1. Prove that $e^{int}$ converges weakly to zero in $L^1[0, 1]$.

2. Let $K$ be a bounded subset of $C[0, 1]$ such that every sequence of elements in $K$ has a subsequence that converges pointwise to some function in $C[0, 1]$. Show that $K$ has compact closure as a subset of $L^2[0, 1]$.

3. (a) A function $f : [0, 1] \to \mathbb{R}$ is said to be absolutely continuous provided . . .

(b) Prove or disprove: Every polynomial is absolutely continuous on $[0, 1]$.

4. Let $(X, \rho)$ be a compact metric space.

   (a) Prove that there exists a compact subset $K$ of $C(X)$ whose linear span is dense in $C(X)$.

   (b) Prove that if $K$ is a compact subset of $C(X)$ whose linear span is dense in $C(X)$, then the pseudometric

   $$d(x, y) := \sup_{f \in K} |f(x) - f(y)|$$

   on $X$ is actually a metric on $X$; moreover, show that $d$ and $\rho$ generate the same topology on $X$.

5. (a) State the Lemma of Fatou.

   (b) Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose that $(f_n)$ is a sequence of non-negative functions which converge pointwise to an integrable function $f$. Suppose that $\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu$. Show that

   $$\lim_{n \to \infty} \int_E f_n \, d\mu = \int_E f \, d\mu$$

   for any measurable $E \subset X$.

6. Let $2 < p \leq \infty$ and let $X$ be a subspace of $L_p[0, 1]$ that is closed in $L_1[0, 1]$.

   (a) Prove that $(X, \| \cdot \|_p)$ is isomorphic to a Hilbert space.

   (b) Prove that if $p = \infty$, then $X$ is finite dimensional.
7. Define $f_n : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = x^{56n}$.

(a) Prove that the linear span of the functions $f_n|_{[0,1]}$ for $n = 1, 2, 3, \ldots$ is dense in $L^1[0,1]$.

(b) Prove that the linear span of the functions $f_n|_{[-1,1]}$ for $n = 1, 2, 3, \ldots$ is not dense in $L^1[-1,1]$.

8. Let $X$ be a Banach space and $Y$ a linear closed subspace of $X$. Show that an extreme point in the unit ball of $Y^*$ can be extended to an extreme point in the unit ball of $X^*$.

9. Assume $f_n \in L^1(-\infty, \infty)$, for $n \in \mathbb{N}$, with

$$\int |f_n(x)| \, dx \leq 2^{-n}.$$ 

Show that $\lim_{n \to \infty} f_n(x) = 0$ a.e.. Is the same conclusion true if $2^{-n}$ is replaced by $\frac{1}{n}$?

10. Assume that $f \in L^1[0,1]$ satisfies for some $C > 0$

$$\int_E |f| \, dx \leq C m^{1/2}(E) \quad \text{whenever } E \subset [0,1] \text{ is measurable}.$$ 

Show that $f \in L^p[0,1]$ for any $1 \leq p < 2$. Give an example which is not in $L^2[0,1]$. 
