Infinite Combinatorics and Applications to Banach space Theory
Course Notes: Math 663-601

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In this course we discuss several results on Infinite Combinatorics, and their applications to Banach space theory.

In the first chapter we present Ramsey’s theorem for infinite subsets of the natural numbers. Assume you color all infinite subsequences of the natural numbers \( \mathbb{N} \) using finitely many colors. Then, under some mild topological condition on this coloring, Ramsey’s theorem states that you can find a subsequence \( N \) of \( \mathbb{N} \), so that all further subsequences \( M \) of \( N \) have the same color. In Chapter 2 we will apply Ramsey’s theorem to obtain several results in Banach space theory, for example Haskell Rosenthal’s celebrated \( \ell_1 \) Theorem: Every semi normalized sequence in a Banach space either contains a weak Cauchy subsequence or it contains a subsequence which is equivalent to the \( \ell_1 \) unit vector basis.

In view of Ramsey’s theorem one might ask whether or not there exist versions of it for Banach spaces of the following kind. Assume you color the vectors of the sphere of a separable, infinite dimensional space \( X \) using finitely many colors. Is it possible, under certain conditions on the coloring, to conclude that \( X \) has an infinite dimensional Banach space whose sphere is monochromatic? The answer to this question is for most spaces negative, and we will present several examples which illustrate this in chapter 3. Nevertheless, parts of Ramsey’s theorem still hold and we will present the following two examples: Gowers’s dichotomy theorem, and its application to solve the homogenous Banach space problem, as well as a combinatorial result, recently obtained by the author in collaboration with E. Odell, and its application to several universality problems.

In chapter 4 we will give a short introduction to ordinal numbers, the arithmetic on them, and the principle of transfinite induction and recursion. We will use them to introduce in chapter 5 several Banach indices, which are isomorphic invariances of Banach spaces and therefore important tools to classify them.
## Contents

1 The Theorem of Ramsey 5
   1.1 Ramsey’s theorem for finite sequences 5
   1.2 Infinite Games 8
   1.3 Ramsey’s theorem for infinite sequences 12

2 Application of Ramsey’s theorem 19
   2.1 Bases of Banach spaces 19
   2.2 Spreading models 25
   2.3 Rosenthal’s $\ell_1$ theorem 33
   2.4 Partial Unconditionality 35

3 Distortion of Banach spaces 41
   3.1 Introduction 41
   3.2 Spaces not containing $\ell_p$ or $c_0$ 47
   3.3 Hilbert space is arbitrarily distortable 63

4 Versions of Ramsey’s theorem in Banach spaces 73
   4.1 Gowers’ game on blocks and the dichotomy theorem 73
   4.2 Trees and Branches in Banach spaces 78
   4.3 Embedding and Universality Theorems 91

5 Ordinal numbers 101
   5.1 Definition of ordinal numbers 101
   5.2 Arithmetic of ordinals 107
   5.3 Classification of countable compacts by the Cantor Bendixson index 107

6 Banach indices 109
   6.1 General definition of indices 109
   6.2 Szlenk’s index 109
   6.3 Classification of $C(K)$, $K$ countable compact 109
Chapter 1

The Theorem of Ramsey

1.1 Ramsey’s theorem for finite sequences

We begin with Ramsey’s original theorem [Ra]. He was only 27 when he died in 1929 and this paper appeared the following year.

Notation. For two sets $X$ and $S$ we denote by $X^S$ the set of all functions $f : S \to X$, or equivalently the set of all families $(x_i)_{i \in S} \subset X$. The powerset of $X$, i.e. the set of all subsets of $X$, is denoted by $\mathcal{P}(X)$.

Assume $\alpha$ is a cardinal number. At the moment we only worry about the cases $\alpha = n \in \mathbb{N}$, $\alpha = \omega$ (the countable infinite cardinal, which can be identified with the set of all natural numbers $\mathbb{N}$), $\alpha = \omega_1$ (the smallest uncountable cardinal) and $\alpha = \omega_c$ (the continuum). We denote by $[X]^\alpha$ the set of all subsets of $X$ of cardinality $\alpha$. $[X]^{<\alpha}$ and $[X]^{\leq \alpha}$ are the set of all subsets of $X$ whose cardinality is less than, respectively at most $\alpha$. In the case $X = \mathbb{N}$ or $X \subset \mathbb{N}$ or any other well ordered set we will identify $[X]^n$ and $X^\omega$ with the set of increasing sequences $X$.

Similarly $X^{<\alpha}$ and $X^{\leq \alpha}$ is the set of all families indexed over cardinalities smaller than $\alpha$, respectively smaller then or equal to $\alpha$.

In particular

- $X^n$, and $X^\omega$, is the set of all sequences in $X$ of cardinality $n$ or all infinite sequences, and $X^{<\omega}$ the set of all finite sequences in $X$.
- $[X]^n$ and $[X]^\omega$, the set of all subsets of cardinality $n$, respectively $\omega$ or the set of all increasing sequences of cardinality $n$ or $\omega$, respectively.

**Theorem 1.1.1.** [Ra] Let $k \in \mathbb{N}$ and $A \subseteq [\mathbb{N}]^k$. Then there exists $M \in [\mathbb{N}]^\omega$ so that either $[M]^k \subseteq A$ or $[M]^k \cap A = \emptyset$.

**Remark.** Note that the case $k = 1$ is simply the Pigeon hole Principle.

**Proof of Theorem 1.1.1.** We give first the proof for $k = 2$. Let $n_1 = 1$. Choose $M_1 \in [\mathbb{N}]^\omega$ with $1 < \min(M_1)$ so that
a) either $(n_1, n) \in \mathcal{A}$ for all $n \in M_1$, or
b) or $(n_1, n) \not\in \mathcal{A}$ for all $n \in M_1$.

Let $n_2 = \min M_1$ and choose $M_2 \subseteq [M_1]^\omega$ so that $n_2 < M_2$ and either

a) $(n_2, n) \in \mathcal{A}$ for all $n \in M_2$, or
b) $(n_2, n) \not\in \mathcal{A}$ for all $n \in M_2$.

Let $n_3 = \min M_2$ and continue in this manner. The alternative a) or b) must occur infinitely many times. Suppose $M = \{n_i : a\}$ holds for $n_i$ is a number.

Thus, assume that for any $k \geq 0$, $M = \{n_i : b\}$ holds for $n_i$ satisfies $[M]^\omega \cap \mathcal{A} = \emptyset$.

Using our induction hypothesis we can choose by induction on $\ell < n$, $\exists M = \{n_i\}$ so that for all $\ell < n$ and choose $M = \{n_i\}$ so that for all $\ell < n$ the same case happens and choose $M = \{n_i : i \in \mathbb{N}\} i \in \mathbb{N}$.

**Definition 1.1.2.** By an $r$-coloring of a set $A$ we mean a partition $(A_i)_{i=1}^r$ of $A$ into $r$ subsets. In that case we call $B \subseteq A$ monochromatic if for some $i \leq r$, $B \subseteq A_i$.

**Corollary 1.1.3.** For any $r \in \mathbb{N}$ and any $r$ coloring of $[\mathbb{N}]^k$ there is an $M \in [\mathbb{N}]^\omega$ so that $[M]^k$ is monochromatic.

Using the Compactness Principle we can deduce the following strengthening of Theorem 1.1.1.

**Corollary 1.1.4.** Given $k$ and $m$ in $\mathbb{N}$ there is an $n = n(k, m) \geq m$ so that the following holds. Assume that $\mathcal{A} \subseteq \{1, 2, 3, \ldots, n\}^k$ then there exists $M \subseteq \{1, 2, 3, \ldots, n\}$ with $\#M = m$ so that

Either $[M]^k \subseteq \mathcal{A}$ or $[M]^k \cap \mathcal{A} = \emptyset$.

**Remark.** For $k = 2$ Corollary 1.1.3 means the following: Given any number $m$ there is a number $n \geq m$ so that: If ones invites any $n$ people there at least $m$ of them who either know each other or who do not know each other.

**Proof of Corollary 1.1.4.** Assume that for some $k$ and $m$ in $\mathbb{N}$ our claim is wrong. Thus, assume that for any $n \in \mathbb{N}$, $n \geq m$ there is an $\mathcal{A}_n \subseteq \{1, 2, \ldots, n\}^k$ so that

\[(*) \quad \forall M \subseteq \{1, 2 \ldots n\}^m \quad [M]^k \not\subseteq \mathcal{A}_n \text{ and } [M]^k \cap \mathcal{A}_n \neq \emptyset.\]
By induction choose $n_1 < n_2 < \ldots$ and infinite sets $\mathbb{N} = N_0 \supset N_1 \supset N_2 \supset \ldots$ so that for $\ell \in \mathbb{N}$

$$n_{\ell} = \min N_{\ell - 1} < \min N_{\ell} \quad \forall n, n' \in N_{\ell}, \quad A_n \cap [\{n_1, n_2, \ldots, n_{\ell}\}]^k = A_{n'} \cap [\{n_1, n_2, \ldots, n_{\ell}\}]^k$$

In order to choose $N_{\ell}$ we can use the Pigeonhole Principle since for any $n \in N_{\ell - 1}$ the set $A_n \cap [\{n_1, n_2, \ldots, n_{\ell}\}]^k$ is an element of the (finite) set $P([\{n_1, n_2, \ldots, n_{\ell}\}]^k)$.

Then put $N = \{n_i : i \in \mathbb{N}\}$ and let

$$A = \bigcup_{\ell=1}^{\infty} A_{n_{\ell}}.$$

Note that for $A \in [N]^k$

$$A \in A \iff \exists \ell \in \mathbb{N} \quad A \in A_{n_{\ell}} \iff \forall \ell \in \mathbb{N}, \max A \leq n_{\ell} \quad A \in A_{n_{\ell}},$$

and, thus, for $\ell \in \mathbb{N}$ it follows that $A \cap [n_1, \ldots, n_{\ell}]^k = A_{n_{\ell}}$.

By Theorem 1.1.1 we can now choose an $L \in [N]^\omega$ so that $[L]^k \subset A$ or $[L]^k \cap A$. Write $L$ as

$$L = \{n_{\ell_1}, n_{\ell_2}, \ldots\},$$

then note that either

$$[\{n_{\ell_1}, n_{\ell_2}, \ldots, n_{\ell_m}\}]^k \subset A \cap [\{1, 2, \ldots, n_{\ell_m}\}]^k = A_{n_{\ell_m}},$$

or

$$[\{n_{\ell_1}, n_{\ell_2}, \ldots, n_{\ell_m}\}]^k \cap A = [\{n_{\ell_1}, n_{\ell_2}, \ldots, n_{\ell_m}\}]^k \cap A_{n_{\ell_m}} = \emptyset$$

which is a contradiction to our assumption. \qed

**Exercise 1.1.5.** For $m \in \mathbb{N}$ let $R(m)$ (sometimes also denoted by $R(2, m)$) be the minimum of all $n \in \mathbb{N}$ so that the conclusion of Corollary 1.1.3 holds, i.e so that for all $A \subset [\{1, 2, 3, \ldots, n\}]^k$ there exists $M \subset \{1, 2, 3, \ldots, n\}$ with $\#M = m$ so that either $[M]^k \subset A$ or $[M]^k \cap A = \emptyset$. Show that $R(3) = 6$.

**Remark.** $R(4) = 18$ (hard, possible only by using computer), $R(5)$ is unknown. It is only known that $R(5) \in [43, 49]$. 
1.2 Infinite Games

Let $X$ be a non empty set and let $\mathcal{A} \subset X^\omega$ we consider the following game $G(\mathcal{A}, X)$.

- Player I: chooses $x_1 \in X$
- Player II: chooses $x_2 \in X$
- Player I: chooses $x_3 \in X$
  
  ... 

Player I has won if the resulting sequence $(x_n)$ is in $\mathcal{A}$. Otherwise Player II is declared winner.

**Remark.** We will use above setup in the case where Player II chooses in his/her moves certain subsets, say $S_1, S_2$ etc. of $S$ (like infinite subsets of $\mathbb{N}$, or closed infinite dimensional subspaces of a Banach space $X$) and Player II chooses elements $s_i$ out of the set $S_i$ (for example $\mathbb{N}$, or a Banach space). Player I has won if he can assure that the resulting sequence $(s_i)$ lies in some set $\mathcal{A} \subset S^\omega$.

In order to reduce that game to the above described game we simply put

$$X = S \times \mathcal{P}, \text{ where } \mathcal{P} \subset \mathcal{P}(X) \text{ are the allowed choices for Player I}$$

$$\tilde{\mathcal{A}} = \{(s_i, S_i) \in X^\omega : (s_{2i}) \in \mathcal{A} \text{ and } \forall i \in \mathbb{N} \ s_{2i} \in S_{2i-1}\}.$$ 

**Exercise 1.2.1.** Assume $S$ is a set and $\mathcal{A} \subset S^\omega$ is closed with respect to the product of the discrete topology. Let $\mathcal{P} \subset \mathcal{P}(X)$

Then show that $\tilde{\mathcal{A}}$, as defined above is closed in the product topology of $(S \times \mathcal{P})^\omega$.

Let $\mathcal{A} \subset X^\omega$, $X$ some non empty set. For Player I having a winning strategy for $G(\mathcal{A}, X)$ means informally that

$$\exists x_1 \in X \ \forall x_2 \in X \ \exists x_3 \in X \ \forall x_4 \in X \ldots \ (x_i) \in \mathcal{A}.$$ 

Given that this is an infinite phrase one might want to be careful.

This is the formal definition:

**Definition 1.2.2.** Let $\mathcal{A} \subset [X]^\omega$, $X \neq \emptyset$. We say that *Player I has a winning strategy for $G(\mathcal{A}, X)$* if

$$(W_I(\mathcal{A}, X)) \text{ There is a sequence of functions } (f_n)_{n=0}^\infty \text{ with } f_n : X^n \rightarrow X \text{ (} f_0 \in X \text{) so that for any sequence } (z_n)_{n=1}^\infty \text{ the sequence } (x_n) \text{ defined by}$$

$$x_{2n-1} = f_{n-1}(z_1, z_2, \ldots z_{n-1}) \text{ and } x_{2n} = z_n \text{ whenever } n \in \mathbb{N}$$

is in $\mathcal{A}$.

We say that *Player II has a winning strategy for $G(\mathcal{A}, X)$* if

$$(W_{II}(\mathcal{A}, X)) \text{ There is a sequence of functions } (g_n)_{n=1}^\infty \text{ with } g_n : X^n \rightarrow X \text{ so that for any sequence } (z_n)_{n=1}^\infty \text{ the sequence } (x_n) \text{ defined by}$$

$$x_{2n-1} = z_n \text{ and } x_{2n} = f_n(z_1, z_2, \ldots z_n) \text{ whenever } n \in \mathbb{N}$$
1.2. INFINITE GAMES

is not in $A$.

In the case that $(W_I(A, X))$ or $(W_{II}(A, X))$ holds we call $(f_n)_{n=0}^\infty$, respectively $(g_n)_{n=1}^\infty$ a winning strategy for Player I, respectively Player II.

**Remark.** It is easy to see that $(W_I(A, X))$ and $(W_{II}(A, X))$ cannot both hold. Indeed, assuming that maps $f_n : X^n \rightarrow X$, $n \in \mathbb{N}_0$, and $g_n : X^n \rightarrow X$, $n \in \mathbb{N}$, existed as in $(W_I(A, X))$ and $(W_{II}(A, X))$, we could choose by induction a sequence $(x_n)$ as follows

$$x_1 = f_0 \text{ and } x_2 = g_1(x_1)$$

assuming $x_1, x_2, \ldots x_{2n}$ have been chosen for some $n \in \mathbb{N}$, we put

$$x_{2n+1} = f_n(x_2, x_4, \ldots, x_{2n}) \text{ and } x_{2n+2} = g_n(x_1, x_3, \ldots, x_{2n+1}).$$

Then $(x_n)$ satisfies the conditions in $(W_I(A, X))$ and as well as in $(W_{II}(A, X))$ which leads to a contradiction since that would imply that $(x_n) \in A$ as well as $(x_n) \not\in A$.

It is not clear whether either $(W_I(A, X))$ or $(W_{II}(A, X))$ have to hold, i.e. whether or not the game is determined.

