SAMPLING AND RECOVERY OF BANDLIMITED FUNCTIONS AND APPLICATIONS TO SIGNAL PROCESSING

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Abstract. Bandlimited functions, i.e square integrable functions on $\mathbb{R}^d$, $d \in \mathbb{N}$, whose Fourier transforms have bounded support, are widely used to represent signals. One problem which arises, is to find stable recovery formulae, based on evaluations of these functions at given sample points. We start with the case of equally distributed sampling points and present a method of Daubechies and DeVore to approximate bandlimited functions by quantized data.

In the case that the sampling points are not equally distributed this method will fail. We are suggesting to provide a solution to this problem in the case of scattered sample points by first approximating bandlimited functions using linear combinations of shifted Gaussians. In order to be able to do so we prove the following interpolation result.

Let $(x_j : j \in \mathbb{Z}) \subset \mathbb{R}$ be a *Rieszbasis sequence*. For $\lambda > 0$ and $f \in PW$, the space of square-integrable functions on $\mathbb{R}$, whose Fourier transforms vanish outside of $[-1, 1]$, there is a unique sequence $(a_j) \in \ell_2(\mathbb{Z})$, so that the function

$$I_\lambda(f)(x) := \sum a_j e^{-\lambda\|x-x_j\|_2^2}, \qquad x \in \mathbb{R}$$

is continuous, square integrable, and satisfies the interpolatory conditions $I_\lambda(f)(x_k) = f(x_k)$, for all $k \in \mathbb{Z}$. It is shown that $I_\lambda(f)$ converges to f in $L_2(\mathbb{R}^d)$ and uniformly on $\mathbb{R}$, as $\lambda \to 0^+$.

1. Introduction

We first introduce some basic notations and recall some results. For $1 \le p \le \infty$ and $B \subset \mathbb{R}^d$, $m(B) > 0$, we let

$$L_p(B) = \Big\{ f : B \to \mathbb{C} \text{ measurable} : \|f\|_p = \Big(\int_B |f| dx \Big)^{1/p} < \infty \Big\},$$

if $1 \le p < \infty$, and

$$L_\infty(B) = \Big\{ f : B \to \mathbb{C} \text{ measurable } : \|f\|_\infty = \sup\{r > 0 : m(|f| \ge r) > 0\} \Big\},$$

$$C_b(\mathbb{R}^d) = \Big\{ f : \mathbb{R}^d \to \mathbb{C} \text{ bounded and continuous} \Big\},$$

$C_0(\mathbb{R}^d) = \big\{ f : \mathbb{R} \to \mathbb{C} \text{ continuous and } \lim_{\|t\|_2 \to \pm\infty} f(t) = 0 \big\}.$

The *Fourier transform of* $f \in L_1(\mathbb{R}^d)$ is defined point wise by

$$\hat{f}(x) = \int_{\mathbb{R}^d} f(t) e^{-i\langle x,t\rangle} dt, \qquad x \in \mathbb{R}^d.$$

Note that

$$\hat{(\cdot)} : L_1(\mathbb{R}^d) \to C_0(\mathbb{R}^d), \quad f \to \hat{f},$$

is well defined, bounded and linear with $\|\hat{(\cdot)}\|_{L(L_1,C_0)} = 1$.

In general $\hat{f} \not\in L_1(\mathbb{R}^d)$. But if $\hat{f} \in L_1(\mathbb{R}^d)$ then *the Inverse Transform* is given by

$$f(t) = [\hat{f}]^\vee = \frac{1}{(2\pi)^d} \int_{\mathbb{R}} \hat{f}(x) e^{i\langle x,t\rangle} dx, \quad t \in \mathbb{R}^d.$$

The linear map

$$L_2(\mathbb{R}^d) \cap L_1(\mathbb{R}^d) \to L_2(\mathbb{R}^d) \cap L_1(\mathbb{R}^d), \quad g \mapsto \hat{g}$$

is an isomorphy with respect to $\|\cdot\|_2$, and $\|\hat{g}\|_2 = (2\pi)^{d/2} \|g\|_2$. We can therefore extend this map to an operator

$$\begin{aligned} \mathcal{F} : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d),\ g \mapsto \mathcal{F}[g], \\ \text{with } \|\mathcal{F}[g]\|_2 = (2\pi)^{1/2} \|g\|_2,\ g \in L_2(\mathbb{R}^d). \end{aligned} \tag{1}$$

From the inverse formula on $L_1(\mathbb{R}^d)$ we deduce that for $g \in L_2(\mathbb{R})$

$$\mathcal{F}^{-1}[g] = \|\cdot\|_2 - \lim_{N\to\infty} H_N, \text{ where } H_N(t) = \frac{1}{(2\pi)^d} \int_{[-N,N]^d} g(x) e^{ixt} dt, \quad t \in \mathbb{R}.$$

Definition 1.1. For $B \subset \mathbb{R}^d$, bounded with positive measure, define

$$PW_B := \big\{ \hat{g} : g \in L_2(\mathbb{R}^d) \text{ and } g = 0 \text{ almost everywhere outside } B \big\}.$$

Elements of PW_B are called *bandlimited* or *Paley Wiener functions.*

On the one hand PW_B is a Hilbert space with respect to the usual scalar product on $L_2(\mathbb{R})$, on the other hand its elements are continuous functions. Moreover

Proposition 1.2. *Let $B \subset \mathbb{R}^d$ be bounded and have positive measure.*

a) *PW_B is a closed linear subspace of $L_2(\mathbb{R})$, and thus a Hilbert space with respect to the usual scalar product on $L_2(\mathbb{R})$.*
b) *The elements of PW_B have an analytic extention onto all of $\mathbb{C}^d$. In particular Dirac measures are well defined continuous functionals on PW_B.*
c) *$f \in PW_B$ then $f_{x_j} \in PW_B$, for $j = 1, 2 \ldots d$.*

Definition 1.3. A sequence (e_j) in a Hilbert space H is called a *Riesz basis* if it is an unconditional basis of H. Usually we will then denote the coordinate for (e_j) by (e_j^*). Thus for every $x \in H$

$$x = \sum_{j=1}^{\infty} \langle e_j^*, x\rangle e_j,$$

and this convergence is unconditional in H.

A sequence $(x_j : j \in \mathbb{N}) \subset \mathbb{R}^d$ is called *Riesz basis sequence* for $L_2(B)$, $m(B) > 0$, if the sequence of exponentials $(e^{-i\langle x_j(\cdot)\rangle} : j \in \mathbb{N})$ is a Riesz basis for $L_2(B)$.

From the boundedness of the coordinate functionals (e_k^*) we can easily deduce that a Riesz basis sequence $(x_j) \subset \mathbb{R}^d$ is *uniformly separated*, which means that

$$q = \inf_k \|x_k - x_{k+1}\| = 2 > 0.$$

Indeed, otherwise we couldchoose a subsequence $k_j \subset \mathbb{Z}$ such that $\lim_{\mathrm{j}\to\infty} \|x_{k_j} - x_{k_j+1}\|_2 = 0$, and thus, we deduce (putting $e_k = e^{i\langle x_k(\cdot)\rangle}$, for $k \in \mathbb{Z}$)

$$\limsup_{j\to\infty} \|e_{k_j}^*\| \le \limsup_{j\to\infty} e_{k_j}^* \frac{e_{k_{j+1}} - e_{k_j}}{\|e_{k_{j+1}} - e_{k_j}\|}\Big) = \infty,$$

which is a contradiction.

If $d = 1$ then we can and will therefore assume that x_j is indexed by $\mathbb{Z}$ and strictly increasing.

Remark 1.4. Note that Riesz bases sequences and *sampling bases* for Paley Wiener functions (see end of this section) are closely connected. Indeed if (x_k) is a Riesz basis sequence for $L_2(B)$ and if $e_k^* \in L_2(B)$, $k \in \mathbb{N}$, are the k-th coordinate functional. Then it follows that (e_k^*) is also a Riesz basis of $L_2(B)$ whose coordinate functionals are $(e^{-i\langle x_k(\cdot)\rangle})$, and we can therefore write for $f \in PW_B$ that

$$\begin{aligned}
\hat{f}(x) &= \sum_{j\in\mathbb{N}} \langle \hat{f}, e^{-i\langle x_j,(\cdot)\rangle}\rangle_B e_j^*(x) \\
&= \sum_{j\in\mathbb{N}} \int_{\mathbb{R}^d} \hat{f}(t) e^{i\langle x_j,t\rangle} dt e_j^*(x) \quad \text{(since supp(}\hat{\text{f}}\text{)} \subset \text{B)} \\
&= (2\pi)^d \sum_{j\in\mathbb{N}} f(x_j) e_j^*(x),
\end{aligned}$$

where $\langle\cdot,\cdot\rangle_B$ denotes the usual scalar product in $L_2(B)$ and where above equalities hold for almost all $x \in B$.

Thus, using again the inverse formula, we obtain

$$f(x) = (2\pi)^d \sum_{j\in\mathbb{Z}} f(x_k)[e_j^*]^\vee(x), \tag{2}$$

which implies that the Dirac measures in x_k, $k \in \mathbb{N}$, are the coordinate functionals for $([e_j^*]^\vee)$ which is a Riesz basis of PW_B since $\mathcal{F}^{-1}$ is an isomorphism.

Remark. Since unconditional bases in Hilbert spaces are unique, up to equivalence (c.f. [AK, Theorem 8.3.5]), every Riesz basis (e_j) of H must be equivalent to an orthonormal basis, and thus, there are numbers $0 < a < b$ so that for every x

$$a^2\|x\|^2 \le \sum_j |\langle x, e_j^*\rangle|^2 \le b^2\|x\|^2. \tag{3}$$

Definition 1.5. A sequence (f_j) in a Hilbert space H is called *frame for H or Hilbert frame for H* if there are $0 < a < b$ so that

$$a^2\|x\|^2 \le \sum_{j\in\mathbb{N}} |\langle x, f_j\rangle|^2 \le b^2\|x\|^2. \tag{4}$$

Reconstruction of $x \in H$ from $(\langle x, f_j\rangle)_{j\in N}$:

$\Theta : H \to \ell_2, \qquad x \mapsto (\langle x, f_j\rangle)_{j\in\mathbb{N}}$ *Analysis operator,*

$\Theta^* : \ell_2 \to H, \qquad (\xi_j)_{j\in\mathbb{N}} \mapsto \sum_{j\in\mathbb{N}} \xi_j f_j$ adjoint

$S = \Theta^* \circ \Theta : H \to H, \qquad x \mapsto \sum_{j\in\mathbb{N}} \langle x, f_j\rangle f_j$ Frame transform.

