}$ be a normalized, weakly null even tree in X^* . By the above observation we can find a normalized, weakly null even tree $(y_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ in X and a full subtree $(y_\alpha^*)_{\alpha \in T_\infty^{\text{even}}}$ of $(x_\alpha^*)_{\alpha \in T_\infty^{\text{even}}}$ such that $y_\alpha^*(y_\alpha) > (1-\eta)/2$ for all $\alpha \in T_\infty^{\text{even}}$. By a further pruning of these trees, we can also assume that $|y_\alpha^*(y_\beta)| < \eta/2^{\max\{|\alpha|, |\beta|\}}$ whenever $\alpha < \beta$ or $\beta < \alpha$.

By assumption, there exist $m_1 < m_2 < \dots$ in \mathbb{N} such that $(y_{(m_1, m_2, \dots, m_{2i})})$ is C -dominated by $(u_{m_{2i-1}})$. Given $(a_i) \in c_{00}$, there exists $(b_i) \in c_{00}$ such that $\|\sum b_i u_{m_{2i-1}}\| = 1$ and $\sum a_i b_i = \|\sum a_i u_{m_{2i-1}}^*\|$. So $\|\sum b_i y_{(m_1, m_2, \dots, m_{2i})}\| \leq C$, and hence

$$\begin{aligned} \left\| \sum a_i y_{(m_1, m_2, \dots, m_{2i})}^* \right\| &\geq \frac{1}{C} \left(\sum a_i b_i \frac{1-\eta}{2} - \sum_{i \neq j} |a_i| |b_j| \eta / 2^{\max\{i, j\}} \right) \\ &> \frac{1}{2C + \varepsilon/D} \left\| \sum a_i u_{m_{2i-1}}^* \right\| \end{aligned}$$

provided η is sufficiently small. Now the branch $(y_{(m_1, m_2, \dots, m_{2i})}^*)$ of $(y_\alpha^*)_{\alpha \in T_\infty^{\text{even}}}$ corresponds to a branch $(x_{(n_1, n_2, \dots, n_{2i})}^*)$ of $(x_\alpha^*)_{\alpha \in T_\infty^{\text{even}}}$, where $n_1 < n_2 < \dots$ and $m_i \leq n_i$ for all $i \in \mathbb{N}$. Since (u_i) is D -right-dominant, it follows that (u_i^*) is D -left-dominant, and hence the above inequality shows that $(x_{(n_1, n_2, \dots, n_{2i})}^*)$ $(2CD + \varepsilon)$ -dominates $(u_{n_{2i-1}}^*)$. \square

We conclude this section with a key combinatorial result. We need to fix some terminology first.

Given a Banach space X , we let $(\mathbb{N} \times S_X)^\omega$ denote the set of all sequences (k_i, x_i) , where $k_1 < k_2 < \dots$ are positive integers, and (x_i) is a sequence in S_X . We equip the set $(\mathbb{N} \times S_X)^\omega$ with the product topology of the discrete topologies of \mathbb{N} and S_X . Given $\mathcal{A} \subset (\mathbb{N} \times S_X)^\omega$ and $\varepsilon > 0$, we let

$$\mathcal{A}_\varepsilon = \left\{ (\ell_i, y_i) \in (\mathbb{N} \times S_X)^\omega : \exists (k_i, x_i) \in \mathcal{A} \quad k_i \leq \ell_i, \quad \|x_i - y_i\| < \varepsilon \cdot 2^{-i} \quad \forall i \in \mathbb{N} \right\},$$

and we let $\overline{\mathcal{A}}$ be the closure of \mathcal{A} in $(\mathbb{N} \times S_X)^\omega$.

Given $\mathcal{A} \subset (\mathbb{N} \times S_X)^\omega$, we say that an even tree $(x_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ in X has a branch in \mathcal{A} if there exist $n_1 < n_2 < \dots$ in \mathbb{N} such that $(n_{2i-1}, x_{(n_1, n_2, \dots, n_{2i})}) \in \mathcal{A}$.

Proposition 5. *Let X be an infinite-dimensional (closed) subspace of a reflexive space Z with an FDD (E_i) . Let $\mathcal{A} \subset (\mathbb{N} \times S_X)^\omega$. Then the following are equivalent.*

- (a) For all $\varepsilon > 0$ every normalized, weakly null even tree in X has a branch in $\overline{\mathcal{A}_\varepsilon}$.
- (b) For all $\varepsilon > 0$ there exist $(K_i) \subset \mathbb{N}$ with $K_1 < K_2 < \dots$, $\bar{\delta} = (\delta_i) \subset (0, 1)$ with $\delta_i \downarrow 0$, and a blocking $F = (F_i)$ of (E_i) such that if $(x_i) \subset S_X$ is a $\bar{\delta}$ -skipped block sequence of (F_n) in Z with $\|x_i - P_{(r_{i-1}, r_i)}^F x_i\| < \delta_i$ for all $i \in \mathbb{N}$, where $1 \leq r_0 < r_1 < r_2 < \dots$, then $(K_{r_{i-1}}, x_i) \in \overline{\mathcal{A}_\varepsilon}$.

Proof. For each $m \in \mathbb{N}$ we set $Z_m = \overline{\bigoplus_{i > m} E_i}$. Given $\varepsilon > 0$, we consider the following game between players S (subspace chooser) and P (point chooser). The game has an infinite sequence of moves; on the n^{th} move ($n \in \mathbb{N}$) S picks $k_n, m_n \in \mathbb{N}$ and P

responds by picking $x_n \in S_X$ with $d(x_n, Z_{m_n}) < \varepsilon' \cdot 2^{-n}$, where $\varepsilon' = \min\{\varepsilon, 1\}$. S wins the game if the sequence (k_i, x_i) the players generate ends up in $\overline{\mathcal{A}_{5\varepsilon}}$, otherwise P is declared the winner. We will refer to this as the $(\mathcal{A}, \varepsilon)$ -game and show that statements (a) and (b) above are equivalent to

(c) For all $\varepsilon > 0$ S has a winning strategy for the $(\mathcal{A}, \varepsilon)$ -game.

Note that statement (b) yields a particular winning strategy for S, so the implication (b) \Rightarrow (c) is clear, however this is included in the sequence of implications (a) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a) which is what we are about to demonstrate.

“(a) \Rightarrow (c)” Assume that for some $\varepsilon > 0$ S does not have a winning strategy for the $(\mathcal{A}, \varepsilon)$ -game. Then there is a winning strategy ϕ for the point chooser P. Thus ϕ is a function taking values in S_X such that for all sequences $(k_i), (m_i)$ in \mathbb{N} if $x_n = \phi(k_1, m_1, k_2, m_2, \dots, k_n, m_n)$ for all $n \in \mathbb{N}$, then $d(x_i, Z_{m_i}) < \varepsilon' \cdot 2^{-i}$ for all $i \in \mathbb{N}$ and $(k_i, x_i) \notin \overline{\mathcal{A}_{5\varepsilon}}$. We will now construct a normalized, weakly null even tree $(x_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ in X to show that (a) fails. This will be a recursive construction which also builds auxiliary trees $(y_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ in X and $(m_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ in \mathbb{N} .

Fix positive integers ℓ , $n_1 < n_2 < \dots < n_{2\ell-1}$. Let $\alpha = (n_1, n_2, \dots, n_{2\ell-1}) \in T_\infty$, and for $1 \leq j \leq \ell$ set $k_j = n_{2j-1}$. Assume that for $1 \leq j < \ell$ we have already defined $x_j = x_{(n_1, n_2, \dots, n_{2j})}$, $y_j = y_{(n_1, n_2, \dots, n_{2j})}$ and $m_j = m_{(n_1, n_2, \dots, n_{2j})}$ such that $y_j = \phi(k_1, m_1, k_2, m_2, \dots, k_j, m_j)$. We will now construct the nodes $(x_{(\alpha, n)})$, $(y_{(\alpha, n)})$ and $(m_{(\alpha, n)})$. Set

$$z_i = \phi(k_1, m_1, \dots, k_{\ell-1}, m_{\ell-1}, k_\ell, i), \quad i \in \mathbb{N}.$$

Note that $z_i \in S_X$ and $d(z_i, Z_i) < \varepsilon' \cdot 2^{-\ell}$ for all $i \in \mathbb{N}$. We now pass to a weakly convergent subsequence: there exist $i_1 < i_2 < \dots$ in \mathbb{N} and $z \in X$ such that $z_{i_n} \xrightarrow{w} z$ as $n \rightarrow \infty$. Note that $\|z\| \leq \varepsilon' \cdot 2^{-\ell}$. For each $n \in \mathbb{N}$ set

$$w_n = \frac{z_{i_n} - z}{\|z_{i_n} - z\|}.$$

Note that (w_n) is a normalized, weakly null sequence in X , and

$$\|z_{i_n} - w_n\| \leq \frac{2\varepsilon' \cdot 2^{-\ell}}{1 - \varepsilon' \cdot 2^{-\ell}} \leq 4\varepsilon \cdot 2^{-\ell}$$

for all $n \in \mathbb{N}$. We now set $x_{(n_1, n_2, \dots, n_{2\ell-1}, n)} = w_n$, $y_{(n_1, n_2, \dots, n_{2\ell-1}, n)} = z_{i_n}$ and $m_{(n_1, n_2, \dots, n_{2\ell-1}, n)} = i_n$ for all $n \in \mathbb{N}$ with $n > n_{2\ell-1}$. This completes the recursive construction.

It follows by induction that $(x_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ is a normalized, weakly null even tree in X and $(y_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ is a normalized even tree in X such that for all $\alpha \in T_\infty^{\text{even}}$ we have $\|x_\alpha - y_\alpha\| \leq 4\varepsilon \cdot 2^{-|\alpha|/2}$. Moreover, given a sequence $n_1 < n_2 < \dots$ in \mathbb{N} , setting $k_j = n_{2j-1}$, $m_j = m_{(n_1, n_2, \dots, n_{2j})}$ and $y_j = y_{(n_1, n_2, \dots, n_{2j})}$ for all $j \in \mathbb{N}$, we have

$$y_n = \phi(k_1, m_1, k_2, m_2, \dots, k_n, m_n) \quad \text{for all } n \in \mathbb{N}.$$

Hence no branch of $(y_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ is in $\overline{\mathcal{A}_{5\varepsilon}}$, and no branch of $(x_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ is in $\overline{\mathcal{A}_\varepsilon}$.

“(c) \Rightarrow (b)” Let (ϕ, ψ) be a winning strategy for S in the $(\mathcal{A}, \varepsilon)$ -game. Thus ϕ and ψ are functions taking values in \mathbb{N} such that for all sequences $(k_i), (m_i)$ in \mathbb{N} and (x_i) in S_X if $d(x_n, Z_{m_n}) < \varepsilon' \cdot 2^{-n}$, $k_n = \phi(x_1, x_2, \dots, x_{n-1})$ and $m_n \geq \psi(x_1, x_2, \dots, x_{n-1})$ for all $n \in \mathbb{N}$, then $(k_i, x_i) \in \overline{\mathcal{A}_{5\varepsilon}}$. For each interval $I \subset \mathbb{N}$ and $\delta > 0$ fix a finite set $S_{I, \delta} \subset S_X$ such that for all $x \in S_{I, \delta}$ we have $\|x - P_I^E x\| < \delta$ and for all $y \in S_X$ if $\|y - P_I^E y\| < \delta$, then there exists $x \in S_{I, \delta}$ such that $\|x - y\| < 3\delta$.

We now construct a blocking (F_i) of (E_i) by recursion. Let $m_1 = \psi()$ and $F_1 = \bigoplus_{i=1}^{m_1} E_i$. Choose any $m_2 > m_1$ and set $F_2 = \bigoplus_{i=m_1+1}^{m_2} E_i$. Assume that for some $n \in \mathbb{N}$, $n \geq 3$, we have already chosen $m_1 < \dots < m_{n-1}$ and we have set

$F_j = \bigoplus_{i=m_{j-1}+1}^{m_j} E_i$ for $1 \leq j < n$ ($m_0 = 0$). We now choose $m_n > m_{n-1}$ such that if $\ell \in \mathbb{N}$, $1 \leq r_0 < r_1 < \dots < r_\ell \leq n$ and

$$x_j \in S_{[m_{r_{j-1}}+1, m_{r_j-1}], \varepsilon' \cdot 2^{-j}} \quad \text{for } 1 \leq j \leq \ell,$$

then $m_n \geq \psi(x_1, x_2, \dots, x_\ell)$. Finally, we set $F_n = \bigoplus_{i=m_{n-1}+1}^{m_n} E_i$. This completes the recursive construction.