In the case that the game has finitely many steps, i.e. if there is an $n \in \mathbb{N}$, say $n$ even, so that $A \subset X^n$ then $(W_I(A, X))$ is equivalent to

$$\exists x_1 \in X \ \forall x_2 \in X \ \exists x_3 \in X \ \forall x_4 \in X \ldots \exists x_{n-1} \in X \ \forall x_n \in X \ (x_i)_{i=1}^n \in A.$$ 

and $(W_{II}(A, X))$ is equivalent to

$$\forall x_1 \in X \ \exists x_2 \in X \ \forall x_3 \in X \ \forall x_2 \in X \ldots \forall x_{n-1} \in X \ \exists x_n \in X \ (x_i)_{i=1}^n \not\in A.$$ 

Since the negation can be pulled through finitely many quantifiers (replacing $\exists$ by $\forall$ and vice versa) we deduce that

$$\neg (W_I(A, X)) \Rightarrow (W_{II}(A, X)) \quad \text{and} \quad (W_{II}(A, X)) \Rightarrow (W_I(A, X))$$

and, thus, that games of finite length are determined. In the case of infinite games the situation is different: one cannot always pull the negation sign through an infinite phrase. A good analogue to this problem in Logic is the fact known from Calculus that infinite sums $\sum$ and $\int$ do not always commute. The determinacy of infinite games will follow under some topological assumption on the set $A$.

**Notation.** For $A \subset X^\omega$ and $x = (x_1, x_2, \ldots, x_n) \in X^{<\omega}$ we write:

$$A(x_1, x_2, \ldots, x_n) = \{(z_i) \in X^\omega : (x_1, x_2, \ldots, x_n, z_1, z_2, \ldots) \in A\}.$$ 

We consider on $X^\omega$ the product topology of the discrete topology on $X$. 
CHAPTER 1. THE THEOREM OF RAMSEY

Proposition 1.2.3. Assume \( A \subset X^\omega \). Then

\[
A \text{ is open } \iff \exists \tilde{A} \subset X^{<\omega} \quad A = \bigcup_{(x_1, x_2, \ldots, x_n) \in \tilde{A}} \{(x_1, x_2, \ldots, x_n)\} \times X^\omega.
\]

\[
A \text{ is closed } \iff \forall (x_i) \in X^\omega \left[ \forall n \in \mathbb{N} \quad A(x_1, x_2, \ldots, x_n) \neq \emptyset \Rightarrow (x_i) \in A \right].
\]

Theorem 1.2.4. [Ma] If \( A \subset X^\omega \) is a Borel set then the game \( G(A, X) \) is determined.

Proof. We will show Theorem 1.2.4 only in the case that \( A \) is either closed or open. The following observations are intuitively clear and can easily be shown using the formal definitions of \( (W_I(A, X)) \) and \( (W_{II}(A, X)) \). For \( A \subset X^\omega \)

(1.1) \( (W_I(A, X)) \iff \exists x \in X \forall y \in X \quad (W_I(A(x, y), X)) \)

(1.2) \( (W_{II}(A, X)) \iff \forall x \in X \exists y \in X \quad (W_{II}(A(x, y), X)) \)

Secondly,

(1.3) \( \neg(W_{II}(A, X)) \iff \exists x \in X \forall y \in X \quad \neg(W_{II}(A(x, y), X)) \)

\( \iff \exists x \in X \forall y \in X \quad A(x, y) \neq \emptyset \)

(1.4) \( \neg(W_I(A, X)) \iff \forall x \in X \exists y \in X \quad \neg(W_I(A(x, y), X)) \)

\( \iff \exists x \in X \forall y \in X \quad A(x, y) \neq \emptyset \)

Assume that \( \neg(W_{II}(A, X)) \) holds. By induction on \( n \in \mathbb{N}_0 \) we can choose \( f_n : X^n \rightarrow X \) so that

(1.5) \( \forall (z_1, z_2, \ldots, z_n, z_{n+1}) \in X^{n+1} \quad \neg(W_{II}(A(f_0, z_1, f_1(z_1), z_2, f_2(z_1, z_2), \ldots, f_n(z_1, 2, \ldots, z_n, z_{n+1}), X))) \)

For \( n = 0 \) this follows from the assumption \( \neg(W_{II}(A, X)) \). Assuming \( f_0, f_1, \ldots, f_{n-1} \) have been chosen, we can apply for a given \( (z_1, z_2, \ldots, z_n) \in X^n \) our induction hypothesis to the game

\( G(A(f_0, z_1, f_1(z_1), z_2, f_2(z_1, z_2), \ldots, f_n(z_1, 2, \ldots, z_n, z_{n+1}), X)) \)

and then apply (1.3) to the set

\( A(f_0, z_1, f_1(z_1), z_2, \ldots, f_{n-1}(z_1, z_2, \ldots, z_{n-1}), z_n) \)

in order to get a \( y \), which we define to be \( f_n(z_1, \ldots, z_n) \), so that,

\( (W_{II}(A(f_0, z_1, f_1(z_1), z_2, f_2(z_1, z_2), \ldots, f_n(z_1, z_2, \ldots, z_n, z_{n+1}), X))) \) for all \( z_{n+1} \in X \).

This finishes the induction step.
We claim that if $A$ is closed $(f_n)$ is a winning strategy for Player I. Indeed, let $(z_i) \in X^\omega$. From (1.3) (second implication) and (1.5) we deduce that for all $n \in \mathbb{N}$ the set $A(f_0, z_1, f_1(z_1), z_2, \ldots z_{n+1})$ is not empty. Proposition (1.2.3) yields that the infinite sequence $(f_0, z_1, f_1(z_1), z_2, f_2(z_1, z_2), z_3, \ldots)$ is in $A$.

If $A$ is open we will assume that $\neg(W_I(A, X))$ and prove in a similar way that $(W_{II}(A, X))$ (see Exercise 1.2.5)

Exercise 1.2.5. Let $A \subset X^\omega$ be open. Show that $\neg(W_I(A, X))$ implies $(W_{II}(A, X))$. 

\qed
1.3 Ramsey’s theorem for infinite sequences

**Proposition 1.3.1.** Let $X$ be a Polish space (completely metrizable and separable). For $A \subset X$ the following statements are equivalent.

- **a)** $A$ is the image of a Borel set $B \subset X \times X$ under the projection onto, say, the first coordinate.
- **b)** There is a Polish space $Y$ so that $A$ is the image of a Borel set $B \subset X \times Y$ under the projection onto the first coordinate.
- **c)** There is a continuous map $f : \mathbb{N}^\omega \to X$ (using the product of the discrete topology) whose image is $A$.
- **d)** $A$ is the image of a Borel set $B \subset X \times \mathbb{N}^\omega$ under the projection onto the first coordinate.
- **e)** For every uncountable Polish space $Y$ there is a $G_\delta$-set (intersection of countably many open sets) $B \in X \times Y$ so that $A$ is the image of $B$ under the projection onto $X$.

We call a set which satisfies above conditions analytic and we call the complement of an analytic set co-analytic.

**Remark.** The set of analytic subsets of a Polish space is not closed under complementation, in particular it is not a $\sigma$-algebra. Nevertheless it is easy to see that analytic sets are closed under countable unions.

We identify $\mathcal{P}(\mathbb{N})$ with the product $\{0, 1\}^\omega$ by

$$N \in [\mathbb{N}]^\omega \leftrightarrow (\varepsilon_n), \text{ where } \varepsilon_n = \begin{cases} 1 \text{ if } n \in N \\ 0 \text{ if } n \notin N. \end{cases}$$

Therefore $\mathcal{P}(\mathbb{N})$ can be endowed with the product topology of the discrete topology on $\{0, 1\}$, which makes it a compact metric, and thus, Polish space, and we note that

$$\mathcal{U} = \{\{F \cup N : N \in \mathcal{P}(\mathbb{N}), \max F < \min N \text{ or } N = \emptyset\} : F \in [\mathbb{N}]^{<\omega}\}$$

$$\equiv \{(\varepsilon_i) \subset \{0, 1\} : \varepsilon_i = 1 \text{ if } i \in F \text{ and } \varepsilon_i = 0 \text{ if } i \leq \max F \text{ and } i \notin F\}$$

is a basis of the topology on $\mathcal{P}(\mathbb{N})$.

For $N_n, n < \omega$, and $N$ in $\mathcal{P}(\mathbb{N})$,

$$N_n \to_{n \to \infty} N \iff \forall k \in \mathbb{N} \exists n_0 \in \mathbb{N} \forall n \geq n_0 \quad N_n \cap \{1, 2, 3 \ldots k\} = N \cap \{1, 2, 3 \ldots k\}.$$ 

On $[\mathbb{N}]^\omega$ we consider the relative topology. $[\mathbb{N}]^\omega$ is actually a dense $G_\delta$ set in $\mathcal{P}(\mathbb{N})$, and therefore it is a Polish space itself.
Exercise 1.3.2. For $B \in [N]<\omega$ and $N \in \mathcal{P}(N)$ we say that $N$ extends $B$ if

$$B = N \cap \{1, 2, \ldots, \max B\}.$$  

Show that $A \subset [N]^{\omega}$ is clopen if and only if there is a $k \in \mathbb{N}$ and a $B \subset [N]^k$ so that

$$A = \{N \in [N]^{\omega} : \exists B \in B \; \text{N extends } B\}.$$  

We can now state the Ramsey theorem for subsets of $[N]^{\omega}$.

Theorem 1.3.3. Assume that $A \subset [N]^{\omega}$ is analytic (with respect to the product topology on $[N]^{\omega} \equiv \{0, 1\}^{\omega}$).

Then there is an $N \in [N]^{\omega}$ so that either $[N]^{\omega} \subset A$ or $[N]^{\omega} \cap A = \emptyset$.

Remark. Note that from Exercise 1.3.2 it follows that Ramsey’s theorem for finite sets, i.e. Theorem 1.1.1, coincides with the statement of Theorem 1.3.3 for clopen sets $A \subset [N]^{\omega}$.

Theorem 1.3.3 was shown by Ellentuck in 1974 [El]. For Borel sets it was proved by Galvin and Prikry in 1973 [GP].

We will prove Theorem 1.3.3 first in the special case that $A \subset [N]^{\omega}$ is closed and then for all Borel sets.

Theorem 1.3.4. Assume $A \subset [N]^{\omega}$ is closed.

If there is an $N \in [N]^{\omega}$ so that $[M]^{\omega} \cap A \neq \emptyset$, for all $M \in [N]^{\omega}$, then there is an $N' \in [N]^{\omega}$ so that $[N']^{\omega} \subset A$.

It is clear that Theorem 1.3.4 implies Theorem 1.3.3 in the case that $A$ is closed. For the proof we will need the following Lemma. Similar to the previous section we define for $F \in [N]<\omega$ and $A \subset [N]^{\omega}$

$$A(F) = \{N \in [N]^{\omega} : \max F < \min N \text{ and } F \cup N \in A\}.$$ 

Lemma 1.3.5. Let $A \subset [N]^{\omega}$ and $N \in [N]^{\omega}$ and assume that for all $M \in [N]^{\omega}$ it follows that $A \cap [M]^{\omega} \neq \emptyset$.

Then for all $N' \in [N]^{\omega}$ there exists an $L \in [N']^{\omega}$ so that $A(\{\ell\}) \cap [M]^{\omega} \neq \emptyset$ for all $\ell \in L$ and all $M \in [L]^{\omega}$.

Proof. Assume our claim were not true for some $N' \in [N]^{\omega}$. Then we could inductively choose $L_0 = N' \supset L_1 \supset L_2 \ldots$ and $\ell_1 < \ell_2 < \ldots$ so that $\ell_k < \min L_k$, $\ell_k \in L_{k-1}$ and $A(\{\ell_k\}) \cap [L_k]^{\omega} = \emptyset$ for all $k \in \mathbb{N}$. But this would imply that $A \cap [M]^{\omega} = \emptyset$ for $M = \{\ell_1, \ell_2, \ldots\}$.

Proof of Theorem 1.3.4. Assume that that there is an $N \in [N]^{\omega}$ so that for all $M \in [N]^{\omega}$ there is an $L \in A \cap [M]^{\omega}$. We need to show that there is an $N' \in [N]^{\omega}$ so that $[N']^{\omega} \subset A$. We will chose recursively $N_0 = N \supset N_1 \supset N_2 \supset \ldots$ and
CHAPTER 1. THE THEOREM OF RAMSEY

$n_1 < n_2 < \ldots$ with $n_k \in N_{k-1}$, if $k \geq 1$, and so that for all $k \geq 0$, all $M \in [N_k]^{\omega}$ and all $F \subset \{n_1, \ldots, n_k\}$ (if $k = 0$ only $F = \emptyset$ is possible)

\[ A(F) \cap [M]^{\omega} \neq \emptyset. \]

For $k = 0$, this is exactly our assumption. Assume that we have chosen $n_1 < n_2 < \ldots < n_k$ and $N_k$. We order all subsets of $\{n_1, \ldots, n_k\}$ into $F_1, F_2, \ldots, F_{2^k}$ and apply Lemma 1.3.5 successively to $B = A(F_i), i = 1, 2, \ldots, 2^k$ in order to get $L_0 = N_k \supset L_1 \supset L_2 \ldots L_{2^k}$ so that for all $i \leq 2^k$ and all $\ell \in L_i$ and all $M \in [L_i]^{\omega}$ the set $(A(F_i))(\{\ell\}) \cap [M]^{\omega}$ is not empty. Finally we choose $n_{k+1} = \min L_{2^k}$ and $N_{k+1} = L_{2^k} \setminus \{n_{k+1}\}$, which finishes our induction step.

Putting now $N' = \{n_1, n_2, \ldots\}$ it follows that for all finite $F \subset N'$ the set $A(F)$ is not empty. Since $A$ is closed we deduce from Proposition 1.2.3 that $[N']^{\omega} \subset A$. \hfill \Box

Later we want to extend results similar to Theorems 1.3.3 and 1.3.4 to Banach spaces and their closed subspaces. Unfortunately only part of the reasonings are transferable. We want therefore reinterpret the statements of Theorems 1.3.3 and 1.3.4.

We consider the following two person game of infinite length. Let $A \subset [N]^{\omega}$ be closed.

Player I: chooses $N_1 \in [N]^{\omega}$

Player II: chooses $n_1 \in N_1$

Player I: chooses $N_2 \in [N]^{\omega}$

Player II: chooses $n_2 \in N_2$

\[ \vdots \]

Player I wins if $\{n_1, n_2, \ldots\} \in A$

It follows from Theorem 1.2.4 that if $A$ Borel, then the game is determined i.e. one of the Players has a winning strategy.

We call above game $G(A)$. If $N \in [N]^{\omega}$ and we demand that Player I can only choose subsets of $N$ we call it the restriction of the $G(A)$ to $N$ and denote it by $G(A, N)$.

Write $W_I(A, N)$ ($W_I(A)$ if $N = N$) if Player I has a winning strategy for the restriction of $A$ to $N$ and we write $W_{II}(A, N)$ ($W_{II}(A)$ if $N = N$) if Player II has a winning strategy (which means that $\{n_1, n_2, \ldots\} \notin A$).

Remark. Using the formal definition (or our intuition) of $W_I(A, N)$ and $W_{II}(A, N)$ for $N \in [N]^{\omega}$, it is clear that $W_I(A, N)$ implies $A \cap [N]^{\omega} \neq \emptyset$ and $W_{II}(A, N)$ implies $A^c \cap [N]^{\omega} \neq \emptyset$.

Therefore Theorem yields the following

Corollary 1.3.6. Let $A \subset [N]^{\omega}$ be closed.
1.3. RAMSEY’S THEOREM FOR INFINITE SEQUENCES

a) Either \( A \) satisfies the condition

\[
\exists N \in [N]^{\omega} \forall M \in [N]^{\omega} \ W_I(A, M),
\]

then it follows

\[
\exists N' \in [N]^{\omega} \quad [N']^{\omega} \subset A.
\]

b) Or \( A \) satisfies

\[
\forall N \in [N]^{\omega} \exists M \in [N]^{\omega} \ W_{II}(A, M)
\]

which implies

\[
\forall N \in [N]^{\omega} \exists M \in [N]^{\omega} \quad A \cap [M]^{\omega} = \emptyset.
\]

Remark. Note that the first part of Corollary 1.3.6 means the following: If Player I has a winning strategy he/she has actually a very simple strategy: at step \( k \) Player I chooses \( N \cap \{n_{k-1} + 1, n_{k-1} + 2, \ldots\} \), where \( n_{k-1} \) was the choice of Player II at step \( k - 1 \).

The second part can be interpreted as follows. If for all \( N \in [N]^{\omega} \) there is an \( M \) so that Player II has a winning strategy for the \((A, M)\)-game, then for any \( N \in [N]^{\omega} \) Player II can find a an \( M \in [N]^{\omega} \) so that in the \((A, M)\)-game he can each step choose any element out of the set Player I suggests, and win the game.

Proof of Corollary 1.3.6. (a) follows immediately from Theorem 1.3.4 and above mentioned remark.