Since

$$a^2\|x\|^2 \le \sum_{j\in\mathbb{N}} |\langle x, f_j\rangle|^2 = \Big\langle x, \sum_{j\in\mathbb{N}} \langle x, f_j\rangle f_j \Big\rangle = \langle x, S(x)\rangle \le b^2\|x\|^2,$$

S is a positive and invertible operator with

$$a\mathrm{Id}_H \le S \le b\mathrm{Id}_H.$$

Thus,

$$x = S \circ S^{-1}(x) = \sum_{j\in\mathbb{N}} \langle S^{-1}(x), f_j\rangle f_j = \sum_{j\in\mathbb{N}} \langle x, S^{-1}(f_j)\rangle f_j, \tag{5}$$

and the series converges unconditionally in L_2. As in the case of unconditional bases we deduce from the Uniform Boundedness Principle that there is a constant R so that

$$\Big\| \sum_{j\in\mathbb{N}} a_j \langle x, S^{-1}(f_j)\rangle f_j \Big\| \le R\|x\| \tag{6}$$

$$\text{for all } x \in H \text{ and all } (a_j),\ |a_j| \le 1, \text{ if } j \in \mathbb{N}.$$

We call (the smallest) R the *unconditional constant of* (f_j)) and we call $S^{-1}(f_j)$ *the coordinate functionals with respect to* (f_j). In the case that $a = b$ we say that (f_i) is a *tight frame*. Note that in this case we obtain

$$x = \frac{1}{a} S(x) = \frac{1}{a} \sum_{j \in \mathbb{N}} \langle x, f_j \rangle f_j. \tag{7}$$

We call the equations (5), and (7) the *expansion of* x *with respect to the frame* $(f_j)_{j\in\mathbb{N}}$.

An important special case is $L_2(B)$), $m(B) > 0$, and we ask when for a given sequence $(x_n : n \in \mathbb{N}) \subset \mathbb{R}^d$ the sequence of exponentials

$$\mathbb{R}^d \to \mathbb{C}, \quad \xi \mapsto e^{-i\langle \xi, x_n \rangle},$$

is a frame for $L_2(B)$ with bounds $0 < a \le b$. In that case we call $(x_n : n \in \mathbb{N})$ a *Fourier Frame for* $L_2(B)$. Using the inverse formula for the Fourier transform we obtain for any $f \in PW_B$

$$f(x) = \frac{1}{(2\pi)^d} \int \hat{f}(\xi) e^{i\langle x, \xi \rangle} d\xi = \frac{1}{(2\pi)^d} \big\langle \hat{f}, e^{-i\langle x, \cdot \rangle} \big\rangle_B.$$

Thus $(x_n) \subset \mathbb{R}^d$ is a Fourier Frame for $L_2(B)$ with bounds $0 < a < b$ if and only if for all $f \in PW_B$

$$\frac{a^2}{(2\pi)^d} \le \sum_n |f(x_n)|^2 \le \frac{b^2}{(2\pi)^d}. \tag{8}$$

Main Goal. Bandlimited functions are used to represent signals. These signal need to be measured, stored and reproduced. It is therefore paramount to find *good bases or frames* for the space PW_B. Here are some properties one seeks:

a) Bases or frames which consist of translations of the same function (*Translation bases*) and/or the corresponding functionals are evaluations at *sampling points* (*Sampling bases*). I.e. there is a strictly increasing sequence $(x_n : n \in \mathbb{Z})$ and a function g on $\mathbb{R}^d$ so that

$$f(x) = \sum_{n \in \mathbb{Z}} f(x_n) g(x - x_n). \tag{9}$$

In case that the *sampling points* (x_n) are not equally distributed this is in general not achievable. So one requires the existence of a matrix

$B = (b_{(m,n)} : m, n \in \mathbb{Z})$ so that

$$f(x) = \sum_{n\in\mathbb{Z}} (B \circ (f(x_j) : j \in \mathbb{Z}))_n g(x - x_n), \quad x \in \mathbb{R}^d. \tag{10}$$

b) *Stability.* Assuming the functions values $f(x_n)$ were measured with some error one seeks to control the error of the function values of f, as represented by (9) or (10). More precisely assume that $\varepsilon > 0$ and assume that $(\tilde{f}_j)$ is such that $\|\tilde{f}_j - f(x_j)\| \le \varepsilon$. Let

$$\tilde{f}(x) = \sum_{n\in\mathbb{Z}} \tilde{f}_n g(x - x_n), \text{ respectively}$$
$$\tilde{f}(x) = \sum_{n\in\mathbb{Z}} (B \circ (\tilde{f}_j) : j \in \mathbb{Z}))_n g(x - x_n).$$

For which bases/frames is it possible to estimate $\|f - \tilde{f}\|_\infty$?

(c) *Quantization.* For which frames can in (9) and (10) the $f(x_j)$'s be replaced by *quantized coefficients,* i.e. integer values of some $q > 0$, and still approximate sufficiently f? Are there quantization algorithms easy to implement, which are easy to implement?

2. Equally distributed sampling points in $\mathbb{R}$

Assume that $f \in PW_\pi$. Thus

$$\mathrm{supp}(\hat{f}) = \{x \in \mathbb{R} : \hat{f}(x) \neq 0\} \subset [-\pi, \pi].$$

We will first derive the *Shannon Whittaker formula.* Since $(g_n)_{n\in\mathbb{Z}}$, with

$$g_n(\xi) = \frac{1}{\sqrt{2\pi}} e^{-in\xi}, \qquad |\xi| \le \pi, n \in \mathbb{Z},$$

is an orthonormal basis of $L_2([-\pi, \pi])$, we can write

$$\hat{f}(\xi) = \sum_{n\in\mathbb{Z}} c_n e^{-in\xi} \text{ for } |\xi| \le \pi, \tag{11}$$

with

$$c_n = \frac{1}{\sqrt{2\pi}} \langle \hat{f}, g_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\xi) e^{in\xi} d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{in\xi} d\xi = f(n).$$

and thus

$$\hat{f}(\xi) = \sum_{n\in\mathbb{Z}} f(n) e^{-in\xi} \chi_{[-\pi,\pi]}(\xi)\,. \tag{12}$$

Using the inverse formula yields the *Shannon Whittaker Formula*:

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n\in\mathbb{Z}} f(n)e^{-in\xi}e^{ix\xi}d\xi \\
&= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} f(n)\int_{-\pi}^{\pi} e^{-in\xi}e^{ix\xi}d\xi \\
&\quad \text{[We can interchange } \textstyle\sum \text{ and } \int \text{ because the inverse} \\
&\quad \text{Fourier transform is a bounded operator on } L_2(\mathbb{R})] \\
&= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} f(n)\Big[\frac{-i}{x-n}e^{i\xi(x-n)}\Big]_{\xi=-\pi}^{\pi} \\
&= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} f(n)\frac{2\sin(x\pi-n\pi)}{x-n} = \sum_{n\in\mathbb{Z}} f(n)\mathrm{sinc}(x\pi-n\pi).
\end{aligned}
\tag{13}
$$

Remark. We obtain a sampling basis for PW_π. But the representation in (12) has bad pointwise properties. It is not pointwise unconditionally converging and highly unstable under errors in measuring $f(n)$.

In order to overcome this highly unstable representation we will pass to a redundant representation and follow the description of the *Analog-Digital Conversion* in [DD].

We fix $\mu_0 > 1$ and let $\mu_0 \le \mu$. We can view $L_2[-\pi,\pi]$ (in a natural way) as a subspace of $L_2[-\mu\pi,\mu\pi]$. The sequence $(g_n^{(\mu)} : n\in\mathbb{Z})$, with

$$
g_n^{(\mu)}(\xi) = \frac{1}{\sqrt{2\mu\pi}}e^{-in\xi/\mu}, \qquad |\xi| \le \mu\pi,\ n\in\mathbb{Z},
$$

is an orthonormal basis for $L_2[-\mu\pi,\mu\pi]$ and as in (12) we obtain

$$
\hat{f}(\xi) = \frac{1}{\mu}\sum_{n\in\mathbb{Z}} f\Big(\frac{n}{\mu}\Big)e^{-in\xi/\mu}\chi_{[-\mu\pi,\mu\pi]}(\xi) \text{ a.s.}
\tag{14}
$$

Remark. We can view the sequence $(g_n^{(\mu)}|_{[-\pi,\pi]})_{n\in\mathbb{Z}}$ as a tight frame for $L_2([-\pi,\pi])$. It is the image of an orthonormal basis of $L_2([-\mu\pi,\mu\pi])$ under the canonical projection from $L_2([-\mu\pi,\mu\pi])$ onto $L_2([-\pi,\pi])$.

Since f was assumed to be in PW_π it follows that the series (14) vanishes almost surely on the set $[-\mu\pi,-\pi]\cup[\pi,\mu\pi]$. We consider a function $g:\mathbb{R}\to\mathbb{R}$ whose Fourier transform $\hat{g}$ has the following properties:

- $\hat{g}$ is C_∞,
- $\hat{g}(\zeta) = 1/2\pi$, if $|\zeta| \le \pi$,
- $\hat{g}(\zeta) = 0$, if $|\zeta| \ge \mu_0\pi$, and

- $0 \le \tilde{g}(\zeta) \le 1/2\pi$ if $\pi \le |\zeta| \le \mu_0\pi$.