For each $n \in \mathbb{N}$ let $\delta_n = \varepsilon' \cdot 2^{-n}$, and let K_n be chosen so that $K_n \geq \phi()$, and if $\ell \in \mathbb{N}$, $1 \leq r_0 < r_1 < \dots < r_\ell \leq n$ and $x_j \in S_{[m_{r_{j-1}}+1, m_{r_j-1}], \varepsilon' \cdot 2^{-j}}$ for $1 \leq j \leq \ell$, then $K_n \geq \phi(x_1, \dots, x_\ell)$. We can of course also ensure that the sequence (K_i) is strictly increasing. Let $\bar{\delta} = (\delta_i)$. We will now verify that (b) holds. Let (y_i) be a $\bar{\delta}$ -skipped block sequence of (F_n) : there exist $1 \leq r_0 < r_1 < r_2 < \dots$ such that

$$\|y_i - P_{[r_{i-1}+1, r_i-1]}^F y_i\| < \varepsilon' \cdot 2^{-i} \quad \text{for all } i \in \mathbb{N},$$

that is to say,

$$\|y_i - P_{[m_{r_{i-1}}+1, m_{r_i-1}]}^E y_i\| < \varepsilon' \cdot 2^{-i} \quad \text{for all } i \in \mathbb{N}.$$

For each $i \in \mathbb{N}$ there exists $x_i \in S_{[m_{r_{i-1}}+1, m_{r_i-1}], \varepsilon' \cdot 2^{-i}}$ such that $\|x_i - y_i\| < 3\varepsilon' \cdot 2^{-i}$. Set

$$k_i = \phi(x_1, \dots, x_{i-1}) \quad \text{for each } i \in \mathbb{N}.$$

Consider the sequence $k_1, m_{r_0}, x_1, k_2, m_{r_1}, x_2, \dots$. We have $x_i \in S_X$ and

$$d(x_i, Z_{m_{r_{i-1}}}) \leq \|x_i - P_{[m_{r_{i-1}}+1, m_{r_i-1}]}^E x_i\| < \varepsilon' \cdot 2^{-i}$$

for all $i \in \mathbb{N}$. Moreover $m_{r_0} \geq m_1 \geq \psi()$, $K_{r_0} \geq K_1 \geq \phi() = k_1$, and given $\ell \in \mathbb{N}$, setting $n = r_\ell$, we have $1 \leq r_0 < r_1 < \dots < r_\ell \leq n$ and $x_i \in S_{[m_{r_{i-1}}+1, m_{r_i-1}], \varepsilon' \cdot 2^{-i}}$ for $1 \leq i \leq \ell$. It follows that ($n \geq 3$ and) $m_{r_\ell} = m_n \geq \psi(x_1, \dots, x_\ell)$ and $k_{\ell+1} \leq K_n = K_{r_\ell}$. So $(k_i, x_i) \in \overline{\mathcal{A}_{5\varepsilon}}$, and hence $(K_{r_{i-1}}, y_i) \in \overline{\mathcal{A}_{8\varepsilon}}$.

“(b) \Rightarrow (a)” Given $\varepsilon > 0$, let (K_i) , $\bar{\delta} = (\delta_i)$ and (F_i) be as in statement (b). First note that if (x_i) is a normalized, weakly null sequence in X , then

$$\forall \eta > 0 \forall p \in \mathbb{N} \exists n \in \mathbb{N} \exists q > p \quad \text{such that} \quad \|x_n - P_{(p,q)}^F x_n\| < \eta.$$

Indeed, the sequence $(P_{[1,p]}^F x_i)$ is weakly null, and hence norm-null, so there exists $n \in \mathbb{N}$ such that $\|P_{[1,p]}^F x_n\| < \eta/2$. One can then choose $q > p$ such that $\|P_{[q,\infty)}^F x_n\| < \eta/2$. The claim now follows by triangle-inequality.

Now let $(x_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ be a normalized, weakly null even tree in X . We choose positive integers $n_1 < n_2 < \dots$ and $1 = r_0 < r_1 < r_2 < \dots$ by recursion. For $\ell \in \mathbb{N}$ we first choose $n_{2\ell-1} > K_{r_{\ell-1}}$ such that $n_{2\ell-1} > n_{2\ell-2}$ ($n_0 = 0$), and then choose $n_{2\ell} > n_{2\ell-1}$ and $r_\ell > r_{\ell-1}$ such that

$$\|x_{(n_1, n_2, \dots, n_{2\ell})} - P_{(r_{\ell-1}, r_\ell)}^F x_{(n_1, n_2, \dots, n_{2\ell})}\| < \varepsilon \cdot 2^{-\ell}.$$

By assumption (b) we have $(n_{2i-1}, x_{(n_1, n_2, \dots, n_{2i})}) \in \overline{\mathcal{A}_\varepsilon}$. \square

3. THE SPACE $Z^V(E)$

Let Z be a space with an FDD $E = (E_i)$, and let V be a space with a 1-unconditional and normalized basis (v_i) . The space $Z^V = Z^V(E)$ is defined to be the completion of $c_{00}(\bigoplus E_i)$ with respect to the following norm $\|\cdot\|_{Z^V}$.

$$\|z\|_{Z^V} = \max_{\substack{k \in \mathbb{N} \\ 1 \leq n_0 < n_1 < n_2 < \dots < n_k}} \left\| \sum_{j=1}^k \|P_{[n_{j-1}, n_j]}^E(z)\|_{Z \cdot v_{n_{j-1}}} \right\|_V \quad \text{for } z \in c_{00}(\bigoplus E_i).$$

Note that if (v_i) is C -block-stable and D -right-dominant, then the projection constant $K(E, Z^V)$ of (E_i) in Z^V satisfies

$$K(E, Z^V) \leq \min\{K(E, Z), C, D, 2\}.$$

Here we allow $C = \infty$ or $D = \infty$ if (v_i) is not block-stable or not right-dominant, respectively. Note also that if $\|\cdot\|$ and $\|\cdot\|'$ are equivalent norms on Z , then the corresponding norms $\|\cdot\|_{Z^V}$ and $\|\cdot\|'_{Z^V}$ are equivalent on $c_{00}(\oplus E_i)$. This often allows us, when examining the space Z^V , to assume that (E_i) is bimonotone in Z .

Our first set of results culminating in Corollary 9 determine when the space $Z^V(E)$ is reflexive.

Lemma 6. *Every normalized block sequence (z_i) of (E_n) in Z^V 1-dominates some block sequence (b_i) of (v_n) that satisfies*

$$1/2 \leq \|b_i\|_V \leq 1 \quad \text{and} \quad \text{ran}(b_i) \subset \text{ran}_E(z_i) \quad \text{for all } i \in \mathbb{N}.$$

(Here the range, $\text{ran}(x)$, of $x = \sum a_i v_i \in V$ is the set $\{i \in \mathbb{N} : a_i \neq 0\}$.)

Proof. Let $z \in S_{Z^V}$ have finite support with respect to (E_i) . Choose k , $1 \leq n_0 < n_1 < \dots < n_k$ in \mathbb{N} such that

$$\|z\|_{Z^V} = \left\| \sum_{j=1}^k \|P_{[n_{j-1}, n_j]}^E(z)\|_{Z \cdot v_{n_{j-1}}} \right\|_V.$$

Without loss of generality we can assume that $n_0 \leq \min \text{supp}_E(z) \leq n_1 - 1$ and that $n_{k-1} \leq \max \text{supp}_E(z) = n_k - 1$. Set $m_0 = \min \text{supp}_E(z)$, $m_j = n_j$ for $1 \leq j < k$, and let $m_k > \max \text{supp}_E(z)$. By the triangle-inequality we have

$$\begin{aligned} \|z\|_{Z^V} &\leq \left\| \|P_{[n_0, n_1]}^E(z)\|_{Z \cdot v_{n_0}} \right\|_V + \left\| \sum_{j=2}^k \|P_{[n_{j-1}, n_j]}^E(z)\|_{Z \cdot v_{n_{j-1}}} \right\|_V \\ &\leq 2 \left\| \sum_{j=1}^k \|P_{[m_{j-1}, m_j]}^E(z)\|_{Z \cdot v_{m_{j-1}}} \right\|_V. \end{aligned}$$

Now let (z_ℓ) be a normalized block sequence of (E_n) in Z^V . It follows from the above that there exist positive integers $1 \leq n_0 < n_1 < \dots$ and $1 = k_1 < k_2 < \dots$ such that $n_{k_\ell - 1} = \min \text{supp}_E(z_\ell)$, $n_{k_{\ell+1} - 2} \leq \max \text{supp}_E(z_\ell)$, and

$$\|z_\ell\|_{Z^V} \leq 2 \left\| \sum_{j=k_\ell}^{k_{\ell+1} - 1} \|P_{[n_{j-1}, n_j]}^E(z_\ell)\|_{Z \cdot v_{n_{j-1}}} \right\|_V \quad \text{for all } \ell \in \mathbb{N}.$$

It follows that

$$b_\ell = \sum_{j=k_\ell}^{k_{\ell+1} - 1} \|P_{[n_{j-1}, n_j]}^E(z_\ell)\|_{Z \cdot v_{n_{j-1}}}$$

satisfies $1/2 \leq \|b_\ell\|_V \leq 1$ and $\text{ran}(b_\ell) \subset \text{ran}_E(z_\ell)$ for all $\ell \in \mathbb{N}$. Moreover, given $(a_i) \in c_{00}$, setting $z = \sum a_i z_i$ we have

$$\begin{aligned} \|z\|_{Z^V} &\geq \left\| \sum_{j=1}^{\infty} \|P_{[n_{j-1}, n_j]}^E(z)\|_{Z \cdot v_{n_{j-1}}} \right\|_V \\ &= \left\| \sum_{\ell=1}^{\infty} \sum_{j=k_\ell}^{k_{\ell+1} - 1} \|P_{[n_{j-1}, n_j]}^E(z)\|_{Z \cdot v_{n_{j-1}}} \right\|_V \\ &= \left\| \sum_{\ell=1}^{\infty} a_\ell b_\ell \right\|_V. \end{aligned}$$

□

Corollary 7. *Let V be a Banach space with a normalized and 1-unconditional basis (v_i) , and let Z be a space with an FDD $E = (E_i)$.*

If the basis (v_i) is boundedly complete, then (E_i) is a boundedly complete FDD for $Z^V(E)$.

Proof. Let (z_i) be a normalized block sequence of (E_n) in Z^V . Let (b_i) be a block sequence of (v_n) given by Lemma 6. Given $\varepsilon > 0$, let (a_i) be a scalar sequence with $|a_i| > \varepsilon$ for all $i \in \mathbb{N}$. Since (v_i) is boundedly complete, and since (z_i) dominates (b_i) it follows that

$$\sup_n \left\| \sum_{i=1}^n a_i z_i \right\|_{Z^V} = \infty.$$

Hence (E_i) is a boundedly complete FDD for $Z^V(E)$. \square

Lemma 8. *Let V be a Banach space with a normalized and 1-unconditional basis (v_i) , and assume that the space Z has an FDD $E = (E_i)$.*

If the basis (v_i) is shrinking and if (E_i) is a shrinking FDD for Z then (E_i) is also a shrinking FDD for $Z^V(E)$.