(b) can be deduced as follows:

\[
\forall N \in [N]^{\omega} \exists M \in [N]^{\omega} \quad W_{II}(A, M) \quad \implies \quad \forall N \in [N]^{\omega} \exists M \in [N]^{\omega} \quad A^c \cap [M]^{\omega} \neq \emptyset
\]

\[
\iff \neg (\exists N \in [N]^{\omega} \quad [N]^{\omega} \subset A = \emptyset)
\]

\[
\implies \neg (\exists N \in [N]^{\omega} \forall M \in [N]^{\omega} \quad [M]^{\omega} \cap A \neq \emptyset)
\]

[Using Theorem 1.3.4]

\[
\iff \forall N \in [N]^{\omega} \exists M \in [N]^{\omega} \quad [M]^{\omega} \cap A = \emptyset.
\]

\( \square \)

We now want to show that the conclusion of Theorem 1.3.3 holds for all Borel sets in \([N]^{\omega}\).

Definition 1.3.7. We call a set \( A \subset [N]^{\omega} \) Ramsey if

\[
(R) \quad \exists M \in [N]^{\omega} \quad [M]^{\omega} \subset A \text{ or } [M]^{\omega} \cap A = \emptyset.
\]

We say that \( A \) is hereditary Ramsey

\[
(RH) \quad \forall N \in [N]^{\omega} \forall F \in [N]^{<\omega} \exists M \in [N]^{\omega} \quad [M]^{\omega} \subset A(F) \text{ or } [M]^{\omega} \cap A(F) = \emptyset.
\]
Remark. If \( \mathcal{A} \) is closed, \( N \in [N]^{\omega} \) and \( F \in [N]^{<\omega} \) then it follows that \( \mathcal{A}(F) \cap [N]^{\omega} \) is also closed in \( [N]^{\omega} \) and therefore it follows from Theorem 1.3.4 that \( \mathcal{A} \) is hereditary Ramsey.

**Theorem 1.3.8.** The set of all subsets of \( [N]^{\omega} \) which are hereditary Ramsey forms a \( \sigma \) algebra. In particular, using above remark, all Borel sets on \( [N]^{\omega} \) are hereditary Ramsey.

**Proof of Theorem 1.3.8.** Let \( \mathcal{R}_h \) be the subsets of \( [N]^{\omega} \) which are hereditary Ramsey. It is clear that \( \mathcal{R}_h \) is closed under taking complements. In order to show that it is closed under finite intersections let \( \mathcal{A} \) and \( \mathcal{B} \) be in \( \mathcal{R}_h \) and let \( N \in [N]^{\omega} \) and \( F \in [N]^{<\omega} \). Then, using (RH) for \( \mathcal{A} \), there is an \( M' \in [N]^{\omega} \) so that \( [M']^{\omega} \subseteq \mathcal{A}(F) \) or \( \mathcal{A}(F) \cap [M']^{\omega} = \emptyset \). Secondly we apply (RH) to \( \mathcal{B} \) and \( M' \) instead of \( N \) and find and \( M \in [M']^{\omega} \) so that \( [M]^{\omega} \subseteq \mathcal{B}(F) \) or \( \mathcal{B}(F) \cap [M]^{\omega} = \emptyset \). Note that \( M \) still satisfies the same alternative with respect to \( \mathcal{A}(F) \) as \( M' \) did. If \( M \) satisfies the first alternative with respect to \( \mathcal{A}(F) \) as well as \( \mathcal{B}(F) \) then, of course \( [M]^{\omega} \subseteq \mathcal{A}(F) \cap \mathcal{B}(F) = (\mathcal{A} \cap \mathcal{B})(F) \). In all the other cases we deduce \( [M]^{\omega} \cap (\mathcal{A} \cap \mathcal{B})(F) = \emptyset \). This shows that \( \mathcal{R}_h \) is an algebra and we deduce that in particular all open sets in \( [N]^{\omega} \) are hereditary Ramsey.

Therefore it is left to show that \( \mathcal{R}_h \) is closed under taking countable intersections.

Assume \( \mathcal{A}_n \in \mathcal{R}_h \), for \( n \in \mathbb{N} \), and let \( F \in [N]^{<\omega} \) and \( N \in [N]^{\omega} \). We need to show that there is an \( M \in [N]^{\omega} \) so that \( [M]^{\omega} \subseteq \bigcap_{n \in \mathbb{N}} \mathcal{A}_n(F) \) or \( [M]^{\omega} \cap \bigcap_{n \in \mathbb{N}} \mathcal{A}_n(F) = \emptyset \). We can assume that \( \mathcal{A}_1 \supset \mathcal{A}_2 \ldots \), otherwise replace \( \mathcal{A}_i \) by \( \bigcap_{j \leq i} \mathcal{A}_j \). We also can assume that \( F = \emptyset \), otherwise replace \( \mathcal{A}_i \) by \( \mathcal{A}_i(F) \).

Using our assumption that \( \mathcal{A}_n \in \mathcal{R}_h \) we can choose by induction \( k_1 < k_2 < \ldots \) and infinite sets \( N = L_0 \supset L_1 \supset L_2 \supset \ldots \) so that for all \( n \in \mathbb{N}, k_n = \min L_{n-1} \)

\[
\forall F \subseteq \{k_1, k_2, \ldots k_n\} \text{ either } [L_n]^{\omega} \subseteq \mathcal{A}_n(F) \text{ or } [L_n]^{\omega} \cap \mathcal{A}_n(F) = \emptyset.
\]

Define \( K = \{k_1, k_2, \ldots \} \). Since the \( \mathcal{A}_n \)'s are decreasing we deduce that for each \( F \in [K]^{<\omega} \), one and only one of the following alternatives can hold (let \( i \in \mathbb{N} \) so that \( k_i = \max F \)):

Either: (A1) \( \forall n \geq i \ [L_n]^{\omega} \subseteq \mathcal{A}_n(F) \)

Or: (A2) \( \exists n(F) \geq i \forall n \geq n(F) \ [L_n]^{\omega} \cap \mathcal{A}_n(F) = \emptyset \).

We pass to a further subsequence \( L = \{k_{\ell_1}, k_{\ell_2}, \ldots \} \) of \( K \) so that the following holds:

\[(1.6) \quad \forall n \in \mathbb{N}, \ \ell_{n+1} \geq 1 + \max \{n(F) : F \subseteq \{k_{\ell_1}, k_{\ell_2}, \ldots, k_{\ell_n}\}, F \text{ is (A2)}\} \]

We define \( \mathcal{F} \) to be all \( F \in [L]^{<\omega} \) for which (A1) holds and put

\[\mathcal{B} = \bigcup_{F \in [L]^{<\omega} \setminus \mathcal{F}} \{F \cup M : M \in [N]^{\omega} \text{ with } \min N > \max F\}.\]
1.3. RAMSEY'S THEOREM FOR INFINITE SEQUENCES

$\mathcal{B}$ is open in $[\mathbb{N}]^\omega$ and therefore we can choose an $M \in [L]^\omega$ so that either $[M]^\omega \subset \mathcal{B}$ or $[M]^\omega \cap \mathcal{B} = \emptyset$.

In the first case we can write any $M' \in [M]^\omega$ as

$$M' = \{k_{i(1)}, k_{i(2)}, \ldots\}, \text{ with } i(1) < i(2) < i(3) \ldots, \text{ in } \{\ell_1, \ell_2, \ldots\}$$

and since $M' \in \mathcal{B}$ we find an initial segment $F$ of $M'$, i.e. either $F = \emptyset$ or $F = \{k_{i(1)}, k_{i(2)}, \ldots, k_{i(n)}\}$ for some $n \in \mathbb{N}$, so that $F \not\in \mathcal{F}$ and, thus, by (1.6) $[L_{i(n+1)}]\cap A_{i(n+1)}(F) = \emptyset$ which implies (recall that $M' \setminus F \in [L_{i(n+1)}]$) that $M' \not\in \bigcap_n A_n$. Since $M' \in [M]^\omega$ was arbitrary we deduce that $[M]^\omega \cap \bigcap_n A_n = \emptyset$.

In the second case, let $M' \in [M]^\omega$ and write again $M'$ as

$$M' = \{k_{i(1)}, k_{i(2)}, \ldots\}, \text{ with } i(1) < i(2) < i(3) \ldots \text{ in } \{\ell_1, \ell_2, \ldots\}.$$  

Let $n \in \mathbb{N}$ be arbitrary. We want to show that $M' \in A_n$. In order to do so choose $j \in \mathbb{N}$ minimal so that $i(j) > n$, and let $F = \{k_{i(1)}, k_{i(2)}, \ldots, k_{i(j-1)}\}$ ($F$ could be empty). Since $M' \not\in \mathcal{B}$ every initial segment of $M'$ has to be in $\mathcal{F}$, in particular $F \in \mathcal{F}$, which implies that $M' \setminus F \in [L_n]^\omega \subset A_n(F)$, and thus $M' \in A_n$. Since $M' \in [M]^\omega$ and $n \in \mathbb{N}$ were arbitrary we deduce that $[M]^\omega \subset \bigcap A_n$. 

**Remark.** An example by Dan Freeman shows that the set $\mathcal{R}$ of subsets of $[\mathbb{N}]^\omega$ which are Ramsey is not closed under taking finitely many intersections:

Let $\mathcal{A} \subset [\mathbb{N}]^\omega$ which is not Ramsey (in Example we will show that such an $\mathcal{A}$ exists). Then take: $\mathcal{B}_1 = [2\mathbb{N}]^\omega \cup \mathcal{A}$ and $\mathcal{B}_2 = [2\mathbb{N} + 1]^\omega \cap \mathcal{A}$, then clearly $\mathcal{B}_1$ and $\mathcal{B}_2$ are Ramsey, but $\mathcal{A} = \mathcal{B}_1 \cap \mathcal{B}_2$ is not.

The Author of these notes does not no whether or not the set of all $\mathcal{A}$ which satisfies the following weak hereditary Ramsey property (RWH) is a $\sigma$-algebra. $\mathcal{A}$ is said to be weak hereditary Ramsey if

$$(\text{RWH}) \quad \forall N \in [\mathbb{N}]^\omega \exists M \in [M]^\omega \quad [M]^\omega \subset \mathcal{A} \text{ or } [M]^\omega \cap \mathcal{A} = \emptyset.$$ 

The next example shows that some conditions on $\mathcal{A} \subset [\mathbb{N}]^\omega$ are needed in order to derive the conclusion in Ramsey's Theorem for infinite sets. We will need tranfinte recursion to do so. We denote the cardinal number of the continuum by $\omega_c$ and we consider on $\omega_c$ an order $<$ which has the property that for $\alpha \in \omega_c$ the set $\{\beta < \alpha\}$ is of strictly smaller cardinality than $\omega_c$. Often instead of $\alpha \in \omega_c$ we write $\alpha < \omega_c$.

Simply extend the wellorder of $\omega_c$ to $\omega_c \cup \{\omega_c\}$ and put and exten the well order onto $\omega_c \cup \{\omega_c\}$ by demanding that $\alpha < \omega_c$ for all $\alpha \in \omega_c$.

**Remark.** Let $S$ be any set and let $\alpha$ be the cardinality of $S$. Then one can think of a well ordering of $S$ as being a bijective map $\alpha \to S$, or equivalently as a family $(s_\beta)_{\beta < \alpha}$ with $s_\beta \not= s_\gamma$ for $\beta < \gamma < \alpha$, and $S = \{s_\beta : \beta < \alpha\}$. 

Example 1.3.9. Let \((M_{\alpha})_{\alpha<\omega_c}\) be a well-ordering of \([\mathbb{N}]^\omega\). By transfinite induction we will choose for each \(\alpha < \omega_c\) a partition of \(M_\alpha\) into sets \(M^{(1)}_\alpha\) and \(M^{(2)}_\alpha\) so that

\[ M^{(1)}_\alpha, M^{(2)}_\alpha \notin \{M^{(1)}_\beta, M^{(2)}_\beta : \beta < \alpha\}. \]

Indeed, assuming we have chosen \(M^{(1)}_\beta\) and \(M^{(2)}_\beta\) for all \(\beta < \alpha\), it follows that the cardinality of \(\{M^{(1)}_\beta, M^{(2)}_\beta : \beta < \alpha\}\) is strictly less than \(\omega_c\), while the cardinality of all partitions of \(M_\alpha\) is \(\omega_c\).

Then we let

\[ A := \{M^{(1)}_\alpha : \alpha < \omega_c\}. \]

Let \(M \in [\mathbb{N}]^\omega\), say \(M = M_\alpha, \alpha < \omega_c\). Since, by choice, \(M^{(2)}_\alpha \notin A\) it follows that \([M]^\omega \notin A\). Since \(M^{(1)}_\alpha \in A\) it also follows that \([M]^\omega \cap A \neq \emptyset\).
Chapter 2

Application of Ramsey’s theorem to Banach spaces

2.1 Bases of Banach spaces

Convention: All of our Banach spaces are considered to be vector spaces over \( \mathbb{R} \).

Notation. If \( X \) is a Banach space with norm \( \| \cdot \| \).

\[
\begin{align*}
B_X &= \{x \in X : \|x\| \leq 1\} \text{unit ball of } X \\
S_X &= \{x \in X : \|x\| = 1\} \text{sphere of } X \\
X^* &= \{f : X \to \mathbb{R} : f \text{ is linear and bounded}\} \text{dual space of } X
\end{align*}
\]

For \( A \subset X \) we denote the linear span of \( A \) by \( \text{span}(A) \) and the closed linear span by \( \text{span}(A) \).

If \( Y \) is a closed linear subspace of a Banach space \( X \) we write \( Y \hookrightarrow X \).
CHAPTER 2. APPLICATION OF RAMSEY’S THEOREM

Special spaces:

For $1 \leq p < \infty$
\[
\ell_p = \{ x = (\xi) \subset \mathbb{R} : \sum_{i=1}^{\infty} |\xi|^p < \infty \} \text{ with } \|x\|_p = \left( \sum_{i=1}^{\infty} |\xi|^p \right)^{1/p},
\]
\[
\ell_{\infty} = \{ x = (\xi) \subset \mathbb{R} : \|x\|_{\infty} = \sup_{n \in \mathbb{N}} |\xi_n| \},
\]
\[
c_0 = \{ x = (\xi) \in \ell_{\infty} : \lim_{n \to \infty} \xi_n = 0 \},
\]

For a measure space $(\omega, \Sigma, \mu)$
\[
L^p(\mu) = \{ f : \Omega \to \mathbb{R} \text{ measurable} \mid \int |f|^p d\mu < \infty \} \text{ with } \|f\|_p = \left( \int |f|^p d\mu \right)^{1/p},
\]
\[
L_{\infty}(\mu) = \{ f : \Omega \to \mathbb{R} \text{ mble} \mid |||f|||_{\text{ess sup}} = \sup\{ r > 0 : \mu(\{|f| \geq r\}) > 0 \} < \infty \},
\]

For a compact space $K$
\[
C(K) = \{ f : K \to \mathbb{R} \text{ continuous} \}, \text{ with } ||f||_{\infty} = \sup_{\xi \in K} |f(\xi)|.
\]

Definition 2.1.1. (Schauder basis)
Let $X$ be a Banach space over . A sequence $(x_i)$ is called (Schauder) basis of $X$ if for every $x \in X$ there is a unique sequence $(a_i) \subset \mathbb{R}$ so that $x = \sum_{i=1}^{\infty} a_i x_i$.

We call $(x_i) \subset X$ basic sequence if it is a basis of its closed linear span.

Proposition 2.1.2. For $(x_i) \subset X$ be a Schauder basis. Define for $n \in \mathbb{N}$
\[
x_n^* : X \to \mathbb{R}, \quad x = \sum a_i x_i \mapsto a_i
\]

(we call the $x_n^*$ the coordinate functionals with respect to $(x_n)$). and define
\[
P_n : X \to X, \quad x \mapsto \left( \sum_{i=1}^{n} x_i \otimes x_i^* \right)(x) = \sum_{i=1}^{n} x_i \otimes x_i^*(x).
\]

Then it follows

a) $x_n^* \in X^*$ for all $n \in \mathbb{N}$.

b) $C_b = \sup_{n \in \mathbb{N}} \|P_n\| < \infty$.

$C$ is called the basis constant of $(x_i)$. We call $(x_n)$ a monotone basis if $C = 1$.

Sketch of Proof: For $x = \sum_{i \in \mathbb{N}} a_i x_i \in X$ define
\[
|||x||| = \sup_{n \in \mathbb{N}} \left\| \sum_{i=1}^{n} a_i x_i \right\|.
\]
Using the Principle of Uniform Boundedness it follows that $||| \cdot |||$ is an equivalent norm on $X$.