Thus, (14) does not change if we multiply both sides by $2\pi\hat{g}$

$$\hat{f}(\xi) = \frac{1}{\mu}\sum_{n\in\mathbb{Z}} f\Big(\frac{n}{\mu}\Big)e^{-in\xi/\mu}2\pi\hat{g}(\xi).$$

Using again the inverse formula for Fourier transforms we arrive to the *Redundant Whittaker Shannon Formula*

$$\begin{aligned} f(x) &= \frac{1}{\mu}\sum_{n\in\mathbb{Z}} f\Big(\frac{n}{\mu}\Big)\int_{-\infty}^{\infty} e^{ix\xi - in\xi/\mu}2\pi\hat{g}(\xi)d\xi \\ &= \frac{1}{\mu}\sum_{n\in\mathbb{Z}} f\Big(\frac{n}{\mu}\Big)g\Big(x-\frac{n}{\mu}\Big). \end{aligned} \tag{15}$$

Note that for $z\in\mathbb{R}$, $z\neq 0$,

$$\begin{aligned} g(z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{g}(\xi)e^{iz\xi}d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{g}'(\xi)\frac{-i}{z}e^{iz\xi}d\xi \\ &= -\frac{1}{z^2}\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{g}''(\xi)e^{iz\xi}d\xi \\ &= \dots \end{aligned}$$

We conclude that $\lim_{z\to\pm\infty}|z|^k g(z) = 0$ for all $k\in\mathbb{N}$ and that the series in (15) converges absolutely and faster than any series of the form $\sum_{n\in\mathbb{N}} n^{-k}$. Moreover, assume that $(\tilde{f}_n : n\in\mathbb{Z})\subset\mathbb{C}$ is a sequence with

$$\varepsilon := \sup_{n\in\mathbb{Z}}\Big|f\Big(\frac{n}{\mu}\Big)-\tilde{f}_n\Big| < \infty,$$

and let

$$\tilde{f}(x) = \sum_{n\in\mathbb{Z}}\tilde{f}_n g\Big(x-\frac{n}{\mu}\Big) \qquad x\in\mathbb{R}.$$

Since for $x\in\mathbb{R}$ and $n\in\mathbb{Z}$ we obtain

$$\begin{aligned} \mu^{-1}\Big|g\Big(x-\frac{n}{\mu}\Big)\Big| &\le \int_{x-\frac{n}{\mu}}^{x-\frac{n-1}{\mu}}|g(\xi)|d\xi + \mu^{-1}\sup_{x-n/\mu\le\xi\le x-(n-1)/\mu}\Big|g\Big(x-\frac{n}{\mu}\Big)-g(\xi)\Big| \\ &\le \int_{x-\frac{n}{\mu}}^{x-\frac{n-1}{\mu}}|g(\xi)|d\xi + \mu^{-1}\int_{x-\frac{n}{\mu}}^{x-\frac{n-1}{\mu}}|g'(\xi)|d\xi, \end{aligned}$$

it follows that

$$|\tilde{f}(x)-f(x)| \le \frac{\varepsilon}{\mu}\sum_{n\in\mathbb{Z}}\Big|g\Big(x-\frac{n}{\mu}\Big)\Big| \tag{16}$$

$$\leq \varepsilon \sum_{n\in\mathbb{Z}} \Big[\int_{x-\frac{n}{\mu}}^{x-\frac{n-1}{\mu}} |g(\xi)| d\xi + \mu^{-1} \int_{x-\frac{n}{\mu}}^{x-\frac{n-1}{\mu}} |g'(\xi)| d\xi \Big]$$
$$\leq \varepsilon \big(\|g\|_{L_1} + \mu^{-1} \|g'\|_{L_1} \big),$$

which proves stability of the representation (15).

We now turn to the problem to *quantize* the representation (15) and to replace the coefficients $f(x_n)$, $n \in \mathbb{N}$, by some integer multiples of a fixed number $q > 0$. The simplest way to do so would be to replace $f(x_n)$ by its closest integer multiple of q, i.e. choose $k_n \in \mathbb{Z}$ so that $|f(n/\mu) - qk_n|$ is minimal. The approximation formula (16) would then give (choosing $\tilde{f}_n = k_n q$)

$$\|\tilde{f}(x) - f(x)\| \leq \sup_{n\in\mathbb{N}} |\tilde{f}_n - f(n/\mu)| \big(\|g\|_{L_1} + \mu^{-1} \|g'\|_{L_1} \big) \\ \leq q \big(\|g\|_{L_1} + \mu^{-1} \|g'\|_{L_1} \big). \tag{17}$$

This approximation does not use the redundancy in (15), i.e. the fact that the representation is not unique. The *First Order Sigma - Delta Procedure*, which we will describe now, takes into account previous errors and tries to cancel them. It thereby achieves a much better approximation.

In order to simplify our exposition we assume that our Paley - Wiener function f is real valued and bounded by 1, i.e $\|f\|_\infty \leq 1$. We will attempt to replace the sampling values $f(n/\mu)$ by either 1 or -1, which corresponds to an A/D conversion (analogue to digital).

Definition 2.1. [First Order Sigma-Delta Algorithm]

Assume $f \in PW_\pi$ is real valued and $\|f\|_\infty \leq 1$. We are defining recursively sequences $(u_n : n \in \mathbb{Z}) \subset \mathbb{R}$ and $(q_n : n \in \mathbb{Z}) \subset \{\pm 1\}$:

First we let $u_0 = 0$. Assuming that for some $n \in \mathbb{N}$ u_{n-1} has been determined we put:

$$q_n = \mathrm{sign}\big(\mathrm{u_{n-1} + f(n/\mu)}\big) \text{ and } \mathrm{u_n = u_{n-1} + f(n/\mu) - q_n}. \tag{18}$$

If $n \in -\mathbb{N}$ and if u_{n+1} has been defined then

$$u_n = u_{n+1} - f((n+1)/\mu) - \mathrm{sign}\big(\mathrm{u_{n+1} - f((n+1)/\mu)}\big). \tag{19}$$

If $n \in -\mathbb{N}$ or $n = 0$

$$q_n = \mathrm{sign}\big(\mathrm{u_{n-1} + f(n/\mu)}\big). \tag{20}$$

Lemma 2.2. *Assume $f \in PW_\pi$ is real valued with $\|f\|_\infty \leq 1$, and that the sequences $(u_n : n \in \mathbb{Z})$ and $(q_n : n \in \mathbb{Z})$ are defined as in Definition 2.1. Then for every $n \in \mathbb{Z}$*

$$|u_n| < 1 \textit{ and} \tag{21}$$

$$u_n - u_{n-1} = f(n/\mu) - q_n. \tag{22}$$

Proof. Assume that $n \in \mathbb{N}$ and that $|u_{n-1}| < 1$. Since $\|f\|_{L_\infty} \le 1$ it follows that $|u_{n-1} + f(n/\mu)| < 2$ and thus

$$|u_n| = |u_{n-1} + f(n/\mu) - q_n| = |u_{n-1} + f(n/\mu) - \mathrm{sign}(f(n/\mu) + u_{n-1})| < 1.$$

(22) follows from definition of u_n in the case that $n \in \mathbb{N}$.

Assuming that $n \in -\mathbb{N}$ and that $|u_{n+1}| < 1$, it follows from the assumption on f that $|u_{n+1} - f((n+1)/\mu)| < 2$ and thus

$$|u_n| = \big|u_{n+1} - f((n+1)/\mu) - \mathrm{sign}\big(u_{n+1} - f((n+1)/\mu)\big)\big| < 1.$$

In order to show (22) we rewriting (19) into

$$u_n = u_{n-1} + f(n/\mu) + \mathrm{sign}\big(u_n - f(n/\mu)\big).$$

Since $|u_n| < 1$ it follows that $\mathrm{sign}\big(u_{n-1} + f(n/\mu)\big) = -\mathrm{sign}\big(u_n - f(n/\mu)\big)$, which yields

$$q_n = \mathrm{sign}\big(u_{n-1} + f(n/\mu)\big) = -\mathrm{sign}\big(u_n - f(n/\mu)\big),$$

and

$$u_n - u_{n-1} = f(n/\mu) + \mathrm{sign}\big(u_n - f(n/\mu)\big) = f(n/\mu) - q_n.$$

This equation is also true for $n = 0$, since

$$\begin{aligned} f(0) - q_0 &= f(0) - \mathrm{sign}(u_{-1} + f(0)) \\ &= f(0) + \mathrm{sign}(u_0 - f(0)) = -u_1 = u_0 - u_1. \end{aligned}$$

□

Proposition 2.3. *If $f \in PW_\pi$ is real valued with $\|f\|_\infty \le 1$ and if for $\mu > 1$ the sequences $(u_n)_{n\in\mathbb{Z}}$ and $(q_n)_{n\in\mathbb{Z}}$ are defined as in (18), (19) and (20) then*

$$\Big| f(t) - \frac{1}{\mu} \sum_{n\in\mathbb{Z}} q_n g\Big(t - \frac{n}{\mu}\Big)\Big| \le \frac{1}{\mu} \|g'\|_{L_1} \text{ for all } t \in \mathbb{R}.$$

Proof. For $t \in \mathbb{R}$

$$\begin{aligned} &\Big| f(t) - \frac{1}{\mu} \sum_{n\in\mathbb{Z}} q_n g\Big(t - \frac{n}{\mu}\Big)\Big| \\ &\quad = \frac{1}{\mu} \Big| \sum_{n\in\mathbb{Z}} \Big[f\Big(\frac{n}{\mu}\Big) - q_n\Big] g\Big(t - \frac{n}{\mu}\Big)\Big| \quad \text{by (16)} \\ &\quad = \frac{1}{\mu} \Big| \sum_{n\in\mathbb{Z}} (u_n - u_{n-1}) g\Big(t - \frac{n}{\mu}\Big)\Big| \quad \text{by (22)} \\ &\quad = \frac{1}{\mu} \Big| \sum_{n\in\mathbb{Z}} u_n \Big[g\Big(t - \frac{n}{\mu}\Big) - g\Big(t - \frac{n+1}{\mu}\Big)\Big]\Big| \end{aligned}$$

$$\leq \frac{1}{\mu}\sum_{n\in\mathbb{Z}}\Big|g\Big(t-\frac{n}{\mu}\Big)-g\Big(t-\frac{n+1}{\mu}\Big)\Big| \quad \text{by (21)}$$

$$\leq \frac{1}{\mu}\sum_{n\in\mathbb{Z}}\int_{t-\frac{n+1}{\mu}}^{t-\frac{n}{\mu}} |g'(x)|dx = \frac{1}{\mu}\|g'\|_{L_1}.$$

□

3. Scattered sampling points

Now we assume that instead of having equally distributed sampling points (i.e $x_n = n$ or more generally $x_n = n/\mu$, for $n \in \mathbb{Z}$) we sample the function at non necessarily equally distributed points $(x_n : n \in \mathbb{Z}) \subset \mathbb{R}$. The wellknown "$\frac{1}{4}$- Theorem of Kadec's states a criterion for the sequence of exponentials $(e^{-ix_n(\cdot)} : n \in \mathbb{Z})$ to be a Riesz basis (i.e. unconditional basis) for $L_2[-1,1]$ (we change now our scaling from the interval $[-\pi,\pi]$ to $[-1,1]$ because this produces shorter formulae).