Proof. Without loss of generality we may assume that (E_i) is bimonotone in Z . We first note that, given positive integers $1 \leq n_0 < n_1 < \dots$ and vectors $z_j^* \in \bigoplus_{i \in [n_{j-1}, n_j]} E_i^*$ with $\|z_j^*\|_{Z^*} \leq 1$ for each $j \in \mathbb{N}$, if $v^* = \sum_{i=1}^{\infty} a_i v_{n_{i-1}}^*$ converges in V^* with $\|v^*\| \leq 1$, then the series $z^* = \sum_{i=1}^{\infty} a_i z_i^*$ converges in $(Z^V)^*$ and $\|z^*\|_{(Z^V)^*} \leq 1$. Indeed, for $p \leq q$ in \mathbb{N} there exists $z \in S_{Z^V}$ with $\text{supp}_E(z) \subset [n_{p-1}, n_q]$ such that

$$\begin{aligned} (1) \quad \left\| \sum_{j=p}^q a_j z_j^* \right\|_{(Z^V)^*} &= \sum_{j=p}^q a_j z_j^*(z) \leq \sum_{j=p}^q |a_j| \cdot \|P_{[n_{j-1}, n_j]}^E(z)\|_Z \\ &\leq \left\| \sum_{j=p}^q a_j v_{n_{j-1}}^* \right\|_{V^*} \cdot \left\| \sum_{j=p}^q \|P_{[n_{j-1}, n_j]}^E(z)\|_Z \cdot v_{n_{j-1}} \right\|_V \\ &\leq \left\| \sum_{j=p}^q a_j v_{n_{j-1}}^* \right\|_{V^*} \cdot \|z\|_{Z^V} \leq \left\| \sum_{j=p}^q a_j v_{n_{j-1}}^* \right\|_{V^*}, \end{aligned}$$

which implies the claim. Next define K to be the union of the following two sets K_1 and K_2 :

$$K_1 = \left\{ \sum_{i=1}^{\infty} a_i z_i^* : 1 \leq n_0 < n_1 < \dots, \quad z_j^* \in \bigoplus_{i \in [n_{j-1}, n_j]} E_i^* \text{ and } \|z_j^*\|_{Z^*} \leq 1 \quad \forall j \in \mathbb{N}, \right. \\ \left. \left\| \sum_{i=1}^{\infty} a_i v_{n_{i-1}}^* \right\|_{V^*} \leq 1 \right\},$$

$$K_2 = \left\{ \sum_{i=1}^{\ell} a_i z_i^* : \ell \in \mathbb{N}, \quad 1 \leq n_0 < n_1 < \dots < n_{\ell} \leq \infty, \quad z_j^* \in \bigoplus_{i \in [n_{j-1}, n_j]} E_i^* \text{ and} \right. \\ \left. \|z_j^*\|_{Z^*} \leq 1 \text{ for } 1 \leq j \leq \ell, \quad \left\| \sum_{i=1}^{\ell} a_i v_{n_{i-1}}^* \right\|_{V^*} \leq 1 \right\}.$$

An element $\sum_{i=1}^{\ell} a_i z_i^*$ of K_2 will also be written as an infinite sum $\sum_{i=1}^{\infty} a_i z_i^*$ by setting $a_i = 0$ and $z_i^* = 0$ for all $i > \ell$. Clearly, K is a Z^V -norming subset (isometrically) of $B_{(Z^V)^*}$. We claim that K is w^* -compact. Indeed, for each $k \in \mathbb{N}$ let $y_k^* = \sum_{i=1}^{\infty} a_i^{(k)} z_{(k,i)}^* \in K$, where for some (finite or infinite) sequence $1 \leq n_0^{(k)} < n_1^{(k)} < n_2^{(k)} < \dots$ in $\mathbb{N} \cup \{\infty\}$ we have $z_{(k,j)}^* \in \bigoplus_{i \in [n_{j-1}^{(k)}, n_j^{(k)}]} E_i^*$ and $\|z_{(k,j)}^*\|_{Z^*} \leq 1$

for all j , and $\left\| \sum_i a_i^{(k)} v_{n_{i-1}}^* \right\|_{V^*} \leq 1$. After passing to a subsequence we can assume that

$$a_i = \lim_{k \rightarrow \infty} a_i^{(k)} \quad \text{exists for all } i \in \mathbb{N}, \text{ and}$$

there exists $\ell \in \mathbb{N} \cup \{0, \infty\}$ such that

$$\begin{aligned} \lim_{k \rightarrow \infty} n_j^{(k)} &= n_j && \text{exists for } 0 \leq j < \ell, \\ \lim_{k \rightarrow \infty} n_\ell^{(k)} &= \infty && \text{if } \ell \in \mathbb{N} \cup \{0\}, \\ \lim_{k \rightarrow \infty} z_{(k,j)}^* &= z_j^* && \text{exists (in norm) for } 0 \leq j < \ell, \\ w^* \text{-} \lim_{k \rightarrow \infty} z_{(k,\ell)}^* &= z_\ell^* && \text{exists if } \ell \in \mathbb{N} \cup \{0\}, \text{ and} \\ w^* \text{-} \lim_{k \rightarrow \infty} \sum_{i \in \mathbb{N}, i > \ell} a_i^{(k)} z_{(k,i)}^* &= 0 && \text{if } \ell \in \mathbb{N} \cup \{0\}. \end{aligned}$$

Consider the case when $\ell = \infty$. We have $1 \leq n_0 < n_1 < \dots$, $z_j^* \in \bigoplus_{[n_{j-1}, n_j]} E_i^*$ and $\|z_j^*\|_{Z^*} \leq 1$ for all $j \in \mathbb{N}$. Moreover, since (v_i^*) is a boundedly complete basis of V^* , the series $\sum_{i=1}^\infty a_i v_{n_{i-1}}^*$ converges and $\left\| \sum_{i=1}^\infty a_i v_{n_{i-1}}^* \right\|_{V^*} \leq 1$. Hence $z^* = \sum_{i=1}^\infty a_i z_i^*$ belongs to K . Finally, given $z \in S_Z$ with finite support with respect to (E_i) , for sufficiently large $r \in \mathbb{N}$ we have

$$\begin{aligned} \lim_{k \rightarrow \infty} y_k^*(z) &= \lim_{k \rightarrow \infty} \sum_{i=1}^r a_i^{(k)} z_{(k,i)}^*(z) \\ &= \sum_{i=1}^r a_i z_i^*(z) = z^*(z). \end{aligned}$$

It follows that $y_k^* \xrightarrow{w^*} z^*$ as $k \rightarrow \infty$. The case $\ell \in \mathbb{N} \cup \{0\}$ is similar. We have $1 \leq n_0 < n_1 < \dots < n_\ell \leq \infty$, $z_j^* \in \bigoplus_{[n_{j-1}, n_j]} E_i^*$ and $\|z_j^*\|_{Z^*} \leq 1$ for $1 \leq j \leq \ell$, and $\sum_{i=1}^\ell a_i v_{n_{i-1}}^* \in B_{V^*}$. So $z^* = \sum_{i=1}^\ell a_i z_i^* \in K$ and $y_k^* \xrightarrow{w^*} z^*$ as $k \rightarrow \infty$. This completes the proof that K is w^* -closed.

We deduce that Z^V is embedded in $C(K)$, the space of continuous functions on K . Let (z_i) be a bounded block sequence of (E_n) in Z^V , and let $z^* \in K$. Using the notation as in the definition of K , if $z^* \in K_1$, then computing as in (1)

$$z^*(z_i) = \sum_{j, n_j \geq \min \text{supp}_E(z_i)} a_j z_j^*(z_i) \leq \|z_i\|_{Z^V} \cdot \left\| \sum_{j, n_j \geq \min \text{supp}_E(z_i)} a_j v_{n_{j-1}}^* \right\|_{V^*},$$

which converges to zero as $i \rightarrow \infty$; and if $z^* \in K_2$, then for all sufficiently large values of i

$$z^*(z_i) = \sum_{j=1}^\ell a_j z_j^*(z_i) = a_\ell z_\ell^*(z_i),$$

which converges to zero as $i \rightarrow \infty$, since (E_i) is assumed a shrinking FDD for Z .

It follows that (z_i) is weakly null in $C(K)$, and thus in Z^V . Since (z_i) was an arbitrary bounded block sequence in Z^V , this finishes the proof that (E_i) is shrinking in Z^V . \square

From Lemma 8 and Corollary 7 we obtain the following result.

Corollary 9. *Assume that V is a reflexive Banach space with a normalized and 1-unconditional basis (v_i) and that Z is a space with a shrinking FDD $E = (E_i)$. Then $Z^V(E)$ is reflexive.*

The idea of the norm $\|\cdot\|_{Z^V}$ is, of course, to introduce a subsequential V -lower-estimate. The next lemma determines when this is the case.

Lemma 10. *Let V be a Banach space with a normalized and 1-unconditional basis (v_i) , and let Z be a Banach space with an FDD $E=(E_i)$.*

If, for some $C \geq 1$, (v_i) is C -block stable, then (E_i) satisfies subsequential $2C$ - V -lower estimates in $Z^V(E)$.

Proof. Let (z_i) be a normalized block sequence in $Z^V(E)$, and for each $i \in \mathbb{N}$ let $m_i = \min \text{supp}_E(z_i)$. By Lemma 6, there exists a block sequence (b_i) of (v_n) with $1/2 \leq \|b_i\|_V \leq 1$ and $\text{ran}(b_i) \subset \text{ran}_E(z_i)$ for all $i \in \mathbb{N}$, which is 1-dominated by (z_i) . Since (v_i) is 1-unconditional and C -block-stable, it follows that (b_i) $2C$ -dominates (v_{m_i}) , which proves the lemma. \square

The final result in this section shows when subsequential U -upper estimates are preserved under $Z \mapsto Z^V$.

Lemma 11. *Let V and U be Banach spaces with normalized, 1-unconditional and block-stable bases (v_i) and (u_i) , respectively, and assume that (v_i) is dominated by (u_i) . Let Z be a Banach space with an FDD (E_i) .*

If (E_i) satisfies subsequential U -upper estimates in Z , then (E_i) also satisfies subsequential U -upper estimates in Z^V .

Proof. Choose constants B_V, B_U, D and C in $[1, \infty)$ such that (v_i) is B_V -block-stable, (u_i) is B_U -block-stable, (v_i) is D -dominated by (u_i) , and (E_i) satisfies subsequential C - U -upper estimates in Z . Let K be the projection constant of (E_i) in Z , and set $\bar{C} = B_V D + B_U C D + 2B_V D K$. We show that for any finite block sequence $(z_i)_{i=1}^\ell$ of (E_n) , and for any k and $n_0 < n_1 < \dots < n_k$ in \mathbb{N} we have (putting $z = \sum_{i=1}^\ell z_i$ and $m_j = \min \text{supp}_E(z_j)$ for $1 \leq j \leq \ell$)

$$(2) \quad \left\| \sum_{j=1}^k \|P_{[n_{j-1}, n_j]}^E(z)\|_{Z \cdot v_{n_{j-1}}} \right\|_V \leq \bar{C} \cdot \left\| \sum_{i=1}^\ell \|z_i\|_{Z^V \cdot u_{m_i}} \right\|_U.$$

Taking then the supremum of the left side of (2) over all choices of k and $n_0 < n_1 < \dots < n_k$ in \mathbb{N} , we obtain

$$\left\| \sum_{i=1}^\ell z_i \right\|_{Z^V} \leq \bar{C} \cdot \left\| \sum_{i=1}^\ell \|z_i\|_{Z^V \cdot u_{m_i}} \right\|_U,$$

and thus that (E_i) satisfies subsequential \bar{C} - U -upper estimates in Z^V . Note that in proving (2) we can of course assume that $n_k \leq \max \text{supp}_E(z_\ell) + 1$.

For $i = 1, 2, \dots, \ell$ put

$$J_i = \{j \in \{1, 2, \dots, k\} : \min \text{supp}_E(z_i) \leq n_{j-1} < n_j \leq \max \text{supp}_E(z_{i+1})\}$$

(with $\min \text{supp}_E(z_{\ell+1}) = \max \text{supp}_E(z_\ell) + 1$) and $J_0 = \{1, 2, \dots, k\} \setminus \bigcup_{i=1}^\ell J_i$.

For $j = 1, 2, \dots, k$ put

$$I_j = \{i \in \{1, 2, \dots, \ell\} : n_{j-1} \leq \min \text{supp}_E(z_i) \leq \max \text{supp}_E(z_i) < n_j\}$$

and $I_0 = \{1, 2, \dots, \ell\} \setminus \bigcup_{j=1}^k I_j$.