We observe that
\[ \sup_{n \in \mathbb{N}} \| P_n \| < \infty = \sup_{x \in B_X} |||x||| < \infty. \]

The converse of Proposition 2.1.2:

**Proposition 2.1.3.** Assume that $(x_i) \subset X$ is linear independent an let $f_i$ the linear (but not necessarily bounded) coordinate functionals defined on the vector space $\text{span}(x_i : i \in \mathbb{N})$. For $n \in \mathbb{N}$ put
\[ P_n = \sum_{i=1}^{n} x_i \otimes f_i : \text{span}(x_i : i \in \mathbb{N}) \to \text{span}(x_i : i \leq n), \quad x \mapsto \sum_{i=1}^{n} x_i \otimes f_i(x). \]

If $P_n$ is a bounded operator for every $n$ and $C = \sup \|P_n\| < \infty$, then $(x_i)$ is a basic sequence.

**Remark.** Enflo [En] showed that there are separable infinite dimensional Banach spaces without a Schauder basis nevertheless we the following result is easy to show

**Proposition 2.1.4.** If $X$ is an infinite dimensional Banach space. Then $X$ contains a basic sequence $(x_n)$ and if $(y_n)$ is a given normalized weakly null sequence $(x_n)$ can be chosen to be a subsequence of a given weakly null sequence.

Moreover, in both cases, $(x_n)$ can be chosen so that the basis constant is arbitrary close to 1.

**Definition 2.1.5.** We denote the vector space of all sequences $(a_i) \subset c_00$ which eventually vanish by $c_00$. We denote its usual unit vector space-basis by $(e_i)$, i.e.
\[ e_i = (0, 0, \ldots, 0, 1, 0, 0, \ldots). \]

For $x = \sum a_i e_i \in c_00$ we call
\[ \text{supp}(x) = \{ i \in \mathbb{N} : a_i \neq 0 \} \]
the support of $x$.

**Remark.** Assume that $(x_i)$ is a basis of $X$ then it is easy to see that its normalization $(e_i)$ with $e_i = x_i/\|x_i\|$, for $i \in \mathbb{N}$, is also a basis. Let $e^*_i$ the coordinate functionals to $(e_i)$.

For $x = \sum a_i e_i \in X$ it follows that $\lim_{i \to \infty} a_i = 0$ an thus we can think of $X$ being the completion of the vector space $c_00$ under some norm $\| \cdot \|$.
**Definition 2.1.6.** Let \((x_n)\) be a basis of a Banach space \(X\). We call

\[
C_p = \sup_{m \leq n} \|P_n - P_m\| = \sup_{m \leq n} \left\{ \left\| \sum_{i=m+1}^{n} a_i x_i \right\| : x = \sum_{i=1}^{\infty} a_i x_i \in B_X \right\},
\]

the projection constant, and note that \(C_p \leq 2C_b\). We call \((x_n)\) **bimonotone** if \(C_p = 1\).

Note that by putting

\[
\|\|x\|\| = \sup_{m \leq n} \|(P_n - P_m)(x)\|,
\]

we can always renorm \(X\) equivalently so that \((x_i)\) becomes a a bimonotone basis.

**Definition 2.1.7.** Let \((x_n)\) and \((y_n)\) be two basic sequences and \(C \geq 1\). We say that \((x_n)\) **\(C\)-dominates** \((y_n)\) or that \((y_n)\) is **\(C\)-dominated by** \((x_n)\) if

\[
\left\| \sum_{i=1}^{n} a_i y_i \right\| \leq C \left\| \sum_{i=1}^{n} a_i x_i \right\| \text{ for all } (a_i) \in c_{00}.
\]

We say that \((x_n)\) and \((y_n)\) are **\(C\)-equivalent** if \((x_n)\) \(C\)-dominates \((y_n)\) and \((y_n)\) \(C\)-dominates \((x_n)\) i.e. if

\[
\frac{1}{C} \left\| \sum_{i=1}^{n} a_i x_i \right\| \leq \left\| \sum_{i=1}^{n} a_i y_i \right\| \leq C \left\| \sum_{i=1}^{n} a_i x_i \right\| \text{ for all } (a_i) \in c_{00}.
\]

**Proposition 2.1.8.** For a sequence \((z_n)\) in a Banach space \(Z\) and \(z \in Z\) the following are equivalent:

a) There is a \(z \in Z\) so that for all \(\varepsilon > 0\) there is a finite \(F \subset \mathbb{N}\) so that \(\|z - \sum_{n \in F'} z_n\| < \varepsilon\) whenever \(F' \subset \mathbb{N}\) is finite with \(F \subset F'\).

b) There is a \(z\) so that for every bijection \(\pi : \mathbb{N} \to \mathbb{N}\) the series \(\sum_{i=1}^{\infty} x_{\pi(i)}\) converges to \(z \in Z\).

c) For every \((\delta_i) \in \{-1, 1\}^\omega\) the series \(\sum_{i=1}^{\infty} \delta_i x_i\) converges.

d) For every \(N \subset [N]^\omega\) the series \(\sum_{i \in N} \delta_i x_i\) converges.

**Definition 2.1.9.** A basic sequence \((x_n)\) is called **unconditional basis** if for all \(x = \sum_{i=1}^{\infty} a_i x_i\) the series is unconditionally converging.

**Proposition 2.1.10.** Let \((x_n)\) be a linearly independent sequence in a Banach space \(X\) and \((x^*_n)\) its coordinate functionals defined on \(\text{span}(x_n)\). For \(A \subset \mathbb{N}\) finite define:

\[
P_A : \text{span}(x_i : i \in \mathbb{N}) \to \text{span}(x_i : i \in A) \ni \sum_{i \in A} a_i x_i \mapsto \sum_{i \in A} a_i x_i.
\]

For \(\sigma = (\sigma_i) i \in N \in \{-1, 1\}^\omega\) define:

\[
T_\sigma : \text{span}(x_i : i \in \mathbb{N}) \to \text{span}(x_i : i \in \mathbb{N}) \ni \sum_{i \in \mathbb{N}} a_i x_i \mapsto \sum_{i \in \mathbb{N}} \sigma_i a_i x_i.
\]

The following are equivalent:
2.1. BASES OF BANACH SPACES

23

a) \( (x_n) \) is an unconditional basic sequence,

b) \( C_u = \sup_{\sigma \in \{x_n\} : \sigma \in N \in \{-1, 1\}^\omega} \|T_\sigma\| < \infty \)

c) \( \tilde{C}_u = \sup_{A \in [N]^{\omega}} \|P_A\| = \sup_{A \in [N]^{\omega}} \sup_{\omega x_{\in \text{span}(x_i : i \in N)} \|x\| \leq 1} \|P_A(x)\| < \infty \)

We say that a basic sequence is \( c \)-unconditional if \( C_u \leq c \) and suppression \( c \)-unconditional if \( \tilde{C}_u \leq c \)

Exercise 2.1.11. Show that always \( \tilde{C}_u \leq C_u \leq 2 \tilde{C}_u \).

Definition 2.1.12. Let \( (x_n) \) be a basic sequence. A Block basis of \( (x_n) \) is a sequence \( (y_n) \subset X \setminus \{0\} \) of the form:

\[
y_n = \sum_{i=k_{n-1}+1}^{k_n} a_i x_i, \text{ with } 0 = k_0 < k_1 < k_2 < \ldots \text{ and } a_i \in \mathbb{R}, \text{ for } i = 1, 2, \ldots.
\]

A subspace spanned by a block basis is called block subspace.

Remark. Note that every normalized block basis of the unit vector basis of \( \ell_p \), \( 1 \leq p < \infty \) or \( c_0 \) is isometrically equivalent to the unit vector basis of \( \ell_p \).

M. Zippin [Z2] proved that a normalized basic sequence \( (x_n) \) which is equivalent to all of its normalized block bases must be equivalent to the unit vector basis of \( \ell_p \), for some \( 1 \leq p < \infty \), or \( c_0 \)

Exercise 2.1.13. Show that block bases of bases are basic sequences.

Proposition 2.1.14. Let \( X \) be an infinite dimensional Banach space with a basis \( (x_n) \) and \( Y \) an infinite dimensional subspace. Given any sequence \( (\varepsilon_n) \subset (0, 1) \) there is a normalized block sequence \( (y_n) \) of \( (x_n) \) and a sequence \( (\tilde{y}_n) \subset Y \) of so that \( \|y_n - \tilde{y}_n\| < \varepsilon_n \) for any \( n \in N \).

Exercise 2.1.15. If \( (e_i) \) is a basis of \( X \) and the sequence of \( (e_i^*) \) its coordinate functionals is also a basic sequence (but not necessarily a basis of \( X^* \) which could for example be non separable) and \( (e_i^*) \) is unconditional if and only if \( (e_i) \) is.

Definition 2.1.16. A basic sequence \( (e_i) \) is called shrinking if the sequence of its coordinate functionals \( (e_i^*) \) is a basis of \( X^* \), and \( (e_i) \) is called boundedly complete if the series \( \sum a_i e_i \) converges whenever \( \sup_{n \in N} \|\sum_{i=1}^{n} e_i\| \).

The proofs of the following results on shrinking and boundedly complete bases can be found in [FHHMPZ]

Proposition 2.1.17. For basis \( (e_i) \) of \( X \) the following are equivalent

a) \( (e_i) \) is shrinking
b) For any \( x^* \in X^* \) it follows that \( \lim_{n \to \infty} \|x^*|_{\text{span}(e_i : i \geq n)} = 0. \)

**Proposition 2.1.18.** For basis \((e_i)\) of \(X\) and \((e^*_i)\) be the coordinate functionals. Then following are equivalent

a) \((e_i)\) is boundedly complete

b) Let \( y = \text{span}(e^*_i : i \in \mathbb{N}) \) then the (canonical) map

\[
T : X \hookrightarrow Y^*, \quad x \mapsto \left[ y = \sum b_i e^*_i \mapsto \sum b_i e^*_i(x) \right],
\]

is an isometry onto \( Y^* \). In particular \( X \) is a dual space.

**Theorem 2.1.19.** ([Ja2], see also [FHHMPZ, Theorem 6.11]) Let \( X \) be a Banach space with basis \((e_i)\). Then \( X \) is reflexive if and only if \((e_i)\) is shrinking and boundedly complete.

**Remark.** Note that the unit vector basis of \( c_0 \) is shrinking but not boundedly complete, and that the unit vector basis of \( \ell_1 \) is boundedly complete but not shrinking.

**Definition 2.1.20.** Let \( X \) be a Banach space. A closed \( Y \hookrightarrow X \) is called complemented in \( X \) and we write \( Y \overset{c}{\hookrightarrow} X \) if there is a bounded projection from \( X \) onto \( Y \), i.e. a bounded map \( P : X \to Y \) with \( P(y) = y \) for all \( y \in Y \).

**Remark.** If \((e_i)\) is basis of a Banach space and \( Y \) be a subspace which is spanned by a subsequence of \((e_i)\) then \( Y \) does not need to be complemented in \( X \) (see Exercise 2.1.22 below). Nevertheless, if \((e_i)\) unconditional then it is easy to see that all closed subspaces spanned by subsequences are complemented.

**Exercise 2.1.21.** Show that every finite dimensional and every cofinite dimensional closed subspace of a Banach space \( X \) is complemented in \( X \).

**Exercise 2.1.22.** Let \( J \) be James’ space, and \((e_i)\) its standard shrinking basis (c.f [FHHMPZ, Page 185]) Then \( Y = \text{span}(e_i : i \in \mathbb{N}, \text{odd}) \) is not complemented in \( J \).

**Exercise 2.1.23.** For a closed subspace \( Y \) of \( X \) the following statements are equivalent.

a) \( Y \overset{c}{\hookrightarrow} X \)

b) There is a closed subspace \( Z \) of \( X \) so that \( X \) is linearly isomorphic to the topological sum

\[
Y \oplus Z = \{(y, z) : y \in Y, z \in Z\}
\]

with the product topology (which is induced by a norm, for example by \( \|(y, z)\| := \|y\| + \|z\| \)).
2.2 Spreading models

The theory of spreading models is due to Brunel and Sucheston in the 70’s [BS]. A consequence is that a normalized weakly null sequence admits a subsequence which is “asymptotically unconditional.”

Let us first state a Corollary of Theorem 1.1.1 which is often called the Ramsey Theorem for Analysts. It shows that Ramsey’s theorem can be seen as a generalization of the fact that in a compact metrizable space every sequence has a convergent subsequence.

**Theorem 2.2.1.** Let \((M,d)\) be a compact metric space, \(k \in \mathbb{N}\),

For any map \(F : [\mathbb{N}]^k \to M\) and any sequence \((\varepsilon_j)\) there is an \(N = \{n_1, n_2, \ldots\} \in [\mathbb{N}]^\omega\), with \(n_1 < n_2 < \ldots\), and an \(m_0 \in M\) so that

\[
d(F(n_{i_1}, n_{i_2}, \ldots n_{i_k}), m_0) < \varepsilon_j \text{ whenever } j \leq i_1 < i_2 < \ldots i_k.
\]

In particular,

\[
\lim_{i_1 \to \infty} \lim_{i_2 \to \infty} \cdots \lim_{i_k \to \infty} F(i_1, i_2, \ldots i_k) = m_0.
\]

Proof. For \(j \in \mathbb{N}\) choose a finite covering \((U^{(j)}_{\ell})_{\ell=1}^{L_j}\) of \(M\) so that, the diameter of each \(U^{(j)}_{\ell}\) does not exceed \(\varepsilon_j / 2\), and so that for any \(j \in \mathbb{N}, j \geq 2,\) and \(1 \leq \ell \leq L_j\) there is an \(1 \leq \ell' \leq L_{j-1}\) so that \(U^{(j)}_{\ell} \subset U^{(j-1)}_{\ell'}\).

Choose a sequence \(N = N_0 \supset N_1 \supset N_2 \ldots\) by induction as follows. If \(j \in \mathbb{N}\) and \(N_{j-1}\) has been chosen, let \((\xi^{(j)}_{\ell})_{\ell=1}^{L_{j}}\) be finite \(\varepsilon_j\)-net of \(M\), then apply Corollary 1.1.3 to the sets \(\mathcal{A}_\ell \subset [N_{j-1}]^\omega\), where

\[
\mathcal{A}_\ell = \{(n_1, n_2, \ldots n_k) \subset [N_{j-1}]^k : F(n_1, n_2, \ldots n_k) \in U^{(j)}_{\ell}, \text{ for } \ell = 1, 2, \ldots L_j\}
\]

in order to get an infinite \(N_j \subset N_{j-1}\) and an \(\ell_j\) so that \([N_j]^k \subset \mathcal{A}_{\ell_j}\).

It follows that \(m_0 \in \bigcap_{\ell} U^{(j)}_{\ell}\) is unique and that the claim follows if we let \(N\) be a diagonal sequence of the \(N_j\)'s. \(\square\)

**Theorem 2.2.2.** Let \((x_n)\) be a normalized basic sequence in \(X\) and let \(\varepsilon_n \downarrow 0\). There exists a subsequence \((y_n)\) of \((x_n)\) and a normalized basis \((e_n)\) for a Banach space \(E\) with the following property. If \(n \in \mathbb{N}\) and \((a_i)_{i=1}^{n} \in [-1,1]^n\) then

\[
(2.1) \quad \left\| \sum_{i=1}^{n} a_i y_k \right\| - \left\| \sum_{i=1}^{n} a_i e_i \right\| < \varepsilon_n \text{ whenever } n \leq k_1 < \ldots < k_n
\]

**Exercise 2.2.3.** Proof Theorem 2.2.2

Hint: Use the fact that

\[
\mathcal{M}_n = \left\{ \| \cdot \| : \mathbb{R}^n \to [0, \infty) : \| \cdot \| \text{ is a norm on } \mathbb{R}^n \text{ and } \|e_1\| = \|e_2\| = \ldots \|e_n\| = 1 \right\},
\]
with the metric
\[ d_n(\| \cdot \|_1, \| \cdot \|_2) = \sup_{\xi \in [-1,1]^n} \left| \|\xi\|_1 - \|\xi\|_2 \right|, \]
is a compact space.

**Definition 2.2.4.** Let \((x_n)\) be a basic normalized sequence in Banach space \(X\). A Banach space \(E\) with a basis \((e_i)\) and norm denoted by \(\| \cdot \|_E\) is called the spreading model of \((x_n)\) if for every \(k \in \mathbb{N}\) and every \((a_i)_{i=1}^k \in \mathbb{R}^k\)

\[ \left\| \sum_{i=1}^k a_i e_i \right\|_E = \lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \ldots \lim_{n_k \to \infty} \left\| \sum_{i=1}^k a_i x_{n_i} \right\|. \]

**Remark.** Note that Theorem 2.2.2 implies that every basic sequence in a Banach space has a subsequence which has a spreading model, and that, moreover, one can pass to a further subsequence for which the stronger statement, more quantitative, (2.1) holds.