Theorem 3.1. [Ka] *Assume that* $(x_n : n \in \mathbb{N}) \subset \mathbb{R}$ *and that*

$$L := \sup_{n\in\mathbb{N}} |x_n - n| < \frac{1}{4}. \tag{23}$$

Then $(e^{ix_n(\cdot)} : n \in \mathbb{Z})$ *is a Riesz basis for* $L_2[-1,1]$.

Corollary 3.2. *Assume that* $(x_n : n \in \mathbb{Z}) \subset \mathbb{R}$ *and that*

$$L := \sup_{n\in\mathbb{N}} |x_n - n| < \frac{1}{4} \tag{24}$$

and that $\mu \geq 1$. *Then* $(e^{ix_n(\cdot)/\mu} : n \in \mathbb{Z})$ *is a frame for* $L_2[-1,1]$.

The following observation follows from the easy fact that the tensor product of Riesz bases is a Riesz basis of the Hilbert space tensor product.

Proposition 3.3. *If* $d \in \mathbb{N}$ *and if for* $j = 1,2,\dots d$ *the sequences* $(x_n^{(j)} : n \in \mathbb{Z})$ *are Riesz bases or Fourier frame for* $L_2[-1,1]$ *then the sequence* $(\overline{x}_{\overline{n}} : \overline{n} \in \mathbb{Z}^d)$, *with* $\overline{x}_{\overline{n}} = (x_{n_1}^{(1)}, x_{n_2}^{(2)}, \dots x_{n_d}^{(d)})$, *for* $\overline{n} = (n_1, n_2, \dots n_d) \in \mathbb{Z}^d$, *is a Riesz bases sequence or Fourier frame for* $L_2[-1,1]^d$.

Remark 3.4. Proposition 3.3 together with Theorem 3.1 prove the existence of Riesz bases sequences and determine geometrical conditions for a sequence $(x_k) \subset \mathbb{R}^d$ to be a Riesz basis sequence for $L_2[-1,1]^d$. By scaling we can get Riesz basis sequences for $L_2(\prod_{j=1}^d [a_j,b_j])$, with $a_j < b_j$, $j = 1,2\dots d$.

We do not know of any Riesz basis sequence for $L_2(B)$ for other sets $B \subset \mathbb{R}^d$, for example we do not know if there is a Riesz basis sequence for $L_2(B_2)$, where B_2 is the Eucledian unit ball in $\mathbb{R}^d$.

Using the following deep Theorem by Beurling, one obtains frames which are not necessarily tensor products of frames on subsets of $\mathbb{R}^d$. We first need the following notation: For a countable set $\Lambda \subset \mathbb{R}^d$ we put

$$r(\Lambda) := \frac{1}{2} \inf_{\mu,\mu'\in\Lambda,\, \mu\neq\mu'} \|\mu - \mu'\|_2, \text{ and}$$
$$R(\Lambda) := \sup_{\xi\in\mathbb{R}} \inf_{\mu\in\Lambda} \|\xi - \mu\|_2.$$

We call Λ *uniformly discrete* if $r(\Lambda) > 0$.

Theorem 3.5. [Be] *If Λ is a uniformly discrete set with $R(\Lambda) < \frac{\pi}{2}$ then it is a Fourier frame for $L_2(B_2)$, with $B_2 = \{x \in \mathbb{R} : \|x\|_2 \le 1\}$, and thus also a Fourier frame for $L_2(B)$ with $B \subset B_2$.*

Moreover the frame bounds only depend on $r(\Lambda)$ and $R(\Lambda)$ and d.

The following result is a generalization of the redundant version of the Whittaker Shannon formula (15) and was recently obtained by my student Aaron Bailey.

Theorem 3.6. [Ba] *Let $B \subset \mathbb{R}^d$ be symmetric, convex and bounded, with $B^\circ \neq \emptyset$. Assume that $(x_n : n \in \mathbb{N}) \subset \mathbb{R}^d$ is a Fourier frame for $L_2(B)$, and assume that S is the corresponding frame operator, i.e.:*

$$S : L_2(B) \to L_2(B) \quad f \mapsto \sum_{n\in\mathbb{N}} \langle f, e^{-ix_n(\cdot)}\rangle_B e^{-i\langle x_n,\cdot\rangle}.$$

Let $\mu \ge \mu_0 > 0$ Then there is a $g \in PW_{\mu_0 B}$ so that for all $f \in PW_{\mu B}$

$$f(x) = \frac{1}{\mu} \sum_{n\in\mathbb{N}} \Big[G \circ \Big(f\Big(\frac{x_k}{\mu}\Big) : k \in \mathbb{N}\Big)\Big]_n g\Big(x - \frac{x_n}{\mu}\Big), \quad x \in \mathbb{R}^d. \tag{25}$$

where G is the Grammian matrix for S with respect to the frame $(e^{-i\langle x_n,\cdot\rangle} : n \in \mathbb{N})$, *i.e. $G = (G_{(n,k)})_{n,k\in\mathbb{N}}$ with*

$$G_{(n,k)} = \big\langle S^{-1}\big(e^{-i\langle x_n,\cdot\rangle}\big), S^{-1}\big(e^{ix_k(\cdot)}\big)\big\rangle_B, \quad n, k \in \mathbb{N}.$$

Remark 3.7. [Ba] The representation (25) is in general not even in the one dimensional case stable under perturbations of $f(x_n)$'s because in general G is not bounded as operator on ℓ_∞.

We will now pass to approximative representations of Paley Wiener functions using translates of Gaussians.

For $\lambda > 0$ we define the *Gaussian function* $g_\lambda : \mathbb{R} \to \mathbb{R}$ by

$$g_\lambda(x) = e^{-\lambda x^2}, \quad \text{for all } x \in \mathbb{R},$$

and recall that

$$\hat{g}_\lambda(u) = \left(\frac{\pi}{\lambda}\right)^{1/2} e^{-u^2/(4\lambda)}, \text{ for all } u \in \mathbb{R}. \tag{26}$$

Let $(x_j)_{j\in\mathbb{Z}}$ be a Riesz basis sequence for $L_2[-1,1]$, $\lambda > 0$ and let $f \in PW_1$. We would like to *interpolate* f *at the points* (x_j) *using Gaussians shifted by* x_j, this means we would like to define $I_\lambda(f)$ to be a function of the form

$$I_\lambda(f) : \mathbb{R} \to \mathbb{R}, \quad \xi \mapsto \sum_{j\in\mathbb{Z}} a_n^{(\lambda)} g_\lambda(\xi - x_k),$$

which satisfies the interpolation condition:

$$I_\lambda(f)(x_j) = f(x_j), \quad j \in \mathbb{Z}.$$

To show the existence of $I_\lambda(f)$ as well as the fact that $I_\lambda(f) \in L_2(\mathbb{R}) \cap C_0(\mathbb{R})$, we need the following result. A sketch of a proof can be found in [Ja]. For the sake of completeness we include a proof in the appendix.

Theorem 3.8. *(See Appendix) Let* $\lambda > 0$*, and let* $(x_j : j \in \mathbb{Z})$ *be a sequence of real numbers satisfying the following condition: there exists a positive number* q *such that* $x_{j+1} - x_j \geq q$ *for every* $j \in \mathbb{Z}$*. Let* $A := A_\mu$ *be a bi-infinite matrix whose entries are given by* $A(j,k) := e^{-\mu(x_j - x_k)^2}$*,* $j,k \in \mathbb{Z}$*. Then there exist positive constants* β_1 *and* γ_1*, depending on* μ *and* q*, such that* $|A^{-1}(s,t)| \leq \beta_1 e^{-\gamma_1|s-t|}$*,* $s,t \in \mathbb{Z}$*.*

Theorem 3.8 allows us to deduce:

Proposition 3.9. *Let* λ*,* $(x_j : j \in \mathbb{Z})$*,* q*, and* A *be as above. The operator* A^{-1} *acts boundedly on every* $\ell_p(\mathbb{Z})$*,* $1 \leq p \leq \infty$*.*

Proof. Thanks to the symmetry of A^{-1} and the M. Riesz Convexity Theorem, or the Riesz-Thorin Interpolation Theorem, it is sufficient to verify that A^{-1} is a bounded operator on $\ell_\infty(\mathbb{Z})$. The latter fact is a consequence of the observation that, for any fixed $s \in \mathbb{Z}$,

$$\sum_{t\in\mathbb{Z}} |A^{-1}(s,t)| \leq \beta_1 \sum_{t\in\mathbb{Z}} e^{-\gamma_1|s-t|} \leq \frac{2}{1 - e^{-\gamma_1}}.$$

□

The interpolation operators, whose study will occupy the rest of the paper, are introduced in the following Proposition; its proof will be given in the next section.

Proposition 3.10. *Let $\lambda>0$, and let $(x_j : j\in\mathbb{Z})\subset\mathbb{R}$ be a Riesz basis sequence for PW_1. For any $f\in PW_1$, there exists a unique square-summable sequence $(a_j^{(\lambda)} : j\in\mathbb{Z})$ such that*

$$\sum_{j\in\mathbb{N}} a_j^{(\lambda)} g_\lambda(x_k-x_j)=f(x_k),\quad k\in\mathbb{Z}. \tag{27}$$

The Gaussian Interpolation Operator $I_\lambda : PW_1\to L_2(\mathbb{R})$, *defined by*

$$I_\lambda(f)(\cdot)=\sum_{j\in\mathbb{N}} a_j^{(\lambda)} g_\lambda((\cdot)-x_j), \tag{28}$$

where $(a_j^{(\lambda)} : j\in\mathbb{N})$ satisfies (27), is a well-defined, bounded linear operator from PW_1 to $L_2(\mathbb{R})$. Moreover, $I_\lambda(f)\in C_0(\mathbb{R})$.

We now state the main result.