Firstly, we have

$$(3) \quad \left\| \sum_{i=1}^\ell \sum_{j \in J_i} \|P_{[n_{j-1}, n_j]}^E(z_i)\|_{Z \cdot v_{n_{j-1}}} \right\|_V = \left\| \sum_{i=1}^\ell b_i \right\|_V$$

(where $b_i = \sum_{j \in J_i} \|P_{[n_{j-1}, n_j]}^E(z_i)\|_{Z \cdot v_{n_{j-1}}}$ for $1 \leq i \leq \ell$)

$$\begin{aligned} &\leq B_V \left\| \sum_{i=1}^{\ell} \|b_i\|_{V \cdot v_{m_i}} \right\|_V \\ &\leq B_V D \left\| \sum_{i=1}^{\ell} \|z_i\|_{Z^V \cdot u_{m_i}} \right\|_U . \end{aligned}$$

Secondly,

$$(4) \quad \left\| \sum_{j \in J_0} \left\| \sum_{i \in I_j} z_i \right\|_{Z \cdot v_{n_{j-1}}} \right\|_V \leq C \left\| \sum_{j \in J_0} \|b_j\|_{U \cdot v_{n_{j-1}}} \right\|_V$$

(where $b_j = \sum_{i \in I_j} \|z_i\|_{Z \cdot u_{m_i}}$ for each $j \in J_0$)

$$\begin{aligned} &\leq CD \left\| \sum_{j \in J_0} \|b_j\|_{U \cdot u_{n_{j-1}}} \right\|_U \\ &\leq CDB_U \left\| \sum_{j \in J_0} b_j \right\|_U \leq CDB_U \left\| \sum_{i=1}^{\ell} \|z_i\|_{Z^V \cdot u_{m_i}} \right\|_U . \end{aligned}$$

Thirdly, given $j \in J_0$ and $i \in I_0$ such that $P_{[n_{j-1}, n_j]}^E(z_i) \neq 0$, we have either

$$(5) \quad n_{j-1} < \min \text{supp}_E(z_i) < n_j \leq \max \text{supp}_E(z_i) , \quad \text{or}$$

$$(6) \quad \min \text{supp}_E(z_i) < n_{j-1} \leq \max \text{supp}_E(z_i) < n_j .$$

Let $J_{0,1}$ be the set of all $j \in J_0$ for which there exists an $i \in I_0$ such that (5) holds and let i_j^1 denote the unique such $i \in I_0$. Similarly, we let $J_{0,2}$ be the set of all $j \in J_0$ for which there exists an $i \in I_0$ such that (6) holds and we denote by i_j^2 the unique such $i \in I_0$. We now obtain

$$\begin{aligned} (7) \quad &\left\| \sum_{j \in J_0} \sum_{i \in I_0} \|P_{[n_{j-1}, n_j]}^E(z_i)\|_{Z \cdot v_{n_{j-1}}} \right\|_V \\ &\leq K \left\| \sum_{j \in J_{0,1}} \|z_{i_j^1}\|_{Z \cdot v_{n_{j-1}}} \right\|_V + K \left\| \sum_{j \in J_{0,2}} \|z_{i_j^2}\|_{Z \cdot v_{n_{j-1}}} \right\|_V \\ &\leq KB_V \left\| \sum_{j \in J_{0,1}} \|z_{i_j^1}\|_{Z \cdot v_{m_{i_j^1}}} \right\|_V + KB_V \left\| \sum_{j \in J_{0,2}} \|z_{i_j^2}\|_{Z \cdot v_{m_{i_j^2}}} \right\|_V \\ &\leq 2KB_V D \left\| \sum_{i=1}^{\ell} \|z_i\|_{Z^V \cdot u_{m_i}} \right\|_U . \end{aligned}$$

Finally, we deduce from (3), (4) and (7) that

$$\begin{aligned}
& \left\| \sum_{j=1}^k \|P_{[n_{j-1}, n_j]}^E(z)\|_{Z \cdot v_{n_{j-1}}} \right\|_V \\
& \leq \left\| \sum_{i=1}^{\ell} \sum_{j \in J_i} \|P_{[n_{j-1}, n_j]}^E(z_i)\|_{Z \cdot v_{n_{j-1}}} \right\|_V \\
& \quad + \left\| \sum_{j \in J_0} \|P_{[n_{j-1}, n_j]}^E(z)\|_{Z \cdot v_{n_{j-1}}} \right\|_V \\
& \leq \left\| \sum_{i=1}^{\ell} \sum_{j \in J_i} \|P_{[n_{j-1}, n_j]}^E(z_i)\|_{Z \cdot v_{n_{j-1}}} \right\|_V \\
& \quad + \left\| \sum_{j \in J_0} \left\| \sum_{i \in I_j} z_i \right\|_{Z \cdot v_{n_{j-1}}} \right\|_V \\
& \quad + \left\| \sum_{j \in J_0} \sum_{i \in I_0} \|P_{[n_{j-1}, n_j]}^E(z_i)\|_{Z \cdot v_{n_{j-1}}} \right\|_V \\
& \leq (B_V D + B_U C D + 2B_V D K) \left\| \sum_{i=1}^m \|z_i\|_{Z^V \cdot u_{m_i}} \right\|_U,
\end{aligned}$$

which finishes the proof of (2). \square

4. EMBEDDING THEOREMS

In this section we will prove and deduce some consequences of

Theorem 12. *Assume that V is a Banach space with a normalized, 1-unconditional and left-dominant basis (v_i) . Let X be a separable, infinite-dimensional, reflexive space with subsequential V -lower tree estimates.*

- (a) *For every reflexive space Z with an FDD $E = (E_i)$ which contains X there is a blocking $H = (H_i)$ of (E_i) , and there exists $N \in \mathbb{N}^{(\omega)}$ such that X naturally isomorphically embeds into $Z^{V_N}(H)$.*
- (b) *There is a space \tilde{Y} with a bimonotone, shrinking FDD $\tilde{G} = (\tilde{G}_i)$, and there exists $N \in \mathbb{N}^{(\omega)}$ such that X is a quotient of $\tilde{Y}^{V_N}(\tilde{G})$.*

Recall that $\mathbb{N}^{(\omega)}$ denotes the set of all infinite subsets of \mathbb{N} , and if V is a Banach space with a normalized, 1-unconditional basis (v_i) , and if $N \in \mathbb{N}^{(\omega)}$, then we write V_N for the closed linear span of $\{v_i : i \in N\}$. When we talk about subsequential V_N -lower estimates, etc., it is with respect to the normalized, 1-unconditional basis $(v_i)_{i \in N}$ of V_N .

Remark. Theorem 12 has a quantitative version. Let $C, D \in [1, \infty)$ and assume that (v_i) is D -left-dominant and that X satisfies subsequential C - V -lower tree estimates.

Then for all $K \in [1, \infty)$ there is a constant $M = M(C, D, K) \in [1, \infty)$ such that in part (a) if $K(E, Z) \leq K$, then in the conclusion X M -embeds into $Z^{V_N}(H)$. Indeed, this follows directly from the proof. What is important is that M depends only on the constants C, D and K .

Also, there exists a constant $L = L(C, D) \in [1, \infty)$ such that in the conclusion of part (b) we get an onto map $\tilde{Q} : \tilde{Y}^{V_N}(\tilde{G}) \rightarrow X$ with $\|\tilde{Q}\| = 1$ and $\tilde{Q}(L \cdot B_{\tilde{Y}^{V_N}}) \supset B_X$. This also follows directly from the proof. However, the proof of part (b) uses [10, Lemma 3.1], which in turn appeals to a theorem of Zippin [18]. The theorem of Zippin we need here states that every separable, reflexive space embeds isometrically into a reflexive space with an FDD. A quantitative version of this result claims

the existence of a universal constant K such that every separable, reflexive space embeds isometrically into a reflexive space with an FDD whose projection constant is at most K . Indeed, if this wasn't true, then for all $n \in \mathbb{N}$ there would be a "bad" space X_n , and then the ℓ_2 -sum of the sequence (X_n) would contradict Zippin's theorem. The existence of this universal constant K gives a quantitative version of (a special case of) [10, Lemma 3.1]: there is a universal constant \bar{K} such that every separable, reflexive space X embeds isometrically into a reflexive space Z with an FDD $E = (E_i)$ with $K(E, Z) \leq \bar{K}$ such that $c_{00}(\oplus_{i=1}^{\infty} E_i) \cap X$ is dense in X . The proof of part (b) now really does give the quantitative version of (b) stated above.

The consequences of all this are quantitative analogues of Corollaries 13 and 14, and of Theorem 15. We shall state (without proof) the quantitative analogue of Theorem 15, and leave the reader to formulate the analogues of Corollaries 13 and 14. The proofs are straightforward: one simply needs to keep track of the various constants in the proofs of the qualitative statements.

Corollary 13. *Assume that V is a reflexive Banach space with a normalized and 1-unconditional basis (v_i) , and that (v_i) is left-dominant and block-stable. Let X be a separable, infinite-dimensional, reflexive space with subsequential V -lower tree estimates.*

Then X is a subspace of a reflexive space Z with an FDD satisfying subsequential V -lower estimates and it is a quotient of a reflexive space Y with an FDD satisfying subsequential V -lower estimates.

Proof. By a theorem of Zippin [18] we can embed X into a reflexive space W with an FDD $E = (E_i)$. Using Theorem 12 (a) we find a blocking $F = (F_i)$ of (E_i) and $L \in \mathbb{N}^{(\omega)}$ such that X embeds into $Z = W^{V_L}(F)$.

Theorem 12 (b) provides a space \tilde{Y} with a shrinking FDD $\tilde{G} = (\tilde{G}_i)$ and $M \in \mathbb{N}^{(\omega)}$ such that X is a quotient of $Y = \tilde{Y}^{V_M}(\tilde{G})$.

By Corollary 9 the spaces Z and Y are reflexive. It follows from Lemma 10 that (F_i) satisfies subsequential V_L -lower estimates in Z , and that (\tilde{G}_i) satisfies subsequential V_M -lower estimates in Y . The result now follows from Lemma 2 (with (u_i) the unit vector basis of $U = \ell_1$). \square

From Corollary 13 and Proposition 3 we deduce in certain instances the inverse implication of Proposition 4.

Corollary 14. *Assume that V is a reflexive Banach space with a normalized, 1-unconditional basis (v_i) , and that (v_i) is left-dominant and block-stable.*

If X is a separable, infinite-dimensional, reflexive space which satisfies subsequential V -lower tree estimates, then X^ satisfies subsequential V^* -upper tree estimates.*

Proof. By Corollary 13 X is a quotient of a reflexive space with an FDD satisfying subsequential V -lower estimates. Hence, by Proposition 3, X^* is the subspace of a reflexive space Z with an FDD (E_i) satisfying subsequential V^* -upper estimates.

Now let $(x_\alpha)_{\alpha \in T_{\infty}^{\text{even}}}$ be a normalized, weakly null even tree in X^* . One can recursively choose $n_1 < n_2 < \dots$ in \mathbb{N} such that

$$\|x_{(n_1, n_2, \dots, n_{2i})} - P_{[n_{2i-1}, n_{2i+1}]^E}(x_{(n_1, n_2, \dots, n_{2i})})\| < 2^{-i} \quad \text{for all } i \in \mathbb{N}.$$

Set

$$z_i = \frac{P_{[n_{2i-1}, n_{2i+1}]^E}(x_{(n_1, n_2, \dots, n_{2i})})}{\|P_{[n_{2i-1}, n_{2i+1}]^E}(x_{(n_1, n_2, \dots, n_{2i})})\|} \quad \text{for all } i \in \mathbb{N}.$$

Then (z_i) is dominated by $(v_{n_{2i-1}}^*)$ since (E_i) satisfies subsequential V^* -upper estimates. It follows that $(x_{(n_1, n_2, \dots, n_{2i})})$ is also dominated by $(v_{n_{2i-1}}^*)$. \square

Theorem 15. *Let V and U be reflexive Banach spaces with 1-unconditional, normalized and block-stable bases (v_i) and (u_i) , respectively. Further assume that (v_i) is left-dominant, (u_i) is right-dominant, and that (v_i) is dominated by (u_i) .*

If X is a separable, infinite-dimensional, reflexive Banach space which satisfies subsequential (V, U) -tree estimates, then X can be embedded into a reflexive Banach space Z with an FDD (G_i) which satisfies subsequential (V, U) -estimates in Z .

Proof. By Proposition 4 X^* satisfies subsequential U^* -lower tree estimates, and we can apply Corollary 13 to deduce that X^* is the quotient of a reflexive space Y^* with an FDD (E_i^*) (Y^* being the dual of a space Y with an FDD (E_i)) satisfying subsequential U^* -lower estimates in Y^* . Thus X is a subspace of the reflexive space Y having an FDD (E_i) which, by Proposition 3, satisfies subsequential U -upper estimates in Y .