**Theorem 2.2.5.** Assume the space \(E\) with a normalized basis \((e_i)\) is the spreading model of a normalized weakly null sequence \((x_n)\) in the Banach space \(X\).

Then \((e_i)\) is suppression-1-unconditional.

We will first prove the following

**Lemma 2.2.6.** Let \((x_i)\) be a weak null sequence.

Then for any \(\varepsilon > 0\), \(m \in \mathbb{N}\) there exists an \(n \in \mathbb{N}\), \(n > m\) so that

\[ \forall x^* \in B_{X^*} \exists i \in (m, n] \quad |x^*(x_i)| < \varepsilon. \]

**Proof.** Assume our claim were not true then we could choose for any \(n \in \mathbb{N}\) an \(x^*_n \in B_{X^*}\) so that \(|x^*_n(x_i)| \geq \varepsilon\) for all \(i \in (m, n]\). We can pass to a subsequence \(x^*_{n_k}\) which converges in \(w^*\) to some \(x^* \in B_{X^*}\). This means that for all \(n \in (m, \infty)\)

\[ |x^*(x_n)| = \lim_{k \to \infty} |x^*_{n_k}(x_n)| \geq \varepsilon. \]

But this contradicts the assumption that \((x_n)\) is weakly null. \(\square\)

Iterating Lemma 2.2.6 yields

**Corollary 2.2.7.** If \((x_n)\) is a weakly null sequence in \(X\) and \((\varepsilon_\ell) \subset (0, 1)\) there is an increasing subsequence \((n_\ell) \subset \mathbb{N}\) so that for all for all \(\ell\) (put \(n_0 = 0\))

\[ \forall x^* \in B_{X^*} \exists i \in (n_{\ell-1}, n_\ell] \quad |x^*(x_i)| < \varepsilon_{\ell}. \]
Proof of Theorem 2.2.5. Let \((\varepsilon_n) \subset (0, 1)\) decrease to 0. According to Theorem 2.2.2 we can assume that

After passing to a subsequence of \((x_n)\) we can assume that

$$\left(\sum_{i=1}^{k} a_i x_{n_i} - \sum_{i=1}^{k} a_i e_i \right) < \varepsilon_k$$

whenever \(k \in \mathbb{N}\), \((a_i) \subset [-1, 1]^n\) and \(k \leq n_1 < n_2 < n_k\).

Let \(k \in \mathbb{N}\), \((a_i)_{i=1}^{k} \in [-1, 1]\), and \(i_0 \in \{1, 2 \ldots n\}\). It is enough to show that

$$\left\| \sum_{i=1, i \neq i_0}^{k} a_i e_i \right\| \leq \left\| \sum_{i=1}^{k} a_i e_i \right\|.$$

Choose \((n_i)\) like in Corollary 2.2.7, then we deduce for any \(m \in \mathbb{N}\)

$$\left\| \sum_{i=1, i \neq i_0}^{k} a_i e_i \right\| \leq \varepsilon_m + \left\| \sum_{i=1, i \neq i_0}^{k} a_i x_{\ell_i} \right\|$$

[By (2.4)]

where \(\ell_i \in (n_{m+i-1}, n_{m+i}]\) for all \(i \in \{1, 2 \ldots k\} \setminus \{i_0\}\)

$$= \varepsilon_m + \sum_{i=1, i \neq i_0}^{k} a_i x^* (x_{\ell_i})$$

[For appropriate \(x^* \in S_{X^*}\) by Hahn Banach]

$$\leq 2\varepsilon_m + \sum_{i=1}^{k} a_i x^* (x_{\ell_i})$$

[For appropriate \(\ell_{i_0} \in (n_{m+i_0-1}, n_{m+i_0}]\), by Corollary 2.2.7]

$$\leq 2\varepsilon_m + \left\| \sum_{i=1}^{k} a_i x_{\ell_i} \right\|$$

$$\leq 3\varepsilon_m + \left\| \sum_{i=1}^{k} a_i e_i \right\|$$

[By (2.4)]

The following example shows that the spreading model can be quite different from the underlying basis.
Example 2.2.8. The Schreier space

Define first the Schreier sets $S_1 \subset [\mathbb{N}]^{<\omega}$ as
$$S_1 = \{ E \in [\mathbb{N}]^{<\omega} : \# E \leq \min E \}.$$  

For example $\{3, 7, 1000\} \in S_1$ but $\{3, 7, 17, 20\} \notin S_1$. Then define for $x = (x_i) \in c_{00}$:
$$\|x\|_S = \max_{E \in S_1} \sum_{i \in E} |x_i|.$$  

Let $S$ be the completion of $c_{00}$ under $\| \cdot \|_S$. It follows for any $k \in \mathbb{N}$ and any $(a_i)_{i=1}^k$ that
$$\lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \ldots \lim_{n_k \to \infty} \left\| \sum_{i=1}^k a_i e_{n_i} \right\|_S = \sum_{i=1}^k |a_i|.$$  

Thus, the spreading model of $(e_i)$ in $S$ is the $\ell_1$ unit vector basis.

On the other hand $S$ is hereditarily $c_0$ (see next exercise).

Exercise 2.2.9. Show that every block subspace of $S$ has a further block which is equivalent to the $c_0$ unit vector basis.

Hint: let $(x_i)$ be a normalized block in $S$, say:
$$x_i = \sum_{j=n_{i-1}+1}^{n_i} a_j^{(i)} e_j, \text{ with } 0 = n_0 < n_1 < n_2 < \ldots.$$  

Then we distinguish between two cases:

If we have
$$\lim \inf_{i \to \infty} \|x_i\|_\infty = \lim \inf_{i \to \infty} \max_{j \in (n_{i-1}, n_i]} |a_j^{(i)}| = 0.$$  

then one can find a subsequence which is $(1 + \varepsilon)$-equivalent to the $c_0$-unit basis for any $\varepsilon > 0$.

Otherwise one can find a normalized block of $x_i$ for which the first case applies.

Remark. Rosenthal’s $\ell_1$ theorem (Theorem 2.3.1 below) yields that every $X$ contains either an isomorph of $\ell_1$ or a normalized weakly null sequence. Thus every $X$ always admits an unconditional spreading model. There are a number of known results and open problems concerning spreading models. They take on one of two forms. Given $X$, perhaps with certain properties, what can one say about the class of spreading models $S(X)$ of $X$. Or given specific information about $S(X)$ what can one say about $X$? For example it can be shown that $X$ is not reflexive if $X$ admits a spreading model that is not unconditional. On the other hand care must be taken since there exists a reflexive space $T$ (Tsirelson’s space) all of whose spreading models are isomorphic to $\ell_1$. Other known results are as follows. If $X \subseteq L_p$ ($1 < p < \infty$) and all spreading
models of $X$ are isomorphic to $\ell_p$ then $X$ embeds into $\ell_p$. There exists a space $X$ so that no spreading model contains an isomorph of $c_0$ or $\ell_p$ ($1 \leq p < \infty$). It is not known if for every $X$ there exists $n$ and a chain of successive spreading models $X_1, \ldots, X_n$ with $X_n$ isomorphic to $c_0$ or $\ell_p$ for some $1 \leq p < \infty$. In [S2] it was shown that if $S(X)$ admits two quite different spreading models of weakly null sequences then one can construct a "nontrivial" operator $T : W \subseteq X \to X$ ($T \neq I + K$, where $K$ is a compact operator, and $I$ the inclusion operator). Recent results and more questions on spreading models can be found in [AOST].

Let us mention some interesting problems. We denote the set of all spreading models generated by normalized weakly null sequences in $X$ by $SP_w(X)$. We identify sequences in $SP_w(X)$ which are equivalent.

We say $SP_w(X)$ is stabilized if $SP_w(Y) = SP_w(X)$ for all infinite-dimensional subspaces $Y$ of $X$.

**Question.** Does every reflexive Banach space $X$ have an infinite-dimensional subspace $Z$ for which $SP_w(Z)$ is stabilized?

It is easy to see that if $SP_w(X)$ is countable, then the answer is positive. It is also not hard to see that one can always stabilize with respect to cardinality, to wit, there is an infinite-dimensional subspace $Z$ such that $\text{card} \ SP_w(Z) = \text{card} \ SP_w(Y)$ for every infinite-dimensional subspace $Y$ of $Z$. But very little is known about which values $\text{card} \ SP_w(Z)$ can assume in this case.

**Question.** Assume that $SP_w(X)$ is stabilized with respect to cardinality. What are the possible values for $\text{card} \ SP_w(X)$?

In the case in which $\text{card} \ SP_w(X) = 1$ the following question is open.

**Question.** Assume that $\text{card} \ SP_w(X) = 1$, i.e. that all spreading models are equivalent. Does it follow that the spreading models are equivalent to the $\ell_p$ unit-vector basis for some $p$, $1 \leq p < \infty$, or to the unit basis in $c_0$?

**Remark.** In [AOST] it shown show that the last question has a positive answer provided there is a constant $C \geq 1$ so that all spreading models are $C$-equivalent.

Of course $\text{card} \ SP_w(\ell_p) = 1$. For the space $S$ constructed by the author in [S1] we find that $\text{card} \ SP_w(S) = \omega_c$, the cardinality of the continuum. No other possible values for $\text{card} \ SP_w(X)$ are known (assuming of course that $\text{card} \ SP_w(X)$ is stabilized). On the other hand, it is not known if any cardinality between 1 and $\omega_c$ can be excluded from being the stabilized cardinality of $SP_w(X)$ for some reflexive space $X$. For example, it is not known if there is a reflexive space $X$ for which $SP_w(X)$ is stabilized and for which $\text{card} \ SP_w(X) = 2$.

We finish the section with an example by Maurey and Rosenthal [MR] exhibits a weakly null basic sequence none of whose subsequences are unconditional. It is one
of the building stones towards constructing a Banach space which does not have a subspace with unconditional basis Gowers and Maurey [GM1].

We start with an exercise

**Exercise 2.2.10.** In $c_0$ let $s_n = \sum_{i=1}^{n} e_i = (1, 1, \ldots, 1, 0, 0, \ldots)$. Show that $(s_n)$ is a monotone basis for $c_0$ which is *conditional* (i.e., not unconditional).

$(s_n)$ is called the *summing basis* for $c_0$.

**Example 2.2.11.** [MR] We will construct a certain family $\mathcal{F} \subset c_0$ of real sequences and define a norm $\| \cdot \|_{\mathcal{F}}$ on $c_{00}$ by setting

$$\|x\|_{\mathcal{F}} = \sup\{|\langle f, x \rangle| : f \in \mathcal{F}\}$$

whenever $x \in c_{00}$, where we put for $f = (f_i) \in c_0$ and $x = (x_i) \in c_{00}$

$$\langle f, x \rangle = \sum_i x_if_i.$$ 

We then let $X_{\mathcal{F}}$ be the completion of $(c_{00}, \| \cdot \|_{\mathcal{F}})$. Note, for example, that $\ell^p$ could be defined this way by taking $\mathcal{F} = B_{\ell^q} \cap c_{00}$ where $\frac{1}{p} + \frac{1}{q} = 1$.

If $E, F \in [\mathbb{N}]^{<\omega}$ we write "$E < F$" if $\max E < \min F$. Let $(m_i)$ be a certain lacunary subsequence of $\mathbb{N}$ (we will specify what we mean later), with $m_1 = 1$. Let

$$\bar{F} = \{(F_i)_1^n : n \in \mathbb{N}, F_1 < F_2 < \ldots F_n \text{ and } F_i \in [\mathbb{N}]^{<\omega} \text{ for } i \in \mathbb{N}\}$$

Let $\phi : \bar{F} \to \{m_i\}_{i=1}^{\infty}$ be one-to-one ($\phi$ is called a coding). We define

$$\mathcal{F} = \left\{ f = \sum_{i=1}^{\infty} \frac{1_{E_i}}{\sqrt{\#E_i}} : E_1 < E_2 < \ldots, \#E_1 = 1 \text{ and } \#E_{i+1} = \phi(E_1, \ldots, E_i) \right\}$$

Note that if $\#E = m_i$, $\#F = m_j$ and, say $i < j$, then

$$\left\langle \frac{1_E}{\sqrt{\#E}}, \frac{1_F}{\sqrt{\#F}} \right\rangle \leq \frac{m_i}{\sqrt{m_i} \sqrt{m_j}} = \frac{\sqrt{m_i}}{\sqrt{m_j}} < \varepsilon_j,$$

where $\varepsilon_j \downarrow 0$ rapidly can be chosen in advance, which determines how $(m_i)$ is chosen.  

Claim 1: $(e_n)$ is a normalized monotone basis for $X_{\mathcal{F}}$. 

(Exercise) 

Claim 2: $(e_n)$ is weakly null.

We think of $\mathcal{F}$ being a subset of the compact space $[0,1]^\omega$ and can therefore think of $X$ being (isometrically) a subspace of $C(K)$ where $K = \overline{\mathcal{F}}$.

Then note that in fact

(2.5) 

$$K = \{ 1_{[1,n]}f : n \in \mathbb{N} \text{ and } f \in \mathcal{F} \} \cup \mathcal{F}.$$
Indeed, let \( f_k \in \mathcal{F} \) for \( k \in \mathbb{N} \), say
\[
f_k = \sum_{i=1}^{\infty} \frac{1_{E_i^{(k)}}}{\sqrt{|E_i|}}.
\]
We can assume also that \( f_k \neq f_{k'} \), for \( k \neq k' \) in \( \mathbb{N} \).

By passing to a subsequence, we can assume that there is an \( \ell \geq 1 \) so that
\[
E_i^{(k)} = E_i^{(k')} =: E_i \quad \text{whenever } i < \ell, \text{ and } k, k' \in \mathbb{N}
\]
\[
E_{\ell}^{(k)} \neq E_{\ell}^{(k')} \quad \text{whenever } k \neq k' \text{ are } \mathbb{N}.
\]
Secondly we can, after passing to a subsequence assume that \( E_{\ell}^{(k)} \) an be written as a union
\[
E_{\ell}^{(k)} = A \cup \tilde{E}_{\ell}^{(k)} , \text{ with } \min \tilde{E}_{\ell}^{(k)} \to \infty.
\]
and, thus, \( f_n \) converges pointwise to
\[
f = \sum_{i=1}^{\ell-1} \frac{1_{E_i}}{\sqrt{|E_i|}} + \frac{1_A}{\sqrt{\phi(E_1, E_2, \ldots E_{\ell-1})}}.
\]
which shows (2.5).

In order to finish the proof of claim 1, we simply observe that for any \( f \in K \) we have \( e_n(f) \to 0 \) for \( n \to \infty \) and recall the fact that in a \( C(K) \)-space, where \( K \) a compact, weak convergence and point wise convergence of sequences are equivalent.

Claim 3: Every subsequence of \( (e_i) \) has a block which is equivalent to the summing basis of \( c_0 \).

Let \( M \in [\mathbb{N}]^\omega \). Then there exists \( (E_i) \) so that \( f = \sum_1^{\infty} (1_{E_i}/\sqrt{|E_i|}) \in \mathcal{F} \) for all and \( \text{supp} f \subseteq M \).

We claim that \( (1_{E_i}/\sqrt{|E_i|})_{i=1}^{\infty} \subseteq X_\mathcal{F} \) is equivalent to the summing basis.

Indeed, it suffices to show that for \( (a_i)_1^n \subseteq \mathbb{R} \)
\[
\sup_{n \leq m} \left| \sum_{i=1}^{n} a_i \right| \leq \left| \sum_{i=1}^{m} a_i \frac{1_{E_i}}{\sqrt{|E_i|}} \right| \leq 3 \sup_{n \leq m} \left| \sum_{i=1}^{n} a_i \right|.
\]
The left hand estimate is easy. To see the right hand estimate let \( f = \sum_1^{\infty} 1_{E_i}/\sqrt{|E_i|} \in \mathcal{F} \).