Theorem 3.11. [ScSi] *Suppose that $(x_j : j\in\mathbb{Z})$ is a Riesz basis sequence for PW_1. Let I_λ, $\lambda>0$, be the associated Gaussian Interpolation Operator. Then for every $f\in PW_1$ we have $f=\lim_{\lambda\to 0^+} I_\lambda(f)$ in $L_2(\mathbb{R})$ and uniformly on $\mathbb{R}$.*

4. Proof of Proposition 3.10 and Theorem 3.11

Our proof of Theorem 3.11 is different from the one given in [ScSi], it allows to be extended to the multidimensional case (see remarks in the next section).

Since (x_k) is a Riesz basis sequence, it is uniformly separated by some $q>0$. For $f\in PW_1$ the sequence

$$\big(f(x_k) : k\in\mathbb{Z}\big)=\big(\langle \hat f, e^{-ix_k(\cdot)}\rangle_{[-1,1]}/2\pi : k\in\mathbb{Z}\big)$$

is in $\ell_2(\mathbb{Z})$ and, thus it follows from Proposition 3.9 that there is a sequence $(a_k^{(\lambda)} : k\in\mathbb{Z})\in\ell_2(\mathbb{Z})$ satisfying (27).

As shown in [NSW, Lemma 2.1] there is a number $\nu>0$ which only depends on λ and q so that $\sum_j g_\lambda(x-x_j)\le\nu$. This implies that the series $\sum_j a_j^{(\lambda)} g_\lambda(x-x_j)$ is uniformly bounded and since each summand is continuous the uniform convergence implies that I_λ is continuous. Since $\lim_{j\to\pm\infty} a_j^{(\lambda)}=0$ it follows moreover that $I_\lambda\in C_0(\mathbb{R})$.

For $m\in\mathbb{N}$ we define a linear bounded operator A_m on $L_2[-1,1]$ as follows: Let $(e_k^*)\subset L_2[-1,1]$ be the coordinate functionals as introduced in Definition 1.5 , *i.e.,* for every $h\in L_2[-1,1]$,

$$h=\sum_{k\in\mathbb{Z}}\langle h, e_k^*\rangle_{[-1,1]} e^{-ix_k(\cdot)}=\sum_{k\in\mathbb{Z}}\int_{-1}^{1} h(\xi)\overline{e_k^*(\xi)}\,d\xi\, e^{ix_k(\cdot)}. \tag{29}$$

Note that for $a\in\mathbb{R}$ we have

$$\begin{aligned}(30)\qquad &\Big\|\sum_{k\in\mathbb{Z}}\langle h, e_k^*\rangle_{[-1,1]}e^{-i(\cdot)x_k}\Big\|_{L_2(a+[-1,1])}\\ &=\Big\|\sum_{k\in\mathbb{Z}}\langle h, e_k^*\rangle_{[-1,1]}e^{-i(\cdot+a)x_k}\Big\|_{L_2[-1,1]}\\ &=\Big\|\sum_{k\in\mathbb{Z}}e^{-iax_k}\langle h, e_k^*\rangle_{[-1,1]}e^{-i(\cdot)x_k)}\Big\|_{L_2[-1,1]}\\ &\le R\|h\|_{L_2[-1,1]},\end{aligned}$$

where R is the unconditional constant of the basis $(e^{-i(\cdot)x_k})$. We can therefore extend h to a locally integrable and almost everywhere defined function on all of $\mathbb{R}$, by simply putting

$$(31)\qquad E(h)(x)=\sum_{k\in\mathbb{Z}}\langle h, e_k^*\rangle_{[-1,1]}e^{-ixx_k},\ x\in\mathbb{R}.$$

Let $m\in\mathbb{N}$, and define $A_m : L_2[-1,1]\to L_2[-1,1]$ by

$$(32)\qquad A_m(h)(\xi)=E(h)(2^m(\xi))\chi_{[-1,-\frac12]\cup[\frac12,1]}(\xi)$$

For $h\in L_2[-1,1]$ it follows from (30) that

$$\begin{aligned}(33)\qquad \|A_m(h)\|^2_{L_2[-1,1]}&=\int_{[-1,-\frac12]\cup[\frac12,1]}|E(h)(2^m u)|^2du\\ &=2^{-m}\int_{2^m[-1,1]\setminus 2^{m-1}[-1,1]}|E(h)(v)|^2dv\\ &\le R^2\|h\|^2_{L_2[-1,1]},\end{aligned}$$

where the last inequality follows form the fact that we need not more than 2^m translates of $[-1,1]$ to cover $2^m[-1,1]\setminus 2^{m-1}[-1,1]$.

Proof of Proposition 3.10. Let $\lambda>0$. By (3) and Proposition 3.9, there is a positive constant κ so that, for each $f\in PW_1$, there is a sequence $(a_j^{(\lambda)})\in\ell_2$ satisfying (27) and the estimate

$$(34)\qquad \|(a_j^{(\lambda)})\|_2\le\kappa\|f\|_2.$$

In order to show that $I_\lambda(f)$ is in $L_2(\mathbb{R})$ we first consider the function

$$w:\mathbb{R}\ni x\mapsto\Big(\frac{\pi}{\lambda}\Big)^{1/2}e^{-x^2/(4\lambda)}\sum_{k\in\mathbb{Z}}a_k^{(\lambda)}e^{-ixx_k},$$

and note that it belongs to $L_2(\mathbb{R})\cap L_1(\mathbb{R})$. Then we notice that, applying $\mathcal{F}^{-1}$ (which is an isomorphism on L_2) gives us $I_\lambda(f)$. Moreover, using (30), (4), and (34), we arrive at the estimate

$$(35)\qquad \|w\|_{L_2(\mathbb{R})}\le C'\|f\|_{L_2(\mathbb{R})},$$

where C' depends only on λ and R. This proves that I_λ is a bounded operator □

Proof of Theorem 3.11. Now fix $f\in PW_1$ and write $I_\lambda(f)$ as

$$I_\lambda(f)(\cdot) = \sum_{j\in\mathbb{N}} a_j^{(\lambda)} g_\lambda((\cdot)-x_j).$$

Recall from the preceding paragraph that the Fourier transform of $I_\lambda(f)$ is given by

$$\mathcal{F}\big[I_\lambda(f)\big](u) = \Big(\frac{\pi}{\lambda}\Big)^{1/2} e^{-u^2/(4\lambda)} \sum_{j\in\mathbb{N}} a_j^{(\lambda)} e^{-ix_j u},\ u\in\mathbb{R}. \tag{36}$$

The proof of Theorem 3.11 proceeds in three steps.
Step 1. We claim that there is a constant $D_1<\infty$ and $\lambda_0>0$, only depending on (x_j), so that

$$\sup_{0<\lambda\le\lambda_0} \|I_\lambda(f)\|_2 \le D_1\|f\|_2.$$

We start by defining

$$H_\lambda(u) = \Big(\frac{\pi}{\lambda}\Big)^{1/2} \sum_{j\in\mathbb{N}} a_j^{(\lambda)} e^{-ix_j u} = e^{u^2/(4\lambda)}\mathcal{F}[I_\lambda(f)](u), \quad u\in\mathbb{R},$$

and let $h_\lambda = H_\lambda|_{[-1,1]}\in L_2[-1,1]$. Thus, $E(h_\lambda)=H_\lambda$.

Suppose that $k\in\mathbb{Z}$. The Inverse formula for Fourier transforms implies that

$$2\pi f(x_k) = \int_{-1}^{1} \mathcal{F}[f](u) e^{ix_k u}\,du = \big\langle \mathcal{F}[f], e^{-ix_k(\cdot)}\big\rangle_{[-1,1]}. \tag{37}$$

On the other hand, equations (27) and (28) assert that

$$\begin{aligned} 2\pi f(x_k) &\\ &= 2\pi I_\lambda(f)(x_k)\\ &= \int_{\mathbb{R}} \mathcal{F}[I_\lambda(f)](u)e^{ix_k u}du \quad \text{(by (5))}\\ &= \int_{\mathbb{R}} e^{-u^2/(4\lambda)} H_\lambda(u) e^{ix_k u}\,du\\ &= \int_{-1}^{1} e^{-u^2/(4\lambda)} H_\lambda(u) e^{ix_k u}du\\ &\qquad + \sum_{m=1}^{\infty} \int_{2^m[-1,1]\setminus 2^{m-1}[-1,1]} e^{-u^2/(4\lambda)} H_\lambda(u) e^{ix_k u}\,du \end{aligned} \tag{38}$$

$$
\begin{aligned}
&= \int_{-1}^{1} e^{-u^2/(4\lambda)} H_\lambda(u) e^{ix_k u}\, du \\
&\quad + \sum_{m=1}^{\infty} 2^m \int_{[-1,-\frac{1}{2}]\cup[\frac{1}{2},1]} e^{-2^m v^2/(4\lambda)} H_\lambda(2^m v) e^{ix_k 2^m v}\, dv \\
&= \int_{-1}^{1} e^{-u^2/(4\lambda)} h_\lambda(u) e^{ix_k u}\, du \\
&+ \sum_{m=1}^{\infty} 2^m \int_{[-1,-\frac{1}{2}]\cup[\frac{1}{2},1]} e^{-2^m v^2/(4\lambda)} A_m(h_\lambda)(v) \overline{A_m(e^{-ix_k(\cdot)})}(v) dv \\
&= \big\langle e^{-(\cdot)^2/(4\lambda)} h_\lambda, e^{-ix_k(\cdot)} \big\rangle_{[-1,1]} \\
&\quad + \sum_{m=1}^{\infty} 2^m \big\langle e^{-2^m(\cdot)/(4\lambda)} A_m(h_\lambda), A_m(e^{-ix_k(\cdot)}) \big\rangle_{[-1,1]} \\
&= \big\langle \mathcal{F}[I_\lambda(f)], e^{-ix_k(\cdot)} \big\rangle_{[-1,1]} \\
&\quad + \sum_{m=1}^{\infty} \big\langle 2^m A_m^*\big(e^{-(2^m(\cdot))^2/4(\lambda)} A_m(h_\lambda)\big), e^{-i\langle x_k,(\cdot)} \big\rangle_{[-1,1]} \\
&= \Big\langle \mathcal{F}[I_\lambda(f)] + \sum_{m=1}^{\infty} 2^m A_m^*\big(e^{-(2^m(\cdot))^2/4(\lambda)} A_m(h_\lambda)\big), e^{-ix_k,(\cdot)} \Big\rangle_{[-1,1]}.
\end{aligned}
$$