Theorem 12 part (a) yields a blocking $F = (F_i)$ of (E_i) and an infinite subset M of \mathbb{N} such that X embeds into $Z = Y^{V_M}(F)$.

By Corollary 9 the space Z is reflexive, and by Lemma 10 (F_i) satisfies subsequential V_M -lower estimates in Z . Since (E_i) satisfies subsequential U -upper estimates in Y , there exists $N \in \mathbb{N}^{(\omega)}$ such that (F_i) satisfies subsequential U_N -upper estimates in Y . Since (u_i) is right-dominant, we may assume after replacing N if necessary that $m_i \leq n_i$ for all $i \in \mathbb{N}$, where m_i and n_i are the i^{th} elements of M and N , respectively. Now $(v_i)_{i \in M}$ is dominated by $(u_i)_{i \in N}$, so by Lemma 11 (F_i) also satisfies subsequential U_N -upper estimates in Z . Finally, since (v_i) is left-dominant, (F_i) satisfies subsequential (V_N, U_N) estimates in Z . An application of Lemma 2 completes the argument. \square

Before proceeding to the proof of Theorem 12 we state the quantitative version of Theorem 15 as promised earlier.

Theorem 16. *For all $B, C, D, L, R \in [1, \infty)$ there exist constants $\overline{C} = \overline{C}(B, D, R)$ and $K = K(C, L, R)$ in $[1, \infty)$ such that the following holds. Let V and U be reflexive Banach spaces with 1-unconditional, normalized and B -block-stable bases (v_i) and (u_i) , respectively. Further assume that (v_i) is L -left-dominant, (u_i) is R -right-dominant, and that (v_i) is D -dominated by (u_i) .*

If X is a separable, infinite-dimensional, reflexive Banach space which satisfies subsequential C - (V, U) -tree estimates, then X can be K -embedded into a reflexive Banach space Z which has a bimonotone FDD (G_i) satisfying subsequential \overline{C} - (V, U) -estimates in Z . \square

Proof of Theorem 12 part (a). Choose constants C and D in $[1, \infty)$ such that X satisfies subsequential C - V -lower tree estimates and (v_i) is D -left-dominant. Let K be the projection constant of (E_i) in Z . Set

$$\mathcal{A} = \{(k_i, x_i) \in (\mathbb{N} \times S_X)^\omega : (v_{k_i}) \text{ is } C\text{-dominated by } (x_i)\},$$

and choose $\varepsilon > 0$ such that

$$\overline{\mathcal{A}}_\varepsilon \subset \{(k_i, x_i) \in (\mathbb{N} \times S_X)^\omega : (v_{k_i}) \text{ is } 2CD\text{-dominated by } (x_i)\}.$$

By Proposition 5 there exist $(K_i) \subset \mathbb{N}$ with $K_1 < K_2 < \dots$, $\bar{\delta} = (\delta_i) \subset (0, 1)$ with $\delta_i \downarrow 0$, and a blocking $F = (F_i)$ of (E_i) such that if $(x_i) \subset S_X$ is a $(2K\bar{\delta})$ -skipped block sequence of (F_n) in Z with $\|x_i - P_{(r_{i-1}, r_i)}^F x_i\| < 2K\delta_i$ for all $i \in \mathbb{N}$, where $1 \leq r_0 < r_1 < r_2 < \dots$, then $(v_{K_{r_{i-1}}})$ is $2CD$ -dominated by (x_i) .

It is easy to see that we can block (F_i) into an FDD $G = (G_i)$ such that there exists $(e_n) \subset S_X$ with

$$\|e_n - P_n^G(e_n)\| < \delta_n/2K \quad \text{for all } n \in \mathbb{N}.$$

Let (v''_i) be a subsequence of (v_i) such that if $(x_i) \subset S_X$ is a $\bar{\delta}$ -skipped block sequence of (G_n) in Z with $\|x_i - P_{(r_{i-1}, r_i)}^G x_i\| < \delta_i$ for all $i \in \mathbb{N}$, where $1 \leq r_0 < r_1 < r_2 < \dots$, then $(v''_{r_{i-1}})$ is $2CD$ -dominated by (x_i) . Note that if $G_j = \bigoplus_{i=m_{j-1}+1}^{m_j} G_i$, $j \in \mathbb{N}$, $0 = m_0 < m_1 < m_2 < \dots$, then $(v''_i) = (v_{K_{m_i}})$ will do.

In order to continue we need the following result from [10], which is due (in a different form) to W. B. Johnson [6].

Proposition 17. *Let X be a Banach space which is a subspace of a reflexive space Z with an FDD $A = (A_i)$ having projection constant K . Let $\bar{\eta} = (\eta_i) \subset (0, 1)$ with $\eta_i \downarrow 0$. Then there exist positive integers $N_1 < N_2 < \dots$ such that the following holds. Given positive integers $1 \leq k_0 < k_1 < \dots$ and $x \in S_X$, there exist $x_i \in X$ and $t_i \in (N_{k_{i-1}-1}, N_{k_{i-1}})$ ($i \in \mathbb{N}$, $N_0 = 0$) such that*

- (a) $x = \sum_{i=1}^{\infty} x_i$, and for all $i \in \mathbb{N}$ we have (putting $t_0 = 0$)
- (b) either $\|x_i\| < \eta_i$ or $\|x_i - P_{(t_{i-1}, t_i)}^A x_i\| < \eta_i \|x_i\|$,
- (c) $\|x_i - P_{(t_{i-1}, t_i)}^A x\| < \eta_i$,
- (d) $\|x_i\| < K + 1$,
- (e) $\|P_{t_i}^A x\| < \eta_i$.

This result is in fact a slight variation of (and follows easily from the proof of) Corollary 4.4 in [10].

We now apply Proposition 17 with $A = G$ and $\bar{\eta} = \bar{\delta}$ to obtain an appropriate sequence $N_1 < N_2 < \dots$ of positive integers. Set $H_j = \bigoplus_{i=N_{j-1}+1}^{N_j} G_i$ for each $j \in \mathbb{N}$ (and with $N_0 = 0$), and let (v'_i) be the subsequence of (v_i) defined by $v'_i = v''_{N_i}$ for all $i \in \mathbb{N}$. Let $N \in \mathbb{N}^{(\omega)}$ be chosen such that $(v_i)_{i \in N}$ is the subsequence (v'_i) of (v_i) .

Fix $x \in S_X$ and a sequence $1 \leq n_0 < n_1 < \dots$ in \mathbb{N} . We will show that

$$(8) \quad \left\| \sum_{i=1}^{\infty} \|P_{[n_{i-1}, n_i]}^H(x)\|_{Z \cdot v'_{n_{i-1}}} \right\|_V \leq 4KD^2C(K + 2\Delta + 2) + K(K + 1) + 3K\Delta,$$

where $\Delta = \sum_{i=1}^{\infty} \delta_i$. Taking then the supremum over all choices of (n_i) , we obtain that the norms $\|\cdot\|_Z$ and $\|\cdot\|_{Z^{V_N}(H)}$ are equivalent when restricted to X , and hence statement (a) follows.

Set $M_i = N_{n_{i-1}}$ for $i = 0, 1, 2, \dots$. We thus have to show that

$$\left\| \sum_{i=1}^{\infty} \|P_{(M_{i-1}, M_i)}^G(x)\|_{Z \cdot v'_{n_{i-1}}} \right\|_V \leq 4KD^2C(K + 2\Delta + 2) + K(K + 1) + 3K\Delta.$$

For each $i \in \mathbb{N}$ choose $x_i \in X$ and $t_i \in (M_{i-1}, N_{n_{i-1}})$ such that (a)–(e) of Proposition 17 hold with $A = G$ and $\bar{\eta} = \bar{\delta}$.

For each $i \in \mathbb{N}$ let $\bar{x}_i = \frac{x_{i+1}}{\|x_{i+1}\|}$ and $\alpha_i = \|x_{i+1}\|$ if $\|x_{i+1}\| \geq \delta_{i+1}$, and let $\bar{x}_i = e_{M_i}$ and $\alpha_i = 0$ if $\|x_{i+1}\| < \delta_{i+1}$. Observe that $\|\bar{x}_i - P_{(t_i, t_{i+1})}^G(\bar{x}_i)\| < \delta_i$ for all $i \in \mathbb{N}$, from which it follows that (v''_{t_i}) is $2CD$ -dominated by (\bar{x}_i) . Hence

$$\begin{aligned} \left\| \sum_{i=1}^{\infty} x_i \right\|_Z &\geq \left\| \sum_{i=1}^{\infty} \alpha_i \bar{x}_i \right\|_Z - \|x_1\|_Z - \Delta \\ &\geq \frac{1}{2CD} \left\| \sum_{i=1}^{\infty} \alpha_i v''_{t_i} \right\|_V - (K + 1) - \Delta \\ &\geq \frac{1}{2CD} \left\| \sum_{i=1}^{\infty} \|x_{i+1}\|_{Z \cdot v''_{t_i}} \right\|_V - \frac{1}{2CD} \Delta - (K + 1) - \Delta, \end{aligned}$$

and thus

$$(9) \quad \left\| \sum_{i=1}^{\infty} \|x_{i+1}\|_Z \cdot v''_{t_i} \right\|_V \leq 2CD(K + 2\Delta + 2) .$$

For each $i \in \mathbb{N}$ we have (putting $t_0 = 0$)

$$\|P_{(M_{i-1}, M_i)}^G(x)\|_Z \leq K \|P_{(t_{i-1}, t_{i+1})}^G(x)\|_Z \leq K(\|x_i\|_Z + \|x_{i+1}\|_Z + 3\delta_i) .$$

It follows that

$$\begin{aligned} & \left\| \sum_{i=1}^{\infty} \|P_{(M_{i-1}, M_i)}^G(x)\|_Z \cdot v'_{n_{i-1}} \right\|_V \\ & \leq K \left\| \sum_{i=1}^{\infty} \|x_i\|_Z \cdot v'_{n_{i-1}} \right\|_V + K \left\| \sum_{i=1}^{\infty} \|x_{i+1}\|_Z \cdot v'_{n_{i-1}} \right\|_V + 3K\Delta \\ & \leq K \left\| \sum_{i=1}^{\infty} \|x_{i+1}\|_Z \cdot v'_{n_i} \right\|_V + K \left\| \sum_{i=1}^{\infty} \|x_{i+1}\|_Z \cdot v'_{n_{i-1}} \right\|_V + K(K+1) + 3K\Delta \\ & \leq 2KD \left\| \sum_{i=1}^{\infty} \|x_{i+1}\|_Z \cdot v''_{t_i} \right\|_V + K(K+1) + 3K\Delta \end{aligned}$$

(since $v'_{n_{i-1}} = v''_{N_{n_{i-1}}}$ and $N_{n_{i-1}} > t_i$ for all $i \in \mathbb{N}$)

$$\leq 4KD^2C(K + 2\Delta + 2) + K(K+1) + 3K\Delta .$$

□

Before we prove part (b) of Theorem 12 we need a blocking result due to Johnson and Zippin.

Proposition 18. [7] *Let $T: Y \rightarrow Z$ be a bounded linear operator from a space Y with a shrinking FDD (G_i) into a space Z with an FDD (H_i) . Let $\varepsilon_i \downarrow 0$. Then there exist blockings $E = (E_i)$ of (G_i) and $F = (F_i)$ of (H_i) so that for all $m < n$ and $y \in S_{\bigoplus_{i \in (m, n)} E_i}$ we have $\|P_{[1, m]}^F Ty\| < \varepsilon_m$ and $\|P_{[n, \infty)}^F Ty\| < \varepsilon_n$.*

Proof of Theorem 12 part (b). By Lemma 3.1 in [10] we can, after renorming X if necessary, regard X^* (isometrically) as a subspace of a reflexive space Y^* (being the dual of a reflexive space Y with bimonotone FDD (E_i) such that $c_{00}(\bigoplus_{i=1}^{\infty} E_i^*) \cap X^*$ is dense in X^*). We have a natural quotient map $Q: Y \rightarrow X$. By a theorem of Zippin [18] we may regard X (isometrically) as a subspace of a reflexive space Z with an FDD (F'_i) . Let K be the projection constant of (F'_i) in Z , and choose constants C and D in $[1, \infty)$ such that X satisfies subsequential C - V -lower tree estimates and (v_i) is D -left-dominant.

