1. Prove that if $E$ is a closed linear subspace of $L^2(0, 1)$ and each element in $E$ is bounded (i.e. $f \in L^\infty(0, 1)$ for all $f \in E$), then $E$ is finite dimensional.

2. Let $f_n := \sum_{k=1}^{2^n} (-1)^k \chi_{(\frac{k-1}{2^n}, \frac{k}{2^n})}$. Prove that $f_n \to 0$ weakly in $L^1(0, 1)$.

3. Let $E$ be a subset of a metric space $X$, and let $B = \{B(r(x), x) : x \in S\}$ be a collection of balls in $X$ which cover $E$ (that is, $E \subset \bigcup_{x \in S} B(r(x), x)$) so that the radii are bounded (that is, sup$r(x) : x \in S$ < $\infty$). Prove that there is a (finite or infinite) sequence $\{B(r(x_i), x_i)\}_{i=1}^N$ of disjoint balls in $B$ so that either
   
   (i) $N = \infty$ and inf$r(x_i) : i = 1, 2, 3, \ldots > 0$, or
   
   (ii) $E \subset \bigcup_{i=1}^N B(5r(x_i), x_i)$. ($N$ can be either finite or infinite in this case.)

4. Prove that every separable Banach space is isometrically isomorphic to a subspace of $C(\Delta)$, where $\Delta$ is the Cantor set. You may use the topological theorem that if $X$ is a compact metric space, then there is a continuous surjection from $\Delta$ onto $X$.

5. For $f \in L^1(0, 1)$ and $y \in [0, 1]$, define $(Tf)(y) = \frac{1}{y} \int_0^y f(x) \, dx$. Show that $T$ defines a bounded linear operator from $L^p(0, 1)$ to $L^p(0, 1)$ for all $1 < p \leq \infty$, but $T$ does not define a bounded linear operator from $L^1(0, 1)$ to $L^1(0, 1)$.

6. For $i = 1, 2$, let $\mu_i$ and $\nu_i$ be finite measures on a measurable space $(X_i, \Sigma_i)$. Assume that $\mu_i << \nu_i$. Prove that $\mu_1 \times \mu_2 << \nu_1 \times \nu_2$.

7. Let $\{f_n\}_{n=1}^\infty$ be a sequence of non negative functions in $L^1(\mathbb{R})$ which converges pointwise almost everywhere to a function $f$ in $L^1(\mathbb{R})$. Prove that if $\int_\mathbb{R} f_n(t) \, dt \to \int_\mathbb{R} f(t) \, dt$, then $\int_\mathbb{R} |f_n(t) - f(t)| \, dt \to 0$.

8. Let $f \in L^1(0, 1)$ be such that for all $n = 1, 2, 3, \ldots$, \[ \int_0^1 f(t) t^{2n} \, dt = 0. \] Prove that $f = 0$ a.e.

9. A sequence $\{x_n\}_{n=1}^\infty$ in a Banach space $X$ is said to be weakly Cauchy provided that for each $x^* \in X^*$, the sequence $\{x^*(x_n)\}_{n=1}^\infty$ or real numbers is convergent. Let $K$ be a compact Hausdorff space. Prove that a sequence $\{f_n\}_{n=1}^\infty$ in $C(K)$ is weakly Cauchy if and only if $\{f_n\}_{n=1}^\infty$ is bounded and pointwise convergent. Deduce that if $\{f_n\}_{n=1}^\infty$ is a weakly Cauchy sequence in $C[0, 1]$, then $\{f_n\}_{n=1}^\infty$ is NORM convergent in $L^1(0, 1)$.

10. Write $\mu$ for Lebesgue measure on the Borel subsets of the unit interval $[0, 1]$. Recall that a Borel probability measure $\nu$ on $[0, 1]$ is singular with respect to $\mu$ (in symbols, $\nu \perp \mu$) if there is a Borel set $D \subseteq [0, 1]$ such that $\nu(D) = 1$ and $\mu(D) = 0$. Let $\nu$ be a Borel probability measure on $[0, 1]$. Show that $\nu \perp \mu$ if and only if for every $\varepsilon > 0$ there is a continuous function $f : [0, 1] \to [0, 1]$ such that $\nu(f) > 1 - \varepsilon$ and $\mu(f) < \varepsilon$. 

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