As $(e^{ix_k(\cdot)} : k \in \mathbb{Z})$ is a frame for $L_2[-1,1]$ (in particular a complete system), equations (37) and (38) lead to the identity
(39)

$$
\mathcal{F}[f] = \mathcal{F}[I_\lambda(f)] + \sum_{m=1}^{\infty} 2^m A_m^*\big(e^{-(2^m(\cdot))^2/4(\lambda)} A_m(h_\lambda)\big) \text{ a.e. on } [-1,1].
$$

Suppose now that $h \in L_2[-1,1]$ and $m \in \mathbb{N}$. We deduce from (33) that

$$
\begin{aligned}
&\|2^m A_m^*\big(e^{-(2^m(\cdot))^2/(4\lambda)} A_m(h)\big)\|_{L_2[-1,1]}^2 \\
&\qquad \le 2^{2m} R^2 \|e^{-(2^m(\cdot))^2/(4\lambda)} A_m(h)\big)\|_{L_2[-1,1]}^2 \\
&\qquad \le 2^{2m} R^2 \|e^{-2^{2m-2}/(4\lambda)} A_m(h)\big)\|_{L_2[-1,1]}^2 \\
&\qquad\quad (\text{because } \mathrm{suppA_m(h)} \subset [-1,-\frac{1}{2}] \cup [\frac{1}{2},1]) \\
&\qquad \le 2^{2m} R^4 e^{-2^{2m-2}/(2\lambda)} \|h\|_{L_2[-1,1]}^2,
\end{aligned}
$$

whence

$$
\|2^m A_m^*\big(e^{-(2^m(\cdot))^2/(4\lambda)} A_m\big)\|_{L_2[-1,1]} \le 2^m R^2 e^{-2^{2m-2}/(4\lambda)}.
$$

Therefore the linear operator

$$\tau_\lambda : L_2[-1,1] \to L_2[-1,1], \quad h \mapsto \sum_{m\in\mathbb{N}} 2^m A_m^*\big(e^{-\|2^m(\cdot)\|_2^2/(4\lambda)} A_m(h)\big)$$

is bounded. In fact, as there are numbers $\lambda_0 > 0$ and D, such that

$$\sum_{m\in\mathbb{N}} 2^m e^{-2^{2m-2}/(4\lambda)} \le De^{-1/(4\lambda)}, \quad \lambda \in (0, \lambda_0], \tag{40}$$

the operator norm of τ_λ obeys the following estimate:

$$\|\tau_\lambda\| \le R^2 De^{-1/(4\lambda)} \text{ whenever } \lambda < \lambda_0. \tag{41}$$

As the operator τ_λ is positive, (39) yields

$$\begin{aligned}\|\mathcal{F}[f]\|_2\, \|h_\lambda\|_2 &\ge \langle \mathcal{F}[f], h_\lambda\rangle_{[-1,1]} \\ &= \langle e^{-(\cdot)^2/(4\lambda)} h_\lambda, h_\lambda\rangle_{[-1,1]} \ge e^{-1/(4\lambda)} \|h_\lambda\|^2_{L_2[-1,1]}.\end{aligned}$$

Consequently,

$$\|h_\lambda\| \le e^{1/(4\lambda)} \|\mathcal{F}[f]\|_2. \tag{42}$$

Thus, from (39) and(41) we get

$$\|\mathcal{F}[I_\lambda(f)]|_{[-1,1]}\|_2 \le \|\mathcal{F}[f]\|_2 + \|\tau_\lambda(h_\lambda)\|_2 \le \|\mathcal{F}[f]\|_2 + DR^2 \|\mathcal{F}[f]\|_2. \tag{43}$$

Our next step is to estimate $\|\mathcal{F}[I_\lambda(f)]\,|_{\mathbb{R}\setminus[-1,1]}\|_2$. Equation (36) implies that

$$\begin{aligned}&\|\mathcal{F}[I_\lambda(f)]|_{\mathbb{R}\setminus[-1,1]}\|_2^2 \\ &= \int_{\mathbb{R}\setminus[-1,1]} e^{-u^2/(2\lambda)} |H_\lambda(u)|^2\, du \\ &= \sum_{m=1}^\infty \int_{2^m[-1,1]\setminus 2^{m-1}[-1,1]} e^{-u^2/(2\lambda)} |H_\lambda(u)|^2\, du \\ &= \sum_{m=1}^\infty 2^m \int_{[-1,-\frac12]\cup[\frac12,1]} e^{-2^{2m}v^2/(2\lambda)} |A_m(h_\lambda)(v)|^2\, dv \\ &\le \sum_{m=1}^\infty 2^m e^{-2^{2m}/(8\lambda)} \|A_m(h_\lambda)\|_2^2 \\ &\qquad \Big(\text{as } \mathrm{suppA_m(h)} \subset \Big[-1, -\frac12\Big] \cup \Big[\frac12, 1\Big]\Big) \\ &\le R^2 \|h_\lambda\|_2^2 \sum_{m=1}^\infty 2^m e^{-2^{2m}/(8\lambda)} \qquad \text{(by (33))}\end{aligned} \tag{44}$$

$$\leq e^{1/(2\lambda)} R^2 \|\mathcal{F}[f]\|_2^2 \sum_{m=1}^{\infty} e^{-2^{2m}/(8\lambda)} 2^m \qquad \text{(by (42))}.$$

By changing λ_0 and D, if need be, one obtains, as in (40),

$$\sum_{m=1}^{\infty} e^{-2^{2m}/(8\lambda)} 2^m \leq D e^{-1/(2\lambda)}, \quad \lambda \in (0, \lambda_0]. \tag{45}$$

Combining (43) and (44) proves our claim.

Step 2. For all $0 < \lambda < \lambda_0$,

$$\begin{aligned} \|f - I_\lambda(f)\|_2 &= (2\pi)^{-1/2} \|\mathcal{F}[f] - \mathcal{F}[I_\lambda(f)]\|_2 \\ &\leq D_2 \|e^{-(1-(\cdot)^2)/(4\lambda)} \mathcal{F}[f]\|_{L_2[-1,1]}, \end{aligned} \tag{46}$$

for some constant D_2.

Remark. Note that (46) and the Dominated Convergence Theorem imply that

$$\lim_{\lambda \to 0^+} I_\lambda(f) = f \text{ in } L_2(\mathbb{R}).$$

Moreover, the inequality in (46) also yields the following for $0 < \beta < 1$:

$$\sup \left\{ \|I_\lambda(f) - f\| : f \in PW_{\beta[-1,1]}, \ \|f\|_2 \leq 1 \right\} \leq D_2 e^{(\beta^2 - 1)/(4\lambda)} \to 0, \quad \text{if } \lambda \to 0^+. \tag{47}$$

To prove (46) we define $\tilde{\tau}_\lambda = e^{1/4\lambda} \tau_\lambda$,

$$M_\lambda : L_2[-1,1] \to L_2[-1,1], \quad h \mapsto e^{-(1-(\cdot)^2)/(4\lambda)} h,$$

and

$$L_\lambda : L_2[-1,1] \to L_2[-1,1], \quad h \mapsto R \circ \mathcal{F} \circ I_\lambda \circ \mathcal{F}^{-1}(h),$$

where $R : L_2(\mathbb{R}) \to L_2[-1,1]$ is the restriction map.

Proposition 4.1. *The map* $\mathrm{Id} + \tilde{\tau}_\lambda \circ \mathrm{M}_\lambda$ *is an invertible operator on* $L_2[-1,1]$*, and* $(\mathrm{Id} + \tilde{\tau}_\lambda \circ \mathrm{M}_\lambda)^{-1} = \mathrm{L}_\lambda$*. Furthermore, there is a constant* $\Delta > 0$*, depending only on* (x_j)*, so that* $\|(\mathrm{Id} + \tilde{\tau}_\lambda \circ \mathrm{M}_\lambda)^{-1}\| \leq \Delta$*, whenever* $0 < \lambda \leq \lambda_0$*.*

Proof. Let $h \in PW_1$. From (39) we obtain

$$\begin{aligned} \mathcal{F}[h] &= \mathcal{F}[I_\lambda(h)] + \tau_\lambda\big(e^{(\cdot)^2/(4\lambda)} \mathcal{F}[I_\lambda(h)]\big) \\ &= \mathcal{F}[I_\lambda(h)] + \tilde{\tau}_\lambda \circ M_\lambda\big(\mathcal{F}[I_\lambda(h)]\big) = \big(\mathrm{Id} + \tilde{\tau}_\lambda \circ \mathrm{M}_\lambda\big) \mathrm{L}_\lambda(\mathcal{F}[\mathrm{h}]). \end{aligned}$$

This implies that $\mathrm{Id} + \tilde{\tau}_\lambda \circ \mathrm{M}_\lambda$ is surjective and is a left inverse of the bounded operator L_λ. Next we show that $\mathrm{Id} + \tilde{\tau}_\lambda \circ \mathrm{M}_\lambda$ is also injective. To that end, let $(\mathrm{Id} + \tilde{\tau}_\lambda \circ \mathrm{M}_\lambda)(\mathrm{h}) = 0$ for some $h \in L_2[-1,1]$. Then

$$0 = \big\langle (\mathrm{Id} + \tilde{\tau}_\lambda \circ \mathrm{M}_\lambda)(\mathrm{h}), \mathrm{M}_\lambda(\mathrm{h}) \big\rangle_{[-1,1]}$$

$$= \big\langle h, M_\lambda(h)\big\rangle_{[-1,1]} + \big\langle \tilde\tau_\lambda(M_\lambda(h)), M_\lambda(h)\big\rangle_{[-1,1]} \geq \big\langle h, M_\lambda(h)\big\rangle_{[-1,1]} \geq 0,$$

the first inequality above being a consequence of the positivity of $\tilde\tau_\lambda$. Hence $\langle h, M_\lambda(h)\rangle_{[-1,1]} = 0$, which implies that $h = 0$, because M_λ is a strictly positive operator. The injectivity of $\mathrm{Id} + \tilde\tau_\lambda \circ \mathrm{M}_\lambda$ follows.