Using Proposition 5 as in the proof of part (a), we find sequences $(K_i) \subset \mathbb{N}$ with $K_1 < K_2 < \dots$, $\bar{\delta} = (\delta_i) \subset (0, 1)$ with $\delta_i \downarrow 0$, and a blocking (F_i) of (F'_i) such that if $(x_i) \subset S_X$ is a $2K\bar{\delta}$ -skipped block sequence of (F_n) in Z with $\|x_i - P_{(r_{i-1}, r_i)}^F x_i\|_Z < 2K\delta_i$ for all $i \in \mathbb{N}$, where $1 \leq r_0 < r_1 < r_2 < \dots$, then $(v_{K_{r_{i-1}}})$ is $2CD$ -dominated by (x_i) , and moreover, using standard perturbation arguments and making $\bar{\delta}$ smaller if necessary, we can assume that if $(z_i) \subset Z$ satisfies $\|x_i - z_i\|_Z < \delta_i$ for all $i \in \mathbb{N}$, then (z_i) is a basic sequence equivalent to (x_i) with projection constant at most $2K$. We also require that

$$(10) \quad \Delta = \sum_{i=1}^{\infty} \delta_i < \frac{1}{7} .$$

Choose a sequence $\bar{\varepsilon} = (\varepsilon_i) \subset (0, 1)$ with $\varepsilon_i \downarrow 0$ and

$$(11) \quad 3K(K+1) \sum_{j=i}^{\infty} \varepsilon_j < \delta_i^2 \quad \text{for all } i \in \mathbb{N} .$$

After blocking (F_i) if necessary, we can assume that for any subsequent blocking D of F there is a sequence (e_i) in S_X such that

$$(12) \quad \|e_i - P_i^D(e_i)\|_Z < \varepsilon_i/2K \quad \text{for all } i \in \mathbb{N} .$$

By Proposition 18 we may assume, after further blocking our FDDs if necessary, that

$$(13) \quad \text{for all } m < n \text{ and } y \in S_{\oplus_{i \in (m, n)} E_i} \text{ we have}$$

$$\|P_{[1, m]}^F \circ Q(y)\| < \varepsilon_m \quad \text{and} \quad \|P_{[n, \infty)}^F \circ Q(y)\| < \varepsilon_n ,$$

and moreover the same holds if one passes to any blocking of (E_i) and the corresponding blocking of (F_i) .

For $i \in \mathbb{N}$ let \tilde{E}_i be the quotient space of E_i determined by Q , i.e., if $y \in E_i$, then the norm of \tilde{y} , the equivalence class of y in E_i , is given by $\|\tilde{y}\| = \|Q(y)\|$. Passing to a further blocking of (E_i) (and the corresponding blocking of (F_i)), we may assume that $\tilde{E}_i \neq \{0\}$ for all $i \in \mathbb{N}$. Given $y = \sum y_i \in c_{00}(\oplus_{i=1}^{\infty} E_i)$, $y_i \in E_i$ for all $i \in \mathbb{N}$, we set $\tilde{y} = \sum \tilde{y}_i \in c_{00}(\oplus_{i=1}^{\infty} \tilde{E}_i)$ and

$$\|\tilde{y}\| = \max_{m < n} \left\| \sum_{i=m}^n Q(y_i) \right\| = \max_{m < n} \|Q \circ P_{[m, n]}^E(y)\| .$$

We let \tilde{Y} be the completion of $c_{00}(\oplus_{i=1}^{\infty} \tilde{E}_i)$ with respect to $\|\cdot\|$. Since (E_i) is a bimonotone FDD in Y , we have $\|\tilde{y}\| \leq \|y\|$ for all $y \in c_{00}(\oplus_{i=1}^{\infty} E_i)$, and hence the map $y \mapsto \tilde{y}$ extends to a norm one map from Y to \tilde{Y} . By the definition of $\|\cdot\|$ we have $\|Qy\| \leq \|\tilde{y}\|$ for any $y \in c_{00}(\oplus_{i=1}^{\infty} E_i)$. It follows that $\tilde{y} \mapsto Q(y)$ extends to a norm one map $\tilde{Q}: \tilde{Y} \rightarrow X$ with $\tilde{Q}(\tilde{y}) = Q(y)$ for all $y \in Y$.

In order to continue our proof we will need the following proposition from [11].

Proposition 19. [11, Proposition 2.6]

- (a) (\tilde{E}_i) is a bimonotone, shrinking FDD for \tilde{Y} .
- (b) \tilde{Q} is a quotient map from \tilde{Y} onto X . More precisely if $x \in X$ and $y \in Y$ is such that $Q(y) = x$, $\|y\| = \|x\|$ and $y = \sum y_i$ with $y_i \in E_i$ for all $i \in \mathbb{N}$, then $\tilde{y} = \sum \tilde{y}_i \in \tilde{Y}$, $\|\tilde{y}\| = \|y\|$ and $\tilde{Q}(\tilde{y}) = x$.
- (c) Let (\tilde{y}_i) be a block sequence of (\tilde{E}_n) in $B_{\tilde{Y}}$, and assume that $(\tilde{Q}(\tilde{y}_i))$ is a basic sequence with projection constant \bar{K} and that $a = \inf_i \|\tilde{Q}(\tilde{y}_i)\| > 0$. Then for all $(a_i) \in c_{00}$ we have

$$\left\| \sum a_i \tilde{Q}(\tilde{y}_i) \right\| \leq \left\| \sum a_i \tilde{y}_i \right\| \leq \frac{3\bar{K}}{a} \left\| \sum a_i \tilde{Q}(\tilde{y}_i) \right\| .$$

To finish the proof of Theorem 12 (b) it suffices to find a constant $L < \infty$, a subsequence (v'_i) of (v_i) , and a blocking $\tilde{G} = (\tilde{G}_i)$ of (\tilde{E}_i) with the following property. For each $x \in S_X$ there exists a $\tilde{y} = \sum \tilde{y}_i \in \tilde{Y}$, $\tilde{y}_i \in \tilde{G}_i$ for all $i \in \mathbb{N}$, such that

$$(14) \quad \|\tilde{Q}(\tilde{y}) - x\| < 1/2 ,$$

$$(15) \quad \left\| \sum_{j=1}^{\infty} \|P_{[n_{j-1}, n_j]}^{\tilde{G}}(\tilde{y})\| \cdot v'_{n_{j-1}} \right\|_V \leq L$$

for any choice of k and $1 \leq n_0 < n_1 < n_2 < \dots$ in \mathbb{N} .

Once this is accomplished, we consider the space $\tilde{Y}^{V_N} = \tilde{Y}^{V_N}(\tilde{G})$, where $N \in \mathbb{N}^{(\omega)}$ is chosen so that $(v_i)_{i \in \mathbb{N}}$ is the subsequence (v'_i) of (v_i) . Given $x = x_0 \in S_X$, the property of \tilde{G} allows us to recursively choose $x_n \in \frac{1}{2^n} B_X$ and $\tilde{y}_n \in \frac{L}{2^{n-1}} B_{\tilde{Y}^{V_N}}$, $n \in \mathbb{N}$, so that $x_n = x_{n-1} - \tilde{Q}(\tilde{y}_n)$ for all $n \in \mathbb{N}$. It follows that $\sum_{n=1}^{\infty} \tilde{y}_n$ converges in \tilde{Y}^{V_N} with $\|\sum_{n=1}^{\infty} \tilde{y}_n\|_{\tilde{Y}^{V_N}} \leq 2L$ and $\tilde{Q}(\sum_{n=1}^{\infty} \tilde{y}_n) = x$. Thus $\tilde{Q}: \tilde{Y}^{V_N} \rightarrow X$ remains surjective, which finishes the proof.

In order to show the existence of a suitable blocking \tilde{G} of \tilde{E} we need the following result from [11].

Lemma 20. [11, Lemma 2.7] *Assume that (13) holds for our original map $Q: Y \rightarrow X$. Then there exist integers $0 = N_0 < N_1 < \dots$ so that if for each $i \in \mathbb{N}$ we define*

$$\begin{aligned} C_i &= \bigoplus_{j=N_{i-1}+1}^{N_i} E_j, & D_i &= \bigoplus_{j=N_{i-1}+1}^{N_i} F_j, \\ L_i &= \left\{ j \in \mathbb{N} : N_{i-1} < j \leq \frac{N_{i-1} + N_i}{2} \right\}, \\ R_i &= \left\{ j \in \mathbb{N} : \frac{N_{i-1} + N_i}{2} < j \leq N_i \right\}, \\ C_{i,L} &= \bigoplus_{j \in L_i} E_j & \text{and} & & C_{i,R} &= \bigoplus_{j \in R_i} E_j, \end{aligned}$$

then the following holds. Let $x \in S_X$, $0 \leq m < n$ and $\varepsilon > 0$, and assume that $\|x - P_{(m,n)}^D(x)\| < \varepsilon$. Then there exists $y \in B_Y$ with $y \in C_{m,R} \oplus \left(\bigoplus_{i \in (m,n)} C_i \right) \oplus C_{n,L}$ (where $C_{0,R} = \{0\}$) and $\|Qy - x\| < K(2\varepsilon + \varepsilon_{m+1})$ (recall that K is the projection constant of (F'_i) in Z).

Let (C_i) and (D_i) be the blockings given by Lemma 20. Note that the sequence (N_i) in the lemma used to define these blockings will not be needed in the sequel, so we can discard it. We now apply Proposition 17 with $(A_i) = (D_i)$ and $\bar{\eta} = \bar{\varepsilon}$ to obtain a sequence $N_1 < N_2 < \dots$ in \mathbb{N} so that the conclusions of the proposition are satisfied. Let (v'_i) be a subsequence of (v_i) such that if $(x_i) \subset S_X$ is a $\bar{\delta}$ -skipped block sequence of (D_n) in Z with $\|x_i - P_{(r_{i-1}, r_i)}^D x_i\|_Z < \delta_i$ for all $i \in \mathbb{N}$, where $1 \leq r_0 < r_1 < r_2 < \dots$, then $(v''_{r_{i-1}})$ is $2CD$ -dominated by (x_i) , and moreover, if $(z_i) \subset Z$ satisfies $\|x_i - z_i\| < \delta_i$ for all $i \in \mathbb{N}$, then (z_i) is a basic sequence equivalent to (x_i) with projection constant at most $2K$. Let (v'_i) be the subsequence of (v_i) defined by setting $v'_i = v''_{N_i}$ for all $i \in \mathbb{N}$. We now come to our final blockings: for each $i \in \mathbb{N}$ set $G_i = \bigoplus_{j=N_{i-1}+1}^{N_i} C_j$ and let $H_i = \bigoplus_{j=N_{i-1}+1}^{N_i} D_j$ ($N_0 = 0$). Put $G = (G_i)$, let $\tilde{G} = (\tilde{G}_i)$ be the corresponding blocking of (\tilde{E}_i) , and set $H = (H_i)$.

Fix a sequence (e_i) in S_X so that (12) holds. Let $x \in S_X$. By the choice of N_1, N_2, \dots , for each $i \in \mathbb{N}$, there are $x_i \in (K+1)B_X$ and $t_i \in (N_{i-1}, N_i)$ such that $x = \sum_{i=1}^{\infty} x_i$ and for all $i \in \mathbb{N}$ either $\|x_i\| < \varepsilon_i$ or $\|P_{(t_{i-1}, t_i)}^D x_i - x_i\| < \varepsilon_i \|x_i\|$ ($t_0 = 0$). For each $i \in \mathbb{N}$ let $\bar{x}_i = \frac{x_{i+1}}{\|x_{i+1}\|}$ and $\alpha_i = \|x_{i+1}\|$ if $\|x_{i+1}\| \geq \varepsilon_{i+1}$, and let $\bar{x}_i = e_{N_i}$ and $\alpha_i = 0$ if $\|x_{i+1}\| < \varepsilon_{i+1}$.