Let \( i_0 = \max \{ i : F_i = E_i \} \). Then
\[
\left| \left( f, \sum_{i=1}^{m} a_i \frac{1_{E_i}}{\sqrt{|E_i|}} \right) \right| \leq \left| \sum_{i=1}^{i_0} a_i \right| + \left| \left( \frac{1_{F_{i_0+1}}}{\sqrt{|F_{i_0+1}|}}, \sum_{i=i_0+1}^{m} a_i \frac{1_{E_i}}{\sqrt{|E_i|}} \right) \right|
\]
\[
\quad + \sum_{j=i_0+2}^{\infty} \sum_{i=i_0+1}^{m} |a_i| \left( \frac{1_{F_j}}{\sqrt{|F_j|}}, \frac{1_{E_i}}{\sqrt{|E_i|}} \right).
\]
We estimate the second term as follows.

\[
| \left\langle \frac{1_{F_{i_0+1}}}{\sqrt{|F_{i_0+1}|}}, \sum_{i=1}^{m} a_i \frac{1_{E_i}}{\sqrt{|E_i|}} \right\rangle |
\]

\[
\leq |a_{i_0+1}| \left| \left\langle \frac{1_{F_{i_0+1}}}{\sqrt{|F_{i_0+1}|}}, \frac{1_{E_{i_0+1}}}{\sqrt{|E_{i_0+1}|}} \right\rangle \right| + \left| \left\langle \frac{1_{F_{i_0+1}}}{\sqrt{|F_{i_0+1}|}}, \sum_{i=i_0+2}^{m} a_i \frac{1_{E_i}}{\sqrt{|E_i|}} \right\rangle \right|
\]

\[
\leq |a_{i_0+1}| + \sum_{i=i_0+2} \left| a_i \right| \varepsilon_i.
\]

The third term can be estimated as follows

\[
\sum_{j=i_0+2}^{\infty} \sum_{i=i_0+1}^{m} |a_i| \left| \left\langle \frac{1_{F_i}}{\sqrt{|F_i|}}, \frac{1_{E_i}}{\sqrt{|E_i|}} \right\rangle \right|
\]

\[
\leq \sum_{j=i_0+1}^{\infty} \sum_{i=i_0+1}^{m} |a_i| \varepsilon_{\max(i,j)}
\]

[Note that for \( i \in \mathbb{N} \) \( \#E_i = m'_i \) for some \( i' \geq i \). Similar observation holds for \( \#F_j \). Also note that \( \#E_i \neq \#F_j \) whenever \( i = i_0 + 1, 2, \ldots \) and \( j = i_0 + 2, i_0 + 3, \ldots \)].

We deduce the claim, assuming we have chosen \( \varepsilon_i \) fast enough decreasing to 0.
2.3 Rosenthal’s $\ell_1$ theorem

One of the prettiest theorems in Banach space theory, Rosenthal’s $\ell_1$ theorem [Ro], can also be proved using Ramsey theory at a certain point. Let us place the theorem in context. In the search for a “nice” infinite dimensional subspace of a given Banach space $X$, one conjecture was: Every Banach space $X$ contains a subspace $Y$ which is either reflexive or isomorphic to $c_0$ or to $\ell_1$. (This was ultimately shown to be false by W.T. Gowers [Go1].) Now if $X$ is not reflexive then $B_X$ is not weakly compact and so by the Eberlein-Smulian theorem there exists $(x_n) \subseteq S_X$ with no weakly convergent subsequence. This splits into two cases:

(I) $(x_n)$ has no weak Cauchy subsequence: for all $(y_n) \subseteq (x_n)$ there exists $x^* \in X^*$ with $(x^*(y_n))_n$ being divergent.

(II) $(x_n)$ has a weak Cauchy subsequence $(y_n)$ but there is no $y \in X$ so that $x^*(y_n) \to x^*(y)$ for all $x^* (X$ is not weakly sequentially complete).

H. Rosenthal proved that (I) yields $\ell_1 \hookrightarrow X$.

**Theorem 2.3.1.** [Ro] Let $(x_n)$ be normalized with no weak Cauchy subsequence. Then there exists $(y_n) \subseteq (x_n)$ which is equivalent to the unit basis of $\ell_1$.

**Definition 2.3.2.** A sequence of pairs of sets $(A_n, B_n)$ is called **Boolean independent** if for any finite and disjoint subsets $I_1 \cap \mathbb{N}$ and $I_2 \cap \mathbb{N}$

$$\bigcap_{n \in I_1} A_n \cap \bigcap_{n \in I_2} B_n \neq \emptyset$$

The following Lemma is easy:

**Lemma 2.3.3.** Let $S$ be a set and let $f_n : S \to [-1, 1]$ and let $A_n = \{f_n = 1\}$, $B_n = \{f_n = -1\}$ then $(A_n, B_n)_{n=1}^\infty$ are Boolean independent.

Then $\|\sum_1^k a_i f_i\|_\infty = \sum_1^k |a_i|$ for all $k$ and $(a_i)_1^k \subseteq \mathbb{R}$, i.e. $f_i$ as sequence in $\ell_\infty(S)$ is isometrically equivalent to the $\ell_1$-unit vector basis.

The following is an isomorphic version of Lemma 2.3.3.

**Lemma 2.3.4.** Let $S$ be a set, $f_n : S \to [-1, 1]$, $r \in \mathbb{R}$, $\delta > 0$, $A_n = \{f_n > r + \delta\}$ and $B_n = \{f_n < r\}$. Assume that $(A_n, B_n)_{n=1}^\infty$ are Boolean independent. Then $(f_n) \subseteq \ell_\infty(S)$ is equivalent to the unit vector basis of $\ell_1$.

**Proof.** Let $(a_i)_1^k \subseteq \mathbb{R}$. It suffices to show that there exists $s \in S$ with

$$|\sum_1^k a_i f_i(s)| \geq \frac{\delta}{2} \sum_1^k |a_i|.$$
Let \( I_1 = \{ i : a_i \geq 0 \} \) and \( I_2 = \{ i : a_i < 0 \} \). Let \( s_1 \in \bigcap_{i \in I_1} A_i \cap \bigcap_{i \in I_2} B_i \) and \( s_2 \in \bigcap_{i \in I_2} A_i \cap \bigcap_{i \in I_1} B_i \). Then it follows that

\[
\sum_{i=1}^{k} a_i (f_i(s_1) - f_i(s_2)) \geq \delta \sum_{i=1}^{k} |a_i|,
\]

and thus,

\[
\max \left( \left| \sum_{i=1}^{k} a_i (f_i(s_1)) \right|, \left| \sum_{i=1}^{k} a_i (f_i(s_2)) \right| \right) \geq \frac{\delta}{2} \sum_{i=1}^{k} |a_i|.
\]

[Note that by choice of \( s_1 \) and \( s_2 \) it follows that \( a_i (f_i(s_1) - f(s_2)) > |a_i| \delta \) \( \square \)]

**Definition 2.3.5.** A sequence of pairs of disjoint sets \((A_n, B_n)\) is said to have no convergent subsequence if for all \( M \in [\mathbb{N}]^\omega \) there exists \( s \in S \) so that \( s \) belongs to infinitely many \( A_n \)'s, \( n \in M \) and to infinitely many \( B_n \)'s, \( n \in M \).

**Lemma 2.3.6.** Let \( A_n \cap B_n = \emptyset \), \( A_n, B_n \subseteq S \), for \( n \in \mathbb{N} \) and assume that \((A_n, B_n)\) has no convergent subsequence.

Then there is a subsequence of \((A_n, B_n)\) which is Boolean independent.

**Proof.** Let

\[
\mathcal{A} = \left\{ L = (\ell_i)_{i=1}^\infty \in [\mathbb{N}]^\omega : \bigcap_{i=1}^{k} A_{\ell_{2i-1}} \cap \bigcap_{i=1}^{k} B_{\ell_{2i}} \neq \emptyset \text{ for all } k \in \mathbb{N} \right\}.
\]

Since

\[
\mathcal{A} = \bigcap_{k \in \mathbb{N}} \left\{ L = (\ell_i)_{i=1}^\infty \in [\mathbb{N}]^\omega : \bigcap_{i=1}^{k} A_{\ell_{2i-1}} \cap \bigcap_{i=1}^{k} B_{\ell_{2i}} \neq \emptyset \text{ for all } k \in \mathbb{N} \text{ with } \ell_{2k} < \ell \right\},
\]

\( \mathcal{A} \) is closed in \([\mathbb{N}]^\omega \) and therefore we can apply Ramsey's Theorem 1.3.3. From the assumption that \((A_n, B_n)\) has no convergent subsequence it follows that there is an \( L \in [\mathbb{N}]^\omega \) with \([L]^\omega \subseteq \mathcal{A} \). Finally take \( M = (\ell_2, \ell_4, \ell_6, \ldots) \). \( \square \)

**Proof of Theorem 2.3.1.** Let \((x_n) \subseteq B_X\) have no w-convergent subsequence. Let \( S = B_X \) and regard \( x_n \) as functions in \( L_\infty(S) \).

For \( r \in [-1, 1] \) \( \delta > 0 \) and \( n \in \mathbb{N} \) we put

\[
A_n(r, \delta) = \{ s \in S : x_n(s) > r + \delta \} \quad \text{and} \quad B_n(r, \delta) = \{ s \in S : x_n(s) < r \}
\]

and claim that for some choice of \( r \in [-1, 1] \) and \( \delta > 0 \) there is an infinite \( M \) so that \((A_m(r, \delta), B_m(r, \delta))_{m \in M}\) has no convergent subsequence. Indeed, if this were not true, let \((r_i, \delta_i)_{i=1}^\infty\) be dense in \([-1, 1] \times (0, 1]\) and inductively choose \( M_{i+1} \subseteq [M_i]^\omega \) so that \((A_n(r_i, \delta_i), B_n(r_i, \delta_i))_{n \in M_i}\), “converges” (every \( s \) belongs to at most finitely many of the \( A_n \)'s or of the \( B_n \)'s). If \( M \) is a diagonal of the \( M_i \)'s then \((x_n)_M\) is point wise convergent on \( S \).

The claim of the Theorem follows now by applying Lemmas 2.3.3, 2.3.4 and 2.3.6. \( \square \)
2.4 Partial Unconditionality

Given a weakly null, normalized sequence in a Banach space $X$, can we pass to a subsequence that is a basic sequence and is in some sense close to being unconditional (recall that Example 2.2.11 shows that not every weakly null sequence may have an unconditional subsequence)?

There are various ways in which one can make this vague question precise, and in many situations one has a positive answer. There are important cases, however, for which the corresponding question is still open. In this section we want to present some of the known results.

The following result of John Elton uses the idea of preserving the positive $\ell_1$ mass, $\sum_I y^*(y_i)^+$, on an arbitrary set $I$ to obtain a weakened form of unconditionality.

**Theorem 2.4.1.** For each $\delta > 0$ there exists $K(\delta) < \infty$ with the following property. Let $(x_n)$ be a normalized weakly null sequence in a Banach space. There exists a subsequence $(y_n)$ of $(x_n)$ such that if $(a_i)_{i=1}^\infty \subseteq [-1,1]$ and $I \subseteq \{i : |a_i| \geq \delta\}$ then $\|\sum_I a_i y_i\| \leq K(\delta) \|\sum_i a_i y_i\|$.

**Remark.** It is an open problem whether or not $K$ in Theorem 2.4.1 could be chosen independently of $\delta > 0$. At first sight this might seem like trying to prove that every weakly null sequence has a subsequence which is unconditional (which is contradicted by Example 2.2.11). But note that we still would have to choose the subsequence $(y_n)$ depending on the given $\delta$. For example in [DKK] it was shown that $K \sim 3$ if we assume that $(x_i)$ does not have a spreading model which is equivalent to the $c_0$-unit vector basis. More about this problem as well as further partial answers can be found in [DOSZ].

**Proof.** We first note that we are trying to bound

$$\inf_{(y_i) \subseteq (x_i)} \sup_{\|\sum_I a_i y_i\| : |a_i| \leq 1 \text{ for all } i} \left\{ \|\sum_I a_i y_i\| : |a_i| \geq \delta \right\}$$

independently of the original sequence $(x_i)$. Let $x^* \in B_{X^*}$ with

$$x^*(\sum_I a_i y_i) = \|\sum_I a_i y_i\|$$

then split $I$ into 4 sets $I_{++}, I_{+-}, I_{-+}, I_{--}$ where for example

$$I_{+-} = \{i \in I : a_i > 0 \text{ and } x^*(y_i) \leq 0\}.$$

We see that for at least one of these sets, say $I_{+-}$, we have $|\sum_{I_{+-}} a_i x^*(y_i)| \geq \|\sum_I a_i y_i\|/4$. It follows (e.g. replace $x^*$ by $-x^*$ in this case) that we can find $x^* \in B_{X^*}$ so that

$$\|\sum_I a_i y_i\| \leq 4 \sum_I a_i^+ x^*(y_i)^+ = 4 \sum_{I_{++}} a_i x^*(y_i)$$
(where \( b^+ = b \lor 0 \) for \( b \in \mathbb{R} \)).

Let us suppose that, given \( \varepsilon > 0 \), we can find \((y_n) \subseteq (x_n)\) so that in this case there exists \( y^* \in B_{X^*} \) with

\[
\sum_{i \in I_*} y^*(y_i)^+ \geq \sum_{i \in I_*} x^*(y_i) - \varepsilon \quad \text{and} \quad \sum_{i : i \notin I_* \text{ or } y^*(y_i) < 0} |y^*(y_i)| \leq \varepsilon \sum_{i \in I_*} y^*(y_i)^+.
\]

Then we can easily derive our claim. Indeed, we then deduce that

\[
\left\| \sum a_i y_i \right\| \geq y^* \left( \sum a_i y_i \right) \\
\geq -\varepsilon \sum_{i \in I_*} y^*(y_i)^+ + \sum_{i \in I_*} a_i y^*(y_i)^+ \\
\geq (\delta - \varepsilon) \sum_{i \in I_*} y^*(y_i)^+ \\
\geq (\delta - \varepsilon) \sum_{i \in I_*} x^*(y_i) - \delta \varepsilon \\
\geq \frac{(\delta - \varepsilon)}{C} \sum_{i \in E} a_i x^*(y_i) - \delta \varepsilon
\]

[where \( C \) is the projection constant of \((y_i)\)]

\[
\geq \frac{(\delta - \varepsilon)}{C^4} \left\| \sum_{i \in I} a - iy_i \right\|.
\]

Since \((x_n)\) is weakly null we can assume, after dropping to a subsequence if necessary, that the basis constant of \((x_n)\) is close to 1, and, thus, projection constant is close to 2. We therefore obtain the result is with \( K(\delta) \lesssim 8/\delta \). Thus, it will suffice to prove the following.

\[\square\]

**Lemma 2.4.2.** Let \( B > 0 \) and \( \varepsilon > 0 \). There exists a subsequence \((y_i)\) of \((x_i)\) so that for all \( A \)

\[
\exists x^* \in B_{X^*} \text{ with } \sum_{j \in A} x^*(y_j)^+ \geq B \\
\implies \exists y^* \in B_{X^*} \text{ with } \sum_{j \in A} y^*(y_j)^+ \geq B - \varepsilon \text{ and } \sum_{j \notin A, y^*(y_j) < 0} |y^*(y_j)| < \varepsilon.
\]

**Proof.** Let \((\varepsilon_i) \subseteq (0,1)\) so that \( \sum \varepsilon_i < \varepsilon \). By induction we choose \( n_1 < n_2 < n_3 < \ldots \) and \( N = N_0 \supset N_1 \supset N_3 \supset \ldots \) so that for \( k \in \mathbb{N} \)

a) \( n_k = \min(N_{k-1}) \) and \( n_k < \min N_k \)

b) For all \( F \subseteq \{n_1, n_2, \ldots n_k\} \), all \( L = \{\ell_0, \ell_1, \ell_2, \ldots \} \subseteq N_k \) and all \( m \in \mathbb{N} \)
2.4. PARTIAL UNCONDITIONALITY

(\ast) \quad \exists x^* \in B_{X*} \text{ with } \sum_{j \in F \cup \{\ell_1, \ell_2, \ldots, \ell_m\}} x^*(x_j)^+ > B

\implies \quad \exists y^* \in B_{X*} \text{ with } \sum_{j \in F \cup \{\ell_1, \ell_2, \ldots, \ell_m\}} y^*(y_j)^+ > B - \varepsilon

|y^*(x_j)| < \varepsilon \text{ for } j = n_i \in \{n_1, n_2, \ldots, n_k\} \setminus F

|y^*(x_j)| < \varepsilon \text{ or } j = n_i \in F \text{ with } y^*(x_j) < 0

\text{and } |y^*(x_{\ell_0})| < \varepsilon_{k+1}

Assume that \( n_1 < n_2 < \ldots < n_{k-1} \) and \( \mathbb{N} = N_0 \supseteq N_1 \supseteq \ldots \supseteq N_{k-1} \) have been chosen. Let \( n_k = \min(N_{k-1}) \) and fix an \( F \subset \{n_1, n_2, \ldots, n_k\} \) and an \( N \in [N_{k-1}]^\omega \). Define

\( A_F = \{L = \{\ell_0, \ell_1, \ldots\} \in [N]^\omega : \forall m \in \mathbb{N}(\ast) \text{ holds }\} \).