Thus $\mathrm{Id}+\tilde\tau_\lambda\circ\mathrm{M}_\lambda$ is invertible, and its inverse is L_λ. As the operators L_λ, $0 < \lambda \leq \lambda_0$, are uniformly bounded (Step 1), the proof is complete. □

Proposition 4.1 provides the following identity on $[-1,1]$:

$$\mathcal{F}[f]-\mathcal{F}[I_\lambda(f)] = \big[\mathrm{Id}-(\mathrm{Id}+\tilde\tau_\lambda\circ\mathrm{M}_\lambda)^{-1}\big](\mathcal{F}[\mathrm{f}]) = (\mathrm{Id}+\tilde\tau_\lambda\circ\mathrm{M}_\lambda)^{-1}\circ\tilde\tau_\lambda\circ\mathrm{M}_\lambda(\mathcal{F}[\mathrm{f}]).$$

Therefore, we conclude via (41) and Proposition 4.1 that

$$\begin{aligned}(48)\quad \|\mathcal{F}[f]-\mathcal{F}[I_\lambda(f)]|_{[-1,1]}\|_2 &\leq \|(\mathrm{Id}+\tilde\tau_\lambda\circ\mathrm{M}_\lambda)^{-1}\|\,\|\tilde\tau_\lambda\|\,\|\mathrm{M}_\lambda(\mathcal{F}[\mathrm{f}])\|_2\\ &\leq \Delta R_B^2 D\|M_\lambda(\mathcal{F}[f])\|_2.\end{aligned}$$

Now the first inequality in (44) yields

$$\begin{aligned}(49)\quad &\big\|\mathcal{F}[I_\lambda(f)]|_{\mathbb{R}\setminus[-1,1]}\big\|_2^2\\
&\leq \sum_{m=1}^\infty 2^m e^{-2^{2m}/(8\lambda)}\|A_m(h_\lambda)\|_2^2\\
&\leq R^2\sum_{m=1}^\infty 2^m e^{-2^{2m}/(8\lambda)}\|e^{(\cdot)^2/(4\lambda)}\mathcal{F}[I_\lambda(f)]|_{[-1,1]}\|_2^2 \qquad \text{by (33)}\\
&= R^2\sum_{m=1}^\infty 2^m e^{(4-2^{2m})/(8\lambda)}\|e^{-(1-(\cdot)^2)/(4\lambda)}\mathcal{F}[I_\lambda(f)]|_{[-1,1]}\|_2^2\\
&\leq DR^2\big[\|e^{-(1-(\cdot)^2)/(4\lambda)}\mathcal{F}[f]\|_2\\
&\quad + \|e^{-(1-(\cdot)^2)/(4\lambda)}\big(\mathcal{F}[I_\lambda(f)]|_{[-1,1]}-\mathcal{F}[f]\big)\|_2\big]^2 \text{ (by (45))}\\
&\leq DR^2\big[\|e^{-(1-(\cdot)^2)/(4\lambda)}\mathcal{F}[f]\|_2 + \|\mathcal{F}[I_\lambda(f)]|_{[-1,1]}-\mathcal{F}[f]\|_2\big]^2\\
&\leq 2DR_B^2\big[\|e^{-(1-(\cdot)^2)/(4\lambda)}\mathcal{F}[f]\|_2^2 + \|\mathcal{F}[I_\lambda(f)]|_{[-1,1]}-\mathcal{F}[f]\|_2^2\big]\\
&\leq 2DR_B^2[1+\Delta^2R^2D]\|M_\lambda(\mathcal{F}(f))\|_2^2.\end{aligned}$$

Combining this with (1) and (48) we obtain

$$\begin{aligned}&\|I_\lambda(f)-f\|_2^2\\
&\quad = 2\pi\|\mathcal{F}[I_\lambda(f)-\mathcal{F}[f]\|_2^2\\
&\quad = 2\pi\big[\|\mathcal{F}[I_\lambda(f)]|_{[-1,1]}-\mathcal{F}[f]\|_2^2 + \|\mathcal{F}[I_\lambda(f)]|_{\mathbb{R}\setminus[-1,1]}\|_2^2\big]\\
&\quad \leq D_2\|M_\lambda(f)\|_2^2,\end{aligned}$$

for some constant D_2 only depending on (x_j).

Step 3. There exist constants $\lambda_1 \in (0, \lambda_0]$ and D_3 so that

$$\big|I_\lambda(f)(x) - f(x)\big| \le D_3 \|e^{((\cdot)^2-1)/(4\lambda)}\mathcal{F}[f]\|_2,$$

for all $x \in \mathbb{R}$. In particular $\lim_{\lambda\to 0^+} I_\lambda(f) = f$ uniformly on $\mathbb{R}$.

Let $x \in \mathbb{R}$. We use (5) to write

$$\begin{aligned}(50)\qquad &\big|I_\lambda(f)(x) - f(x)\big| \\ &= \frac{1}{2\pi}\Big|\Big[\int_{-1}^{1}\big[\mathcal{F}[I_\lambda(f)](u) - \mathcal{F}[f](u)\big]e^{ixu}du \\ &\qquad + \int_{\mathbb{R}\setminus[-1,1]} \mathcal{F}[I_\lambda(f)](u)e^{ixu}du\Big]\Big| \\ &\le \frac{1}{2\pi}\big[\|\mathcal{F}[I_\lambda(f)]|_{[-1,1]} - \mathcal{F}[f]\|_1 + \big\|\mathcal{F}[I_\lambda(f)]|_{\mathbb{R}\setminus[-1,1]}\big\|_1\big].\end{aligned}$$

The first term converges to 0 if λ approaches 0 because $\|\cdot\|_1 \le m[-1,1]\|\cdot\|_2$ on $L_1[-1,1]$ and because of the Step 2. That the second term converges to 0, can be shown with arguments similar to the arguments which proved that $\big\|\mathcal{F}[I_\lambda(f)]|_{\mathbb{R}\setminus[-1,1]}\big\|_2$ converges to 0. □

5. Multidimensional Versions of Theorem 3.11

We consider now the multidimensional Gaussian interpolation operator. Let $d \in \mathbb{N}$, and let $(x_j : j \in \mathbb{N}) \subset \mathbb{R}^d$ be a Rieszbasis sequence for $L_2(B)$, where we assume that B is bounded, convex, symmetric, and has a non empty open kernel.

The *d-dimensional Gaussian function with scaling parameter* $\lambda > 0$ is defined by

$$g_\lambda^{(d)}(x_1, x_2, \dots, x_d) = e^{-\lambda\|x\|^2} = e^{-\lambda\sum_{j=1}^d x_j^2}, \quad x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d.$$

The existence of the more dimensional interpolation operator is cannot be established in such an elementary way as in the one dimensional case but can be deduced from the following Theorem

Theorem 5.1. cf. [NW, Theorem 2.3]
Let λ and q be fixed positive numbers. There exists a number θ, depending only on d, λ, and q, such that the following holds: if (x_j) is any sequence in $\mathbb{R}^d$ with $\|x_j - x_k\|_2 \ge q$ for $j \ne k$, then $\sum_{j,k}\xi_j\overline{\xi}_k g_\lambda(\|x_j - x_k\|_2) \ge \theta\sum_j|\xi_j|^2$, for every sequence of complex numbers (ξ_j).

Using Theorem 5.1 we can deduce the existence of the Interpolation operator similarly as in the 1 dimensional case

Theorem 5.2. *Let $d \in \mathbb{N}$, let λ be a fixed positive number, and let $(x_j : j \in \mathbb{N})$ be a Riesz-basis sequence for $L_2(B)$. For any $f \in PW_B$*

there exists a unique square-summable sequence $(a(j,\lambda) : j \in \mathbb{N})$ *such that*

$$\sum_{j\in\mathbb{N}} a(j,\lambda) g_\lambda^{(d)}(x_k - x_j) = f(x_k), \quad k \in \mathbb{N}. \tag{51}$$

The Gaussian Interpolation Operator $I_\lambda : PW_B \to L_2(\mathbb{R}^d)$*, defined by*

$$I_\lambda(f)(\cdot) = \sum_{j\in\mathbb{N}} a(j,\lambda) g_\lambda^{(d)}(\cdot - x_j),$$

where $(a(j,\lambda) : j \in \mathbb{N})$ *satisfies* (51)*, is a well-defined, bounded linear operator from* PW_B *to* $L_2(\mathbb{R}^d)$*. Moreover,* $I_\lambda(f) \in C_0(\mathbb{R}^d)$*.*

We can now extend Theorem 3.11 in the following special situations First let $B = Q = [-1,1]^d$ and assume that for $j = 1,\dots d$ the sequence $(x_k^{(j)} : k \in \mathbb{Z}) \subset \mathbb{R}$ is a Riesz basis sequence for $L_2[-1,1]$, and put $x_k = (x_{k_1}^{(1)}, x_{k_2}(2), \dots x_{k_d}^{(d)})$, for $k = (k_1,k_2,\dots k_d) \in \mathbb{Z}^d$. It is then easy to see that $(x_k)_{k\in\mathbb{Z}^d}$ is a Riesz basis sequence for Q. We can therefore define I_λ for that sequence.

Theorem 5.3. *Under the above assumptions it follows that for all* $f \in PW_Q$

$$f = \lim_{\lambda\to 0} I_\lambda(f) \text{ in } L_2(\mathbb{R}^d) \text{ and uniformly.}$$

The second case we can extend is the case that $B = B_2$, the Euclidean unit ball in $\mathbb{R}^d$.

Theorem 5.4. *Assume that* (x_k) *is a Riesz basis sequence for* $L_2(B_2)$ *and let* I_λ *be the interpolation operator defined above for* (x_k)*, then for all* $f \in PW_{B_2}$ *it follows that*

$$f = \lim_{\lambda\to 0} I_\lambda(f) \text{ in } L_2(\mathbb{R}^d) \text{ and uniformly.}$$

Remark. Theorem 5.4 can essentially be shown the same way as Theorem 3.11. Nevertheless there is one problem though. We do not know whether a there exists a Riesz basis sequence for $L_2(B_2)$.

We do not know whether Theorem 5.4 still holds if we only assume that (x_k) is a uniformly separated Fourier frame, If so, Beurling's Theorem would provide easy to satisfy conditions on (x_k) for which it holds.

6. Appendix: Proof of Theorem 3.8

Theorem 3.8 will follow from the following result on bi-infinite matrices which appears to be folkloric. A sketch of a proof can be found in [Ja], but for the sake of completeness we include a self-contained argument.