Since

$$(16) \quad \|\bar{x}_i - P_{(t_i, t_{i+1})}^D(\bar{x}_i)\| < \varepsilon_{i+1} \quad \text{for all } i \in \mathbb{N},$$

there exists $(y_i) \subset B_Y$ with $y_i \in C_{t_i, R} \oplus \left(\bigoplus_{j \in (t_i, t_{i+1})} C_j \right) \oplus C_{t_{i+1}, L}$ and

$$(17) \quad \|Q(y_i) - \bar{x}_i\| < 3K\varepsilon_{i+1}, \quad i \in \mathbb{N}.$$

Also, if $\|x_1\| < \varepsilon_1$, then set $y_0 = 0$, and if $\|x_1\| \geq \varepsilon_1$, then choose $y_0 \in (K+1)B_Y$ such that $y_0 \in \left(\bigoplus_{j \in (0, t_1)} C_j \right) \oplus C_{t_1, L} \subset G_1$ and $\|Q(y_0) - x_1\| < 3K(K+1)\varepsilon_1$.

Set $\bar{x} = x_1 + \sum_{i=1}^{\infty} \alpha_i \bar{x}_i$, and note that (this series converges and) by (10) and (11)

$$(18) \quad \|x - \bar{x}\| \leq \sum_{i=2}^{\infty} \varepsilon_i < \frac{1}{4}.$$

As a $\bar{\delta}$ -skipped block sequence of (D_i) (this follows from (16) and (11)), (\bar{x}_i) is a basic sequence with projection constant at most $2K$ that $2CD$ -dominates (v''_{t_i}) . Since, by (17), $\|\tilde{Q}(\tilde{y}_i) - \bar{x}_i\| < 3K\varepsilon_{i+1} < \delta_i$ for all $i \in \mathbb{N}$, the sequence $(\tilde{Q}(\tilde{y}_i))$ is also a basic sequence with projection constant at most $2K$ and is equivalent to (\bar{x}_i) . Furthermore, we have $\inf_i \|\tilde{Q}(\tilde{y}_i)\| \geq \inf_i (\|\bar{x}_i\| - \delta_i) > 6/7$, and thus, by Proposition 19 (c),

$$(19) \quad \left\| \sum a_i \tilde{Q}(\tilde{y}_i) \right\| \leq \left\| \sum a_i \tilde{y}_i \right\| \leq 7K \left\| \sum a_i \tilde{Q}(\tilde{y}_i) \right\| \quad \text{for all } (a_i) \in c_{00}.$$

Thus (\tilde{y}_i) is a basic sequence equivalent to (\bar{x}_i) and, in particular, $\sum_{i=1}^{\infty} \alpha_i \tilde{y}_i$ converges. Putting $\tilde{y} = \tilde{y}_0 + \sum_{i=1}^{\infty} \alpha_i \tilde{y}_i$ we have

$$\begin{aligned} \|\tilde{Q}\tilde{y} - \bar{x}\| &\leq \|\tilde{Q}\tilde{y}_0 - x_1\| + \sum_{i=1}^{\infty} |\alpha_i| \cdot \|\tilde{Q}\tilde{y}_i - \bar{x}_i\| \\ &\leq 3K(K+1) \sum_{i=1}^{\infty} \varepsilon_i < 1/4, \end{aligned}$$

and hence, by (18), $\|\tilde{Q}\tilde{y} - x\| < 1/2$, so we have (14).

We now fix integers $1 \leq n_0 < n_1 < n_2 < \dots$. We have $\tilde{y}_i \in \tilde{G}_i \oplus \tilde{G}_{i+1}$ for each $i \in \mathbb{N}$, and $\tilde{y}_0 \in \tilde{G}_1$. It follows that

$$(20) \quad \left\| \sum_{s=1}^{\infty} \|P_{[n_{s-1}, n_s]}^{\tilde{G}}(\tilde{y})\| \cdot v'_{n_{s-1}} \right\|_V \leq \|\tilde{y}_0\| + \left\| \sum_{s=1}^{\infty} \alpha_{n_{s-1}-1} \cdot v'_{n_{s-1}} \right\|_V \\ + \left\| \sum_{s=1}^{\infty} \left\| \sum_{i=n_{s-1}}^{n_s-1} \alpha_i \tilde{y}_i \right\| \cdot v'_{n_{s-1}} \right\|_V,$$

where we put $\alpha_0 = 0$ in case $n_0 = 1$. We now show how to bound each of the three terms of the right-hand side of the above inequality, and hence obtain (15) with $L = 126CD^2K^3$.

We already have $\|\tilde{y}_0\| \leq K+1$. Since (\bar{x}_i) $2CD$ -dominates (v''_{t_i}) we get

$$\begin{aligned} 2CD^2 \left\| \sum_i \alpha_i \bar{x}_i \right\|_Z &\geq D \left\| \sum_i \alpha_i \cdot v''_{t_i} \right\|_V \\ &\geq \left\| \sum_i \alpha_i \cdot v'_{i+1} \right\|_V \end{aligned}$$

(since $v'_{i+1} = v''_{N_{i+1}}$ and $N_{i+1} > t_i$ for all $i \in \mathbb{N}$)

$$\geq \left\| \sum_{s=1}^{\infty} \alpha_{n_{s-1}-1} \cdot v'_{n_{s-1}} \right\|_V.$$

Moreover, it follows from (18) that

$$(21) \quad \left\| \sum_i \alpha_i \bar{x}_i \right\|_Z = \|\bar{x} - x_1\|_Z \leq K + 3.$$

This yields the bound of $2CD^2(K+3)$ for the second term of (20).

For each $s \in \mathbb{N}$ let $\tilde{w}_s = \sum_{i=n_{s-1}}^{n_s-1} \alpha_i \tilde{y}_i$ and $b_s = \sum_{i=n_{s-1}}^{n_s-1} \alpha_i \bar{x}_i$. Note that by (16) and (11)

$$(22) \quad \begin{aligned} \|b_s - P_{(t_{n_{s-1}}, t_{n_s})}^D(b_s)\| &\leq \sum_{i=n_{s-1}}^{n_s-1} |\alpha_i| \cdot 2K \cdot \|\bar{x}_i - P_{(t_i, t_{i+1})}^D \bar{x}_i\| \\ &< 2K(K+1) \sum_{i=n_{s-1}}^{n_s-1} \varepsilon_{i+1} < \delta_s^2 \quad \text{for all } s \in \mathbb{N}. \end{aligned}$$

For each $s \in \mathbb{N}$ set $\bar{b}_s = \frac{b_s}{\|b_s\|}$ and $\beta_s = \|b_s\|$ if $\|b_s\| \geq \delta_s$, and set $\bar{b}_s = \bar{x}_{n_{s-1}}$ and $\beta_s = 0$ if $\|b_s\| < \delta_s$. It follows from (22) and (16) that $(\bar{b}_s) \subset S_X$ is a $\bar{\delta}$ -skipped block sequence of (D_i) in Z with $\|\bar{b}_s - P_{(t_{n_{s-1}}, t_{n_s})}^D(\bar{b}_s)\| < \delta_s$ for all $s \in \mathbb{N}$, and hence it is a basic sequence that $2CD$ -dominates $(v''_{t_{n_{s-1}}})$.

From (17) and (19) we have

$$(23) \quad \begin{aligned} \|\tilde{Q}(\tilde{w}_s) - b_s\| &\leq \sum_{i=n_{s-1}}^{n_s-1} |\alpha_i| \cdot \|\tilde{Q}(\tilde{y}_i) - \bar{x}_i\| \\ &< 3K(K+1) \sum_{i=n_{s-1}}^{n_s-1} \varepsilon_{i+1} < \delta_s \end{aligned}$$

and

$$(24) \quad \|\tilde{w}_s\| \leq 7K \|\tilde{Q}(\tilde{w}_s)\| \quad \text{for all } s \in \mathbb{N}.$$

We now obtain the following sequence of inequalities.

$$\begin{aligned} \left\| \sum_{s=1}^{\infty} \|\tilde{w}_s\| \cdot v'_{n_{s-1}} \right\|_V &\leq 7K \left\| \sum_{s=1}^{\infty} \|\tilde{Q}(\tilde{w}_s)\| \cdot v'_{n_{s-1}} \right\|_V && \text{(from (24))} \\ &\leq 7K \left\| \sum_{s=1}^{\infty} \|b_s\| \cdot v'_{n_{s-1}} \right\|_V + 7K\Delta && \text{(from (23))} \\ &\leq 7K \left\| \sum_{s=1}^{\infty} \beta_s v'_{n_{s-1}} \right\|_V + 14K\Delta \\ &\leq 7KD \left\| \sum_{s=1}^{\infty} \beta_s v''_{t_{n_{s-1}}} \right\|_V + 14K\Delta \end{aligned}$$

(as (v_i) is D -left-dominant)

$$\leq 14CD^2K \left\| \sum_{s=1}^{\infty} \beta_s \bar{b}_s \right\| + 14K\Delta$$

(since (\bar{b}_s) $2CD$ -dominates $(v''_{t_{n_{s-1}}})$)

$$\leq 14CD^2K \left\| \sum_{i=n_0}^{\infty} \alpha_i \bar{x}_i \right\| + 14CD^2K\Delta + 14K\Delta.$$

Finally, since (\bar{x}_i) is a basic sequence with projection constant at most $2K$, it follows from (21) that

$$\left\| \sum_{i=n_0}^{\infty} \alpha_i \bar{x}_i \right\|_Z \leq 2K \left\| \sum_{i=1}^{\infty} \alpha_i \bar{x}_i \right\|_Z \leq 2K(K+3).$$

This provides an upper bound of $116CD^2K^3$ for the third term of (20), which leads (15) with $L = 126CD^2K^3$, as claimed. This completes the proof of part (b) of Theorem 12. \square

5. UNIVERSAL CONSTRUCTIONS AND APPLICATIONS

Let V and U be reflexive spaces with normalized, 1-unconditional, block-stable bases (v_i) and (u_i) , respectively, such that (v_i) is left-dominant, (u_i) is right-dominant and (v_i) is dominated by (u_i) . For each $C \in [1, \infty)$ let $\mathcal{A}_{V,U}(C)$ denote the class of all separable, infinite-dimensional, reflexive Banach spaces that satisfy subsequential C - (V,U) -tree estimates. We also let

$$\mathcal{A}_{V,U} = \bigcup_{C \in [1, \infty)} \mathcal{A}_{V,U}(C),$$

which is the class of all separable, infinite-dimensional, reflexive Banach spaces that satisfy subsequential (V,U) -tree estimates.

Theorem 21. *The class $\mathcal{A}_{V,U}$ defined above contains an element which is universal for the class.*

More precisely, for all $B, D, L, R \in [1, \infty)$ there exists a constant $\overline{C} = \overline{C}(B, D) \in [1, \infty)$ and for all $C \in [1, \infty)$ there is a constant $K(C) = K_{B,D,L,R}(C) \in [1, \infty)$ such that if (v_i) is B -block-stable and L -left-dominant, if (u_i) is B -block-stable and R -right-dominant, and if (v_i) is D -dominated by (u_i) , then there exists $Z \in \mathcal{A}_{V,U}$ such that for all $C \in [1, \infty)$ every $X \in \mathcal{A}_{V,U}(C)$ $K(C)$ -embeds into Z , and moreover Z has a bimonotone FDD satisfying subsequential \overline{C} - (V,U) estimates in Z .

Proof. By a result of Schechtman [16] there exists a space W with a bimonotone FDD $E = (E_i)$ with the property that any bimonotone FDD is naturally almost isometric to a subsequence $\overline{\bigoplus_{i=1}^{\infty} E_{k_i}}$ which is 1-complemented in W . More precisely, given a Banach space X with a bimonotone FDD (F_i) and given $\varepsilon > 0$, there is a subsequence (E_{k_i}) of (E_i) and a $(1+\varepsilon)$ -embedding $T: X \rightarrow W$ such that $T(F_i) = E_{k_i}$ for all $i \in \mathbb{N}$, and $\sum_{i=1}^{\infty} P_{k_i}^E$ is a norm-1 projection of W onto $\overline{\bigoplus_{i=1}^{\infty} E_{k_i}}$.

We shall now modify the norm on W in two stages. We first consider the space $(W^{(*)})^{U^*}(E^*)$. By Corollary 7 the sequence (E_i^*) is a boundedly complete (and bimonotone) FDD for this space. It follows that (E_i) is a bimonotone, shrinking FDD for a space Y with $Y^* = (W^{(*)})^{U^*}(E^*)$. By Lemma 10 and Proposition 3 (E_i) satisfies subsequential $2B$ - U -upper estimates in Y .