Since

\( A_F = \bigcap_m A_{F,m}, \text{ where } A_{F,m} = \{L = \{\ell_0, \ell_1, \ldots\} \in [N_{k-1}]^\omega : (\ast) \text{ holds }\}, \)

and containance of an \( L \) in \( A_{F,m} \) only depends on the first \( m + 1 \) elements of \( L \) it follows that we can apply the Theorem of Ramsey 1.3.3.

We claim that we are in Case 1 of Theorem 1.3.3 (meaning that we can choose an \( L_F \in [N]^\omega \) so that \( [L]^\omega \subset A \)). Indeed, assume that there is an \( L \in [N]^\omega \) so that for all \( [L]^\omega \cap A = \emptyset \), say \( L = \{\ell_1, \ell_2, \ell_3, \ldots\}, \ell_1 < \ell_2 < \ell_3 < \).

Fix an \( n \in \mathbb{N} \) and define for \( i = 1, 2, \ldots \)

\( L^{(i)} = \{\ell_i, \ell_{n+1}, \ell_{n+2}, \ell_{n+3}, \ldots\}. \)

Since \( L^{(i)} \not\in A \) we can find for \( i = 1, 2, \ldots, n \), a number \( m_i \in \mathbb{N} \), and an \( x_i^* \in B_{X*} \) so that

\( \sum_{j \in F \cup \{\ell_{n+1}, \ell_{n+2}, \ldots, \ell_{n+m_i}\}} x_i^*(x_j)^+ > B \)

(2.7)

\( \forall y^* \in B_{X*} \text{ with } \sum_{j \in F \cup \{\ell_{n+1}, \ell_{n+2}, \ldots, \ell_{n+m_i}\}} y^*(y_j)^+ > B - \varepsilon \)

(2.8)

one of the following \textbf{does not hold}

\( |y^*(x_j)| < \varepsilon \text{ for } j = n_i \in \{n_1, n_2, \ldots, n_k\} \setminus F \)

\( |y^*(x_j)| < \varepsilon \text{ for } j = n_i \in F \text{ with } y^*(x_j) < 0 \)

\text{and } |y^*(x_{\ell_0})| < \varepsilon_{k+1}

For each \( i \in \{1, \ldots, n\} \) an \( y_i^* \in B_{X} \) we define

\( j(i) = \min\{0 \leq j \leq k : \forall j', j_n \in F \text{ and } x_i^*(x_j') > 0\}, \)

\( j(i) = k_{i;j} \quad (i) \quad k_{i;j} \)
and put $y_j^* = x_i^*$ if $j(i) = 0$. Otherwise we apply our induction hypothesis to $F' = \{1, 2, \ldots, n_{j(i)-1}\} \cap F$ and to $L' = \{n_{j(i)}, n_{j(i)+1}, \ldots, \ell_{n+1}, \ell_{n+2}, \ldots\} \subset N_{j(i)-1}$ to find a $y_j^*$ so that

$$(2.9) \quad \sum_{j \in F \cup \{\ell_{n+1}, \ell_{n+2}, \ldots, \ell_{n+m}\}} y_i^*(y_j)^* > B - \varepsilon$$

$$|y_i^*(x_j)| < \varepsilon_i \text{ for } j = n_i \in \{n_1, n_2, \ldots n_k\} \setminus F$$

$$|y_i^*(x_j)| < \varepsilon_i \text{ for } j = n_i \in F \text{ with } y_i^*(x_j) < 0.$$ 

Note that (2.9) is also satisfied if $j(i) = 0$. From (2.8) we deduce therefore for each $i = 1, 2 \ldots n$ that

$$|y_i^*(x_{\ell_i})| \geq \varepsilon_{k+1}.$$ 

Letting $m = \max_{i \leq n} m_i$ and choosing $i_0 \leq n$ so that $m_{i_0} = m$

$$(2.10) \quad \sum_{j \in F \cup \{\ell_{n+1}, \ell_{n+2}, \ldots, \ell_{n+m}\}} y_{i_0}^*(x_j)^* > B - \varepsilon$$

$$|y_{i_0}^*(x_j)| < \varepsilon_i \text{ for } j = n_i \in \{n_1, n_2, \ldots n_k\} \setminus F$$

$$|y_{i_0}^*(x_j)| < \varepsilon_i \text{ for } j = n_i \in F \text{ with } y^*(x_j) < 0.$$ 

which implies that for all $i \in \{1, 2 \ldots n\}$

$$(2.11) \quad |y_{i_0}^*(x_{\ell_i})| \geq \varepsilon_{k+1}.$$ 

We ended up proving the following: for any $n \in \mathbb{N}$ there is a $z_n^* \in B_{X^*}$ (namely $z_n^* = y_{i_0}^*$ from (2.11)) so that $z_n^*(x_{\ell_i}) \geq \varepsilon_{k+1}$ for all $i \leq n$. Let $z^*$ be an accumulation point of the sequence $(z_n^*)$ then it follows that $z^*(x_i) \geq \varepsilon_k$ for all $i \in \mathbb{N}$. But this is a contradiction of the assumption that $(x_i)$ is a weak null sequence, and finishes our claim.

Write $P(\{n_1, \ldots, n_k\})$ as $\{F_1, F_2, \ldots, F_{2^k}\}$ we can apply our proven claim to all $F = F_i$ successively and get $N_{k-1} \supset L_1 \supset L_2 \supset \ldots L_{2^k}$ so that $[L_i]^{\omega} \subset A_{F_i}$. Finally we chose $N_k = L_{2^k}$ which finishes our induction step.

The claim of our Lemma now follows if we take the subsequence $(y_i) = (x_{n_i})$. \qed

A second type of partial unconditionality is the following result due to E. Odell. We first need the following Definition

**Theorem 2.4.3.** [O2] (Schreier Unconditionality) Let $(x_i)$ be a normalized weakly null sequence in a Banach space $X$ and let $\varepsilon > 0$.

Then there exists a basic subsequence $(y_i)$ of $(x_i)$ which is $(2 + \varepsilon)$ Schreier unconditional, which means the projections:

$$P_A : \text{span}(y_i) \ni \sum a_iy_i \mapsto \sum_{i \in A} a_iy_i,$$

are of norm not exceeding $2 + \varepsilon$ provided $A \in S_1$. $S_1$ denotes the Schreier sets in $[N]^{<\omega}$ introduced in Example 2.2.8.
2.4. PARTIAL UNCONDITIONALITY

The proof is sketched in the following Exercise.

**Exercise 2.4.4.** Let \((x_i)\) be a normalized weakly null sequence in \(X\)

a) Let \(\varepsilon > 0\), \(k \in \mathbb{N}\) and \(I_1, I_2, \ldots I_k\) be subintervals of \([-1, 1]\). Show that there is a subsequence \((y_i)\) of \((x_i)\) so that

\[\ast\text{ If there exists an } x^* \in B_{X^*} \text{ so that } x^*(y_{m_i}) \in I_i, \text{ for } i = 1, 2 \ldots k, \text{ for some choice of } m_1 < m_2 < \ldots m_k \in \mathbb{N} \text{ then there is also a } y^* \in B_{X^*} \text{ so that } y^*(y_{m_i}) \in I_i, \text{ for } i = 1, 2 \ldots k, \text{ and } \sum_{j \in \mathbb{N} \setminus \{m_1, m_2, \ldots m_k\}} |y^*(y_{m_i})| < \varepsilon.\]

**Hint:** Let \((\varepsilon_m) \subset (0, 1)\), with \(\sum \varepsilon_m < \varepsilon\). By induction choose \(n_1 < n_2 < \ldots\) and \(N = N_0 \supset N_1 \supset N_2\) so that for all \(m \in \mathbb{N}\)

\[n_m = \min \mathbb{N}_{m-1}\] and

for all \(L = \{\ell_0, \ell_1, \ell_2 \ldots\} \subset N_m\), all \(k' \leq k\) and all \(F = \{f_1, f_2 \ldots f_{k'}\} \subset \{n_1, n_2, \ldots n_m\}\) the following implication holds:

\[\exists x^* \in B_{X^*} \]

\[\ast (x^*(x_{f_1}), x^*(x_{f_2}), \ldots x^*(x_{f_{k'}}), x^*(x_{\ell_1}), x^*(x_{\ell_2}) \ldots x^*(x_{\ell_{k-k'}})) \in I_1 \times I_2 \times \ldots I_k \]

\[\Rightarrow \exists y^* \in B_{X^*} \]

\[\ast (y^*(x_{f_1}), y^*(x_{f_2}), \ldots y^*(x_{f_{k'}}), y^*(x_{\ell_1}), y^*(x_{\ell_2}) \ldots y^*(x_{\ell_{k-k'}})) \in I_1 \times I_2 \times \ldots I_k \]

\[|y^*(x_{m_i})| < \varepsilon_i \text{ if } n_i \in \{n_1, n_2, \ldots n_m\} \setminus F \text{ and } |y^*(x_{\ell_0})| < \varepsilon_{m+1} \]

b) Let \(k \in \mathbb{N}\) and \(\varepsilon > 0\). Show that there is a subsequence \((y_i)\) of \((x_i)\) so that for all \((a_i) \subset c_{00}\) and all \(I \subset N\), with \(#I \leq k\):

\[\left\| \sum_{i \in I} a_i y_i \right\| \leq (1 + \varepsilon) \left\| \sum a_i y_i \right\|.\]

c) Prove Theorem 2.4.3.
Chapter 3

Distortion of Banach spaces

3.1 Introduction

We are interested in the following type of problem:

Let $X$ be an infinite dimensional and separable Banach space and let $(A_i)_{i=1}^r$ be an $r$-coloring of the sphere $S_X$. Can we choose an infinite dimensional subspace $Y$ so that $S_Y$ is monochromatic?

As stated above it is easy to find a counterexample:

**Example 3.1.1.** Let $X$ be a space with a normalized basis $(x_i)$ and $(x_i^*)$ the corresponding coordinate functionals. For each $x = \sum_{i=1}^{\infty} x_i^*(x) x_i \in S_X$ let $i_0(x) \in \mathbb{N}$ be the minimum of all $i$'s in $\mathbb{N}$ for which $|x_i^*(x)| = \max_{j \in \mathbb{N}} x_j^*(x)$.

Then let

$$A = \{ x \in S_X : x_{i_0(x)}^*(x) > 0 \} \quad \text{and} \quad B = \{ x \in S_X : x_{i_0(x)}^*(x) < 0 \}.$$ 

Then $X$ does not contain an infinite dimensional (actually not even a one dimensional) subspace which is monochromatic with respect to the coloring $(A, B)$.

As usual in Analysis, one has to allow arbitrary small perturbations in order to come to the right question.

**Problem 3.1.2.** Let $X$ be Banach space and let $(A_i)_{i=1}^r$ be a coloring of $S_X$. Does there exist for any $\varepsilon > 0$ an infinite dimensional subspace $Y$ and an $i \leq r$ so that

$$S_Y \subset (A_i)_\varepsilon := \{ x \in S_X : \text{dist}(x, A_i) < \varepsilon \}?$$

**Definition 3.1.3.** Let $f : S_X \to \mathbb{R}$. We say that $f$ stabilizes on infinite dimensional subspaces if for all infinite dimensional $Y \hookrightarrow X$ and $\varepsilon > 0$ there exists an infinite dimensional $Z \hookrightarrow Y$ so that

$$\text{osc}(f, S_Z) \equiv \sup \{ f(z_1) - f(z_2) : z_1, z_2 \in S_Z \} < \varepsilon.$$
Closely connected to the notion of stabilization is the notion of distortion of norms.

Definition 3.1.4. Let $X$ be an infinite dimensional and separable Banach space and let $\lambda > 1$. $X$ is $\lambda$-distortable if there exists an equivalent norm $\| \cdot \|$ on $X$ so that for all $Y \subseteq X$,

$$\sup \left\{ \frac{|y_1|}{|y_2|} : y_1, y_2 \in S_Y(\| \cdot \|) \right\} \geq \lambda.$$ 

In that case we call $\| \| \cdot \| \|$ a $\lambda$-distortion.

$X$ is distortable if $X$ is $\lambda$-distortable for some $\lambda > 1$. $X$ is arbitrarily distortable if $X$ is $\lambda$-distortable for all $\lambda > 1$.

Exercise 3.1.5. Let the Banach space $X$ have a basis $(x_n)$ and let $f : S_X \to \mathbb{R}$ be Lipschitz continuous.

Prove that $f$ stabilizes if and only if for every block subspace $Y$ of $X$ and for every $\varepsilon > 0$, there is a blocksubspace $Z$ of $Y$ so that $\text{osc}(f, S_Z) < \varepsilon$.

Hint: Proposition 2.1.14

Exercise 3.1.6. Define for $x = (\xi_i) \in \ell_2$

$$\|\|x\|\| = \left( \sum |\xi_i|^2 \right)^{1/2} + \max_{i \in \mathbb{N}} |\xi_i|.$$ 

Show that $\|\| \cdot \|\|$ is an equivalent norm on $\ell_2$, but not a distortion of the usual norm.

Proposition 3.1.7. For a separable Banach space $X$ the following are equivalent.

a) Every Lipschitz function $f : S_X \to \mathbb{R}$ stabilizes.

b) Every symmetric Lipschitz function $f : S_X \to \mathbb{R}$ ($f(x) = f(-x)$ for $x \in S_X$) stabilizes.

c) For any $r$-coloring $(A_i)_{i=1}^r$, any infinite dimensional $Y \hookrightarrow X$ and any $\varepsilon > 0$ there is an infinite dimensional $Z \hookrightarrow Y$ and an $i \leq r$ so that $S_Y \subseteq (A_i)_\varepsilon$.

d) For any covering $(A_i)_{i=1}^r$ with symmetric sets of $S_X$ any infinite dimensional $Y \hookrightarrow X$ and any $\varepsilon > 0$ there is an infinite dimensional $Z \hookrightarrow Y$ and an $i \leq r$ so that $S_Y \subseteq (A_i)_\varepsilon$.

Proof. (a)$\Rightarrow$(b) clear.

(b)$\Rightarrow$(a) Note that a Lipschitz function $f : S_X \to \mathbb{R}$ can be written as

$$f = \frac{1}{2} (f(x) + f(-x)) + \frac{1}{2} (f(x) - f(-x)).$$ 

If (b) holds then $\frac{1}{2} (f(x) + f(-x))$ as well $\frac{1}{2} (f(x) - f(-x))$ stabilizes. But $\frac{1}{2} (f(x) - f(-x))$ can only stabilize at 0, i.e. for all infinite dimensional subspaces $Y \hookrightarrow X$ and all $\varepsilon > 0$ there exists an infinite dimensional $Z \hookrightarrow Y$ so that $\frac{1}{2} (f(x) - f(-x)) < \varepsilon$ and
we can find a further infinite dimensional $U \hookrightarrow Z$ so that \( \text{osc}(\frac{1}{2}(f(x) + f(-x)), S_U) < \varepsilon \), which yields that

\[
\text{osc}(f, S_U) \leq \text{osc}(\frac{1}{2}(f(x) + f(-x)), S_U) + \text{osc}(\frac{1}{2}(f(x) - f(-x)), S_U) < 2\varepsilon.
\]

(a) $\Rightarrow$ (c) Let \((A_i)_{i=1}^r\), and apply (a) to the Lipschitz functions \(f_i(x) = \text{dist}(A_i, x)\) and note that \(\min f_i = 0\).