First let us observe that for any $\mu > 0$ Bochner's Theorem implies that g_μ is a positive definite function which means that for any finite sequence $(\xi_j)_{j-1}^{\ell} \subset \mathbb{R}$ the matrix

$$A = (e^{-\mu(x_k - x_\ell)^2} : 1 \le k, \ell n)$$

is positive definite.

This implies that also for any uniformly separated sequence $(x_j : j \in \mathbb{Z})$ the bi-infinite matrix

$$A = (e^{-\mu(x_k - x_\ell)^2} : k, \ell \in \mathbb{Z})$$

bounded as operator on $\ell_2(\mathbb{Z})$ (because of the fast decay of all rows and colums) and positive.

Theorem 6.1. *Suppose that $(A(j,k))_{j,k\in\mathbb{Z}}$ is a bi-infinite matrix which, as an operator on $\ell^2(\mathbb{Z})$, is self adjoint, positive and invertible. Assume further that there exist positive constants τ and γ such that $|A(j,k)| \le \tau e^{-\gamma|j-k|}$ for every pair of integers j and k. Then there exist constants $\tilde{\tau}$ and $\tilde{\gamma}$ such that $|A^{-1}(s,t)| \le \tilde{\tau} e^{-\tilde{\gamma}|s-t|}$ for every $s, t \in \mathbb{Z}$.*

For the proof of Theorem 6.1 we shall need the following pair of lemmata.

Lemma 6.2. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, and let $A : H \to H$ be a bounded linear operator satisfying the following conditions: (i) $A = A^*$, and (ii) $\inf\{\langle x, Ax\rangle : \|x\| = 1\} > 0$. Let $R := I - \frac{A}{\|A\|}$, where I denotes the identity. Then $R = R^*$, $\langle x, Rx\rangle \ge 0$ for every $x \in H$, and $\|R\| < 1$.*

Proof. The symmetry of R is evident. If $\|x\| = 1$, then

$$\langle x, Rx\rangle = \|x\|^2 - \Big\langle x, \frac{A}{\|A\|} x \Big\rangle.$$

By assumption (ii) and the BCS inequality we see that the term on the right of the preceding equation is between 0 and 1. Therefore $\|R\| = \sup\{\langle x, Rx\rangle : \|x\| = 1\} \le 1$. If $\|R\| = 1$, then there is a sequence $(x_n : n \in \mathbb{N})$ such that $\|x_n\| = 1$ for every n, and

$$1 = \lim_{n\to\infty} \langle x_n, Rx_n\rangle = \lim_{n\to\infty} \left(1 - \Big\langle x_n, \frac{A}{\|A\|} x_n \Big\rangle\right).$$

But the second term on the right cannot converge to zero because of assumption (ii). □

Lemma 6.3. *Suppose that $(R(s,t))_{s,t\in\mathbb{Z}}$ is a bi-infinite matrix satisfying the following condition: there exist positive constants C and γ such that $|R(s,t)| \le e^{-\gamma|s-t|}$ for every pair of integers s and t. Given*

$0<\gamma'<\gamma$, there is a constant $C(\gamma,\gamma')$, depending on γ and γ', such that $|R^n(s,t)|\le C^nC(\gamma,\gamma')^{n-1}e^{-\gamma'|s-t|}$ for every $s,t\in\mathbb{Z}$.

Proof. Suppose firstly that $s\ne t\in\mathbb{Z}$, and assume without loss that $s<t$. Note that

(52)
$$\begin{aligned}\sum_{u=-\infty}^{\infty} e^{-\gamma|s-u|}e^{-\gamma'|t-u|} &= \sum_{u=s}^{t} e^{-\gamma(u-s)}e^{-\gamma'(t-u)}\\ &\quad+\sum_{u=-\infty}^{s-1} e^{-\gamma(s-u)}e^{-\gamma'(t-u)}+\sum_{u=t+1}^{\infty} e^{-\gamma(u-s)}e^{-\gamma'(u-t)}\\ &=:\Sigma_1+\Sigma_2+\Sigma_3.\end{aligned}$$

Now

$$\begin{aligned}\Sigma_1 &= e^{-\gamma'(t-s)}e^{(\gamma-\gamma')s}\sum_{u=s}^{t} e^{-u(\gamma-\gamma')}\\ &= e^{-\gamma'(t-s)}e^{(\gamma-\gamma')s}\sum_{v=0}^{t-s} e^{-s(\gamma-\gamma')-v(\gamma-\gamma')}\\ &= e^{-\gamma'(t-s)}\sum_{v=0}^{t-s} e^{-v(\gamma-\gamma')}\le \frac{e^{-\gamma'(t-s)}}{1-e^{-(\gamma-\gamma')}}.\end{aligned}\tag{53}$$

Moreover,

$$\begin{aligned}\Sigma_2 &= \sum_{v=1}^{\infty} e^{-\gamma v}e^{-\gamma'(t-s+v)}\\ &= e^{-\gamma'(t-s)}\sum_{v=1}^{\infty} e^{-v(\gamma+\gamma')}\le \frac{e^{-\gamma'(t-s)}}{1-e^{-(\gamma+\gamma')}},\end{aligned}\tag{54}$$

whereas

$$\begin{aligned}\Sigma_3 &= \sum_{v=1}^{\infty} e^{-\gamma' v}e^{-\gamma(v+t-s)}\\ &= e^{-\gamma(t-s)}\sum_{v=1}^{\infty} e^{-(\gamma+\gamma')v}\le \frac{e^{-\gamma'(t-s)}}{1-e^{-(\gamma+\gamma')}}.\end{aligned}\tag{55}$$

If $s=t$, then

$$\sum_{u=-\infty}^{\infty} e^{-\gamma|s-u|}e^{-\gamma'|t-u|} = \sum_{u=-\infty}^{\infty} e^{-(\gamma+\gamma')|s-u|}\le \frac{2}{1-e^{-(\gamma+\gamma')}}.\tag{56}$$

From (52)-(56) we conclude that

$$|R^2(s,t)| \le C^2\Big[\frac{1}{1-e^{-(\gamma-\gamma')}} + \frac{2}{1-e^{-(\gamma+\gamma')}}\Big] =: C^2 C(\gamma,\gamma').$$

The general result follows from this via induction. □

We are now ready to prove Theorem 6.1 and Theorem 3.8

Proof of Theorem 6.1. We begin with the remark that assumptions (i) and (ii) of Lemma 6.2 ensure that A is, in fact, boundedly invertible. Let $R = I - \frac{A}{\|A\|}$ be the matrix given in that lemma. As

$$R(j,k) = \frac{A(j,k)}{\|A\|} \text{ if } j \neq k, \quad \text{and} \quad R(k,k) = \frac{\|A\| - A(k,k)}{\|A\|},$$

there is some constant C such that $|R(s,t)| \le Ce^{-\gamma|s-t|}$ for every pair of integers s and t. As $A = \|A\|\,(I-R)$, and $r := \|R\| < 1$ (Lemma 6.2), the standard Neumann series expansion yields the relations

$$\begin{aligned} A^{-1} &= \|A\|^{-1}\sum_{n=0}^{\infty} R^n \\ &= \|A\|^{-1}\sum_{n=0}^{N-1} R^n + \|A\|^{-1}R^N\sum_{n=0}^{\infty} R^n \\ &= \|A\|^{-1}\sum_{n=0}^{N-1} R^n + R^N A^{-1}, \end{aligned} \tag{57}$$

for any positive integer N. As $R^0(s,t) = I(s,t) = 0$ if $s \neq t$, $s,t \in \mathbb{Z}$, we see from (57) that

$$A^{-1}(s,t) = \|A\|^{-1}\sum_{n=1}^{N-1} R^n(s,t) + [R^N A^{-1}](s,t), \quad s \neq t. \tag{58}$$

Choose and fix a positive number $\gamma' < \gamma$, and recall from Lemma 6.3 that there is a constant $C(\gamma,\gamma')$ such that $|R^n(s,t)| \le C^n C(\gamma,\gamma')^{n-1}e^{-\gamma'|s-t|}$ for every positive integer n, and every pair of integers s and t. So we may assume that there is some constant $D := D(\gamma,\gamma') > 1$ such that $|R^n(s,t)| \le D^n e^{-\gamma'|s-t|}$ for every positive integer n, and every pair of integers s and t. Using this bound in (58) provides the following estimate for every $s \neq t$:

$$\begin{aligned} |A^{-1}(s,t)| &\le \|A\|^{-1}e^{-\gamma'|s-t|}\sum_{n=1}^{N-1} D^n + \|A^{-1}\|r^N \\ &\le \|A\|^{-1}e^{-\gamma'|s-t|}\frac{D^N}{D-1} + \|A^{-1}\|r^N. \end{aligned} \tag{59}$$

Let m be a positive integer such that $e^{-\gamma'}D^{1/m} < 1$, and let $s, t \in \mathbb{Z}$ with $|s-t| \geq m$. Writing $|s-t| = Nm + k$, $0 \leq k \leq m-1$, we find that

$$e^{-\gamma'|s-t|}D^N = \left[e^{-\gamma'}D^{\frac{1}{m+(k/N)}}\right]^{|s-t|} \leq [e^{-\gamma'}D^{1/m}]^{|s-t|}. \tag{60}$$

Further,

$$r^N = \left[r^{\frac{1}{m+(k/N)}}\right]^{|s-t|} \leq [r^{1/2m}]^{|s-t|}, \tag{61}$$

and combining (60) and (61) with (59) leads to the following bounds for every $|s-t| \geq m$:

(62)

$$|A^{-1}(s,t)| \leq \frac{\|A\|^{-1}}{D-1}[e^{-\gamma'}D^{1/m}]^{|s-t|} + \|A^{-1}\|[r^{1/2m}]^{|s-t|} = O(e^{-\tilde{\gamma}|s-t|}).$$

On the other hand, if $|s-t| < m$, we obtain

$$|A^{-1}(s,t)| \leq \|A^{-1}\| \leq (\|A^{-1}\|e^{m\tilde{\gamma}})e^{-\tilde{\gamma}|s-t|}, \tag{63}$$

and combining (62) with (63) finishes the proof. □

Proof of Theorem 3.8. It was observed that A satisfies the assumptions of Lemma 6.2. Moreover, the hypothesis $x_{j+1} - x_j \geq q$, $j \in \mathbb{Z}$, implies that $|x_j - x_k| \geq |j-k|q$ for every pair of integers j and k. So an appeal to Theorem 6.1 yields the required result. □

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