We now let $Z = Y^V(E)$. By Corollary 9 Z is reflexive, by Lemma 10 (E_i) satisfies subsequential $2B$ - V -lower estimates in Z , and by Lemma 11 (E_i) also satisfies subsequential $(3BD + 2B^2D)$ - U -upper estimates in Z . Thus (E_i) is a bimonotone FDD satisfying subsequential \overline{C} - (V,U) -estimates in Z , where $\overline{C} = 3BD + 2B^2D$. It remains to show that Z is universal for $\mathcal{A}_{V,U}$.

Let $C \in [1, \infty)$ and let $X \in \mathcal{A}_{V,U}(C)$. By Theorem 16 there exist constants $K_1 = K_1(B, D, R)$ and $K_2 = K_2(C, L, R)$ in $[1, \infty)$ such that X K_2 -embeds into a reflexive space \tilde{X} which has a bimonotone FDD (F_i) satisfying subsequential K_1 - (V,U) estimates in \tilde{X} . Now we can find a subsequence (E_{k_i}) of (E_i) and a 2-embedding $T: \tilde{X} \rightarrow W$ such that $T(F_i) = E_{k_i}$ for all $i \in \mathbb{N}$ and $\sum_i P_{k_i}^E$ is a norm-1 projection of W onto $\overline{\bigoplus_i E_{k_i}}$. It follows in particular that (E_{k_i}) satisfies subsequential $2K_1$ - (V,U) estimates in W , i.e., if (w_i) is a normalized block sequence of (E_{k_n}) in W with $\min \text{supp}_E(w_i) = k_{m_i}$ for all $i \in \mathbb{N}$, then (w_i) $2K_1$ -dominates (v_{m_i}) and is $2K_1$ -dominated by (u_{m_i}) . Hence by Proposition 3 $(E_{k_i}^*)$ satisfies subsequential $2K_1$ - (U^*, V^*) estimates in $W^{(*)}$. (Note that the dual of the subspace $\overline{\bigoplus_i E_{k_i}}$ of W is naturally isometrically isomorphic to the subspace $\overline{\bigoplus_i E_{k_i}^*}$ of $W^{(*)}$.) We shall

now use this to show that the norms $\|\cdot\|_W$, $\|\cdot\|_Y$ and $\|\cdot\|_Z$ are all equivalent when restricted to $c_{00}(\oplus_i E_{k_i})$, which implies that \tilde{X} and hence also X embed into Z .

Fix $w^* \in c_{00}(\oplus_i E_{k_i}^*)$. Clearly we have $\|w^*\|_{W^{(*)}} \leq \|w^*\|_{Y^*}$. Choose $1 \leq m_0 < m_1 < \dots$ in \mathbb{N} such that

$$\|w^*\|_{Y^*} = \left\| \sum_{i=1}^{\infty} \|P_{[m_{i-1}, m_i]}^{E^*} w^*\|_{W^{(*)}} \cdot u_{m_{i-1}}^* \right\|_{U^*}.$$

We may assume that $m_0 = 1$ and $P_{[m_{i-1}, m_i]}^{E^*} w^* \neq 0$ for all $i \in \mathbb{N}$. Then there exist $j_1 < j_2 < \dots$ in \mathbb{N} such that $k_{j_i} = \min \text{supp}_{E^*} P_{[m_{i-1}, m_i]}^{E^*} w^*$ for all $i \in \mathbb{N}$. Since (u_i^*) is B -block-stable and R -left-dominant, and since $(E_{k_i}^*)$ satisfies subsequential $2K_1$ - U^* -lower estimates in $W^{(*)}$, we have

$$\begin{aligned} (25) \quad \|w^*\|_{Y^*} &\leq B \left\| \sum_{i=1}^{\infty} \|P_{[m_{i-1}, m_i]}^{E^*} w^*\|_{W^{(*)}} \cdot u_{k_{j_i}}^* \right\|_{U^*} \\ &\leq BR \left\| \sum_{i=1}^{\infty} \|P_{[m_{i-1}, m_i]}^{E^*} w^*\|_{W^{(*)}} \cdot u_{j_i}^* \right\|_{U^*} \\ &\leq 2BRK_1 \|w^*\|_{W^{(*)}}. \end{aligned}$$

This shows that $\|\cdot\|_{W^{(*)}}$ and $\|\cdot\|_{Y^*}$ are equivalent on $c_{00}(\oplus_i E_{k_i}^*)$. It is easy to verify that $\sum_i P_{k_i}^{E^*}$, which defines a norm-1 projection of $W^{(*)}$ onto $\overline{\bigoplus_i E_{k_i}^*}$, is also a norm-1 projection of Y^* onto $\overline{\bigoplus_i E_{k_i}^*}$. It follows that $\frac{1}{2BRK_1} \|w\|_W \leq \|w\|_{Y^*} \leq \|w\|_W$ for all $w \in c_{00}(\oplus_i E_{k_i})$.

A very similar argument shows that $\|y\|_Y \leq \|y\|_Z \leq 2BLK_1 \|y\|_Y$ for all $y \in c_{00}(\oplus_i E_{k_i})$. Indeed, the first inequality is clear from the definition of $\|\cdot\|_Z$, whereas the second one is obtained by a computation similar to the one in (25).

We have thus shown that the 2-embedding $T: \tilde{X} \rightarrow W$ becomes a $8B^2LRK_1^2$ -embedding viewed as a map $\tilde{X} \rightarrow Z$. Hence X K -embeds into Z , where $K = 8B^2LRK_1^2K_2$. \square

We conclude this paper with two applications of our embedding theorems. The first one is the observation that our results here give an alternative proof to the main theorem in [11].

Theorem 22 ([11]). *Let X be a separable, reflexive Banach space and let $1 \leq q \leq p \leq \infty$. The following are equivalent.*

- (a) X satisfies (p, q) tree estimates.
- (b) X is isomorphic to a subspace of a reflexive space Z having an FDD which satisfies (p, q) estimates.
- (c) X is isomorphic to a quotient of a reflexive space Z having an FDD which satisfies (p, q) estimates.

Here an FDD (E_n) of a Banach space Z is said to satisfy (p, q) estimates if there is a constant $C > 0$ such that for every block sequence (x_i) of (E_n) we have

$$C^{-1} \left(\sum \|x_i\|^p \right)^{1/p} \leq \left\| \sum x_i \right\| \leq C \left(\sum \|x_i\|^q \right)^{1/q},$$

and a Banach space X is said to satisfy (p, q) estimates if every normalized, weakly null tree $(x_\alpha)_{\alpha \in T_\infty}$ in X has a branch that dominates the unit vector basis of ℓ_p and that is dominated by the unit vector basis of ℓ_q . The family $(x_\alpha)_{\alpha \in T_\infty}$ in X is called a normalized, weakly null tree if for all $\alpha \in T_\infty \cup \{\emptyset\}$ the sequence $(x_{(\alpha, n)})$ is

normalized and weakly null, and a branch of $(x_\alpha)_{\alpha \in T_\infty}$ is a sequence $(x_{(n_1, n_2, \dots, n_i)})$, where $n_1 < n_2 < \dots$.

The second, and main, application concerns the existence of universal spaces for the classes C_α defined in the Introduction. Recall that for each countable ordinal α the class C_α consists of all separable, reflexive spaces X such that both X and its dual X^* have Szlenk index at most α . Szlenk introduced his index to show that there is no separable, reflexive space that contains isomorphic copies of every separable, reflexive space [17].

The Szlenk index $\text{Sz}(\cdot)$ has the following properties [17]: for a separable space X , $\text{Sz}(X) < \omega_1$ if and only if X^* is separable (so $\bigcup_{\alpha < \omega_1} C_\alpha$ is the class of all separable, reflexive spaces); if Y embeds into X , then $\text{Sz}(Y) \leq \text{Sz}(X)$; for all $\alpha < \omega_1$ there exists a separable, reflexive space X such that $\text{Sz}(X) > \alpha$. From these properties it follows immediately that if a separable space Z contains isomorphic copies of every separable, reflexive space, then Z^* is not separable, and so Z cannot be reflexive. (Later J. Bourgain showed that such a space Z must contain $C[0, 1]$, and hence all separable Banach spaces [1].)

It seems natural to ask if there is, for each countable ordinal α , a separable, reflexive space that is universal for C_α . This question was indeed raised by Pelczyński motivated by the results of [11], which imply an affirmative answer for $\alpha = \omega$. In [13] we show that Pelczyński's question has an affirmative answer for all $\alpha < \omega_1$.

Theorem 23. *For each countable ordinal α there is a separable, reflexive space Z which is universal for the class C_α .*

This is a simplified version of our result which also includes estimates on embedding constants and determines the class C_β in which the universal space Z lives. The proof, which is given in [13], splits into two parts. We first prove that if $X \in C_\alpha$, then there exists $\gamma < \alpha$ such that X satisfies subsequential $((T_{\gamma, \frac{1}{2}})^*, T_{\gamma, \frac{1}{2}})$ tree estimates, where $T_{\gamma, \frac{1}{2}}$ is the Tsirelson space of order γ . The ingredient for the second part of the proof is Theorem 21 from this paper. We fix a sequence (α_n) of ordinals with $\alpha = \sup_n (\alpha_n + 1)$, and for each $n \in \mathbb{N}$ we let Z_n be a separable, reflexive space which is universal for the class $\mathcal{A}_{((T_{\alpha_n, \frac{1}{2}})^*, T_{\alpha_n, \frac{1}{2}})}$. The ℓ_2 -direct sum Z of the sequence (Z_n) is then the required universal space for C_α .

REFERENCES

- [1] J. Bourgain, 'On separable Banach spaces, universal for all separable reflexive spaces', *Proc. Amer. Math. Soc.* 79 (1980) no. 2 241–246.
- [2] S. F. Bellenot, R. Haydon, E. Odell, 'Quasi-reflexive and tree spaces constructed in the spirit of R. C. James', *Banach space theory (Iowa City, IA, 1987)*, 19–43, Contemp. Math., 85, Amer. Math. Soc., Providence, RI, 1989.
- [3] P. Dodos, V. Ferenczi, 'Some strongly bounded classes of Banach spaces', preprint.
- [4] D. Gale and F.M. Stewart, 'Infinite games with perfect information', Contributions to the theory of games, *Annals of Math. Studies* no. 28 Princeton University Press (1953) 245–266.
- [5] W. T. Gowers, 'An infinite Ramsey theorem and some Banach-space dichotomies', *Ann. of Math.* (2) 156 (2002) no. 3 797–833.
- [6] W. B. Johnson, 'On quotients of L_p which are quotients of l_p ', *Compositio Math.* 34 (1977) no. 1 69–89.
- [7] W. B. Johnson and M. Zippin, 'On subspaces of quotients of $(\sum G_n)_{l_p}$ and $(\sum G_n)_{c_0}$ ', Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces, Jerusalem, 1972, *Israel J. Math.* 13 (1972) 311–316.
- [8] D. A. Martin, 'Borel determinacy', *Ann. of Math.* (2) 102 (1975) no. 2 363–371.
- [9] B. Maurey, V. D. Milman and N. Tomczak-Jaegermann, 'Asymptotic infinite-dimensional theory of Banach spaces', *Oper. Theory: Adv. Appl.* 77 (1994) 149–175.
- [10] E. Odell and Th. Schlumprecht, 'Trees and branches in Banach spaces', *Trans. Amer. Math. Soc.* 354 (2002) no. 10 4085–4108.
- [11] E. Odell and Th. Schlumprecht, 'A universal reflexive space for the class of uniformly convex Banach spaces', *Mathematische Annalen* (to appear).

- [12] E. Odell, Th. Schlumprecht and A. Zsák, 'On the structure of asymptotic ℓ_p spaces', submitted to LMS.
- [13] E. Odell, Th. Schlumprecht and A. Zsák, 'Banach spaces with bounded Szlenk index', preprint.
- [14] C. Rosenthal, 'Infinite asymptotic games', preprint.
- [15] H. P. Rosenthal, 'Block-determined finite-dimensional decompositions', preprint.
- [16] G. Schechtman, 'On Pełczyński's paper "Universal bases"', *Israel J. Math.* 22 (1975) no. 3-4 181–184.
- [17] W. Szlenk, 'The non-existence of a separable reflexive Banach space universal for all separable reflexive Banach spaces', *Studia Math.* 30 1968 53–61.
- [18] M. Zippin, 'Banach spaces with separable duals', *Trans. Amer. Math. Soc.* 310 no. 1 (1988) 371–379.