(c) $\Rightarrow$ (d) clear

(d) $\Rightarrow$ (b) Let \(f : S_X \rightarrow [0, 1]\) (range of \(f\) is bounded) be symmetric and Lipschitz and \(\varepsilon > 0\) and apply (d) to the sets

\[A_i = [i - 1]\varepsilon, i\varepsilon], \text{for } i = 1, 2, \ldots \left\lceil \frac{1}{\varepsilon} \right\rceil.\]

\[\Box\]

**Definition 3.1.8.** A Banach space is called *uniformly convex* if for all \(\varepsilon\) there exists a \(\delta = \delta(\varepsilon)\) so that

\[\forall x, y \in S_X \quad \|x - y\| \geq \varepsilon \Rightarrow \left\| \frac{x + y}{2} \right\| \leq 1 - \delta.\]

In that case we call

\[\delta_X : (0, 1) \rightarrow (0, 1), \quad \delta_X(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : x, y \in S_X, \|x - y\| \geq \varepsilon \right\},\]

the *modulus of uniform convexity of \(X\).*

**Lemma 3.1.9.** Assume that \(X\) is uniform convex. Then there exists for every \(0 < \varepsilon < 1\) a \(\eta = \eta(\varepsilon)\) so that for any \(x \in S_X\)

\[\text{dist}(x, \text{co}(B_X \setminus B_\varepsilon(x))) \geq \eta(\varepsilon),\]

where \(\text{co}(A)\) for \(A \subset X\) denotes the convex hull \(A\), i.e. the set

\[\text{co}(A) = \left\{ \sum_{i=1}^n \alpha_i x_i : x_i \in A, \alpha_i \geq 0, \sum \alpha_i = 1 \right\} = \bigcap \left\{ C \subset X : C \text{ convex and } A \subset C \right\}.\]

**Proof.** Assume our claim is not true for some \(\varepsilon > 0\). Then we can find sequences \((x_n)\) and \((z_n)\) with \(z_n \in \text{co}(B_\varepsilon(x_n)))\), for \(n \in \mathbb{N}\), so that \(\lim_{n \to \infty} \|x_n - z_n\| = 0\). Write each \(z_n\) as

\[z_n = \sum_{i=1}^{k_n} \alpha_i^{(n)} y_i^{(n)}, \text{ with } k_n \in \mathbb{N}, \alpha_i^{(n)} \geq 0, \sum_{i=1}^{k_n} \alpha_i^{(n)} = 1, \text{ and } y_i^{(n)} \in B_X \setminus B_\varepsilon(z),\]
and choose for \( n \in \mathbb{N} \), \( x_n^* \in S_{X^*} \) with \( x_n^*(x_n) = 1 \). It follows that

\[
\lim_{n \to \infty} \sum_{i=1}^{k_n} \alpha_i^{(n)} x_n^*(y_i^{(n)}) = 1,
\]

and thus we can choose for \( n \in \mathbb{N} \) an \( i_n \in \{1, 2, \ldots k_n\} \) so that

\[
\lim_{n \to \infty} x_n^*(y_{i_n}^{(n)}) = 1.
\]

But this is a contradiction to the assumed uniform convexity since on one hand \( \|x_n - y_{i_n}^{(n)}\| \geq \varepsilon \) for each \( n \in \mathbb{N} \) and on the other hand

\[
\limsup_{n \in \mathbb{N}} \|x_n + y_{i_n}^{(n)}\| \geq \limsup_{n \in \mathbb{N}} x_n^*(x_n + y_{i_n}^{(n)}) = 2.
\]

Proposition 3.1.10. If there is a Lipschitz function \( f \) on \( S_X \) which does not stabilize, then there is a distortion on an infinite dimensional subspace \( Y \) of \( X \).

Proof. By Proposition 3.1.7. there is a symmetric set \( A \subset S_X \), an infinite dimensional subspace \( Y \) of \( X \) and an \( 0 < \varepsilon < 1/4 \) so that for every infinite dimensional subspace \( Z \) of \( Y \)

(3.1) \[ A \cap S_Z \neq \emptyset \text{ and } S_Z \setminus A_{\varepsilon} \neq \emptyset. \]

Now define \( B \) to be the closed convex hull of \( A \cup \frac{1}{4} B_{X,\|\cdot\|} \) and let \( |||\cdot||| \) be the Minkowski functional with respect to \( ||\cdot|| \), i.e.

\[
|||x||| = \inf \{ r : x \in rB \} \text{ for } x \in X.
\]

Since \( A \) is symmetric it follows that \( |||\cdot||| \) is a norm and, since \( \frac{1}{4} B_{X,\|\cdot\|} \subset B \subset B_{X,\|\cdot\|} \) it follows for any \( x \in X \),

\[
||x|| \leq |||x||| \leq 4\|x\|.
\]

We claim that \( |||\cdot||| \) is a distortion on the space \( Y \). Indeed, let \( Z \hookrightarrow Y \) be infinite dimensional. By the first part of (3.1) there is an \( z_1 \in S_Z \cap A \), and thus \( |||z_1||| = \|z_1\| = 1 \). By the second part of (3.1) there is an \( z_2 \in S_Z \setminus A_{\varepsilon} \). It follows that (let \( \eta(\varepsilon) \) chosen as in Lemma)

\[
|||z_2||| = \inf \{ r : z_2 \in rB \} \text{ for } x \in X \geq \inf \{ r : z_2 \in r\text{co}(B_X \setminus B_{\varepsilon}(z_2)) \} \text{ for } x \in X \text{ [Since } A \cup \frac{1}{4} B_X \subset B_X \setminus B_{\varepsilon}(z_2) \text{]}
\]

\[
\geq 1 + \frac{1}{2} \eta(\varepsilon)
\]

[Otherwise \( z_2 \in (1 + \frac{1}{4} \eta(\varepsilon))\text{co}(B_X \setminus B_{\varepsilon}(z_2)) \)]

and, thus, \( \text{dist}(z_2, \text{co}(B_X \setminus B_{\varepsilon}(z_2))) < \eta(\varepsilon) \).

It follows that \( |||\cdot||| \) is a \((1 + \eta(\varepsilon))\) distortion. \(\square\)
Theorem 3.1.11. [Ja2] $\ell_1$ and $c_0$ are not distortable.

Proof. We first consider $\ell_1$ and denote the usual norm on $\ell_1$ by $\| \cdot \|$. Let $||| \cdot |||$ be an equivalent norm on $\ell_1$.

Put

$$r = \lim_{n \to \infty} r_n$$

with $r_n = \inf \{ |||x||| : x \in \text{span}(e_i : i \geq n) : \|x\| = 1 \}$ for $n \in \mathbb{N}$,

and let $0 < \varepsilon < r/10$. Then choose $n \in \mathbb{N}$ so that $r - \varepsilon/2 \leq r_n \leq r$ and then choose a block basis $(x_i)$ of $(e_i)_{i \geq n}$ which is normalized with respect to $\| \cdot \|$ and so that

$$r - \varepsilon \leq |||x_i||| \leq r + \varepsilon,$$

Then it follows for any $(a_i) \in c_0$ with $\sum |a_i| = 1$, by the triangular inequality

$$||| \sum a_i x_i ||| \leq \sum |a_i|||x_i||| \leq r + \varepsilon.$$ 

On the other hand

$$||| \sum a_i x_i ||| \geq \inf \{ |||x||| : x \in \text{span}(e_i : i \geq n) : \|x\| = 1 \} = r_n \geq r - \varepsilon/2,$$

thus for all $y_1, y_2 \in \text{span}(x_i : i \in \mathbb{N})$, $\|y_1\| = \|y_2\| = 1$ it follows that

$$\frac{|||y_1|||}{|||y_2|||} \leq \frac{r + \varepsilon}{r - \varepsilon/2},$$

which can be made arbitrary close to 1 providing we choose $\varepsilon > 0$ small enough.

Secondly we consider the space $c_0$ and denote its usual norm by $\| \cdot \|$.

Let $||| \cdot |||$ be an equivalent norm on $c_0$, define

$$r = \lim_{n \to \infty} r_n$$

with $r_n = \sup \{ |||x||| : x \in \text{span}(e_i : i \geq n) : \|x\| = 1 \}$ for $n \in \mathbb{N}$,

and let $\varepsilon > 0$.

Choose $n \in \mathbb{N}$ large enough so that $r \leq r_n \leq r + \varepsilon$, and choose $(x_i)$ to be a block sequence of $(e_i)_{i \geq n}$, which is normalized with respect to $\| \cdot \|$, so that

$$r - \varepsilon \leq |||x_i||| \leq r_n \leq r + \varepsilon.$$

Let $(a_i)_{c_0}$ with $1 = |a_{i_0}| = \max |a_i|$. Then

$$||| \sum a_i x_i ||| \leq r_n \leq r + \varepsilon = (r + \varepsilon) ||| a_i x_i |||.$$

On the other hand

$$||| \sum a_i x_i ||| + ||| \sum_{i \neq i_0} a_i x_i - a_{i_0} x_{i_0} ||| \geq 2 ||| x_{i_0} ||| \geq 2(r - \varepsilon),$$

and thus

$$||| \sum a_i x_i ||| \geq 2(r - \varepsilon) - \sum_{i \neq i_0} ||| a_i x_i - a_{i_0} x_{i_0} ||| \geq r - 3\varepsilon = (r - 3) ||| a_i x_i |||.$$ 

Since $\varepsilon > 0$ is arbitrary small
Exercise 3.1.12. Show the following finite version of James’ Theorem for \(\ell_1\).

Let \(n,k \in \mathbb{N}\) and \(C > 1\) and assume that \((f_i)_{i=1}^n\) is a normalized sequence which is \(C\)-equivalent to the unit vector basis of \(\ell_1^n\). Show that there is a normalized block of \((f_i)_{i=1}^n\) of length \(n\) which is \(C^{1/k}\)-equivalent to the unit vector basis of \(\ell_1^n\).

The following consequence of Dvoretzky’s Theorem is due to Milman and Schechtman.

**Theorem 3.1.13.** (see [MS, p.6]) For every \(\varepsilon > 0\) and any \(k \in \mathbb{N}\) there is an \(n = n(\varepsilon,k)\) so that if \(E\) is an \(n\) dimensional \(f : S_X \to \mathbb{R}\) is Lipschitz with constant 1, there is a \(k\) dimensional subspace \(F \hookrightarrow E\) so that \(\text{osc}(f,S_F) \leq \varepsilon\).

Let us finish this section by giving a short overview about the history of distortion:

- **(1964)** James [Ja2] showed that \(\ell_1\) and \(c_0\) are not distortable.
- **(1971)** At this time it was not yet known whether or not there existed an infinite dimensional Banach space which does not contain \(\ell_p\), for some \(1 \leq p < \infty\), or \(c_0\). Milman [M] showed: An infinite dimensional Banach space \(X\) which does not contain \(\ell_p\), for some \(1 \leq p < \infty\), must have a distortable infinite dimensional subspace.
- **(1974)** Tsirelson [T] constructed the first known space not containing \(\ell_p\) for some \(1 \leq p < \infty\), or \(c_0\).
- **(1974)** Figiel and Johnson [FJ] discrition of the dual of Tsirelson’s space and construction of a uniform convex space not containing \(\ell_p\) for some \(1 < p < \infty\).
- **(1990)** Haydon, Odell, Rosenthal and Schlumprecht (see [OS3]) proved the following refinement of Milman’s result: If \(X\) is a separable space not containing \(\ell_p\) or \(c_0\), then there exists a \(z \in Z\) so that the following norm \(|||\cdot|||_z\) distorts an infinite dimensional subspace:

\[
|||\cdot|||_z : X \to [0,\infty), \quad x \mapsto |||x|||_z := \|z\|_x + \|x\| + \|z\|_x - x.
\]

- **(1991)** Gowers [Go1] showed that all Lipschitz functions on the sphere of \(c_0\) stabilize.
- **(1991)** First arbitrarily distortable Banach space was constructed by author of these notes [S1]
- **(1993)** Gowers and Maurey: space without any unconditional basic sequence.
- **(1994)** By Odell and the author of these notes [OS1] it was shown that all \(\ell_p\)’s, \(1 < p < \infty\) are arbitrarily distortable.
- **(1995)** Maurey [Mau] showed that every asymptotic \(\ell_p\) space has an arbitrarily distortable subspace.

**Still Open:** Is Tsirelson space arbitrarily distortable? More generally, is every distortable space arbitrarily distortable?
3.2 Spaces not containing $\ell_p$ or $c_0$

In 1974 Tsirelson [T] constructed the first known Banach space which did not contain any subspace isomorphic to $\ell_p$, $1 \leq p < \infty$, or $c_0$ in this notes we will the dual of Tsirelson’s original construction, a space which constructed by Figiel and Johnson [FJ], and which does not contain copies $\ell_p$, $1 \leq p < \infty$, or $c_0$, either.

**Notation.** For $A, B \in [\mathbb{N}]^{<\omega}$ we will write $A < B$ if $\max A < \min B$. For $n \in \mathbb{N}$ we write $n \leq A$ if $n \leq \min A$.

We introduce the convention that $\min \emptyset = \infty$ and $\max \emptyset = 0$, which implies that $\emptyset < A$ and $A > \emptyset$ for any $A \in [\mathbb{N}]^{<\omega}$.

For $x = \sum_{i=1}^{\infty} a_i e_i \in c_0$ and $E \in [\mathbb{N}]^{<\omega}$ we write $E(x) = \sum_{i \in E} a_i e_i$.

**Definition 3.2.1.** Recall the Schreier sets

$$\mathcal{S}_1 := \{ A \subset [\mathbb{N}]^{<\omega} : \# A \leq \min A \}. $$

A finite sequence $(E_i)_{i=1}^{n} \subset [\mathbb{N}]^{<\omega}$ is called $\mathcal{S}_1$-admissible if $E_1 < E_2 < \ldots < E_n$ and if $\{ \min E_i : i = 1, 2, \ldots, n \} \in \mathcal{S}_1$, i.e. if $n \leq \min E_1$.

**Definition 3.2.2.** For each $n \in \mathbb{N}_0$ we define by induction a norm $|\cdot|_n$ on $c_0$.

$$|x|_0 = ||x||_\infty$$

for $x \in c_0$

and assuming $|\cdot|_{n-1}$ has been defined for some $n \in \mathbb{N}$

$$|x|_n = \max \left( |x|_{n-1}, \max_{(E_i)_{i=1}^{n} \text{S}_1-\text{admissible}} \frac{1}{2} \sum_{i=1}^{n} |E_i(x)|_{n-1} \right).$$

Note that for any $x \in c_0$ the sequence $(|x|_n)$ is increasing and must become constant, and we put

$$\|x\| = \max_n |x|_n.$$

$T$ (sometimes also denoted by $T_{1/2}$) is then the completion of under the norm $\| \cdot \|$.

First some easy observations.

**Proposition 3.2.3.** The unit vector basis is a 1-unconditional basis $(e_i)$ of $T$.

**Proposition 3.2.4.** The norm on $T$ satisfies the following implicit equation. For $x = (\xi_i) \in T$

$$\|x\| = \max \left( \|x\|_\infty, \sup_{n \in \mathbb{N}, (E_i)_{i=1}^{n} \text{S}_1-\text{admissible}} \frac{1}{2} \sum_{i=1}^{n} \|E_i(x)\| \right)$$

(if $x \in c_0$ then above “sup” can be replaced by “max”).
CHAPTER 3. DISTORTION OF BANACH SPACES

Proof. We only need to show the claim for $x \in c_{00}$ since for general $x \in T$ it follows by density of $c_{00}$ in $T$.

Let $x \in c_{00}$. If $\|x\| = \|x\|_\infty$ then clearly the right side of (3.2) is at least as large as the right side. Otherwise there is an $k \in \mathbb{N}$, and an $S - 1$-admissible collection $(E_i)_{i=1}^n \subset [\mathbb{N}]^{<\omega}$ so that

$$\|x\| = |x|_n = \frac{1}{2} \sum_{i=1}^n |E_i(x)|_{k-1} \leq \frac{1}{2} \sum_{i=1}^n \|E_i(x)\|.$$

To show the converse let $(E_i)_{i=1}^n$ be $S_1$-admissible. We can choose appropriate $k_1, k_2, \ldots k_n$ so that

$$\frac{1}{2} \sum_{i=1}^n \|E_i(x)\| = \frac{1}{2} \sum_{i=1}^n |E_i(x)|_{k_i} \leq \frac{1}{2} \sum_{i=1}^n |E_i(x)|_{\max_i \leq n(k_i)} \leq |x|_1 + \max_i \leq n(k_i) \leq \|x\|.$$

The following is an alternate way of constructing $T$, via normalising functionals

Proposition 3.2.5. Define by induction on $n = 0, 1 \ldots$ the following sets $A_n \subset c_{00}$:

$A_0 = \{ \pm e_i : i \in \mathbb{N} \}$,

and assuming $A_{n-1}$ has been defined for some $n \in \mathbb{N}$ we put

$$A_n = A_{n-1} \cup \left\{ \frac{1}{2} \sum_{i=1}^{k} f_i : k \in \mathbb{N}, f_i \in A_{n-1}, \text{ for } i \leq n, \text{ (supp}(f_i))_{i=1}^k \text{ is } S_1\text{-admissible} \right\}$$

Finally put $A = \bigcup A_i$. Let $\| \cdot \|$ be the norm on $T$ then

$$\|x\| = \sup_{f \in A} \langle f, x \rangle.$$

Proof. First we proof by induction that $A_n \subset B_{T^*}$. For $n = 0$ this is clear and, assuming $A_{n-1} \subset B_{T^*}$ and

$$f = \frac{1}{2} \sum_{i=1}^{k} f_i \in A_n,$$

then for $x \in T$

$$|\langle f, x \rangle| = \frac{1}{2} \sum_{i=1}^{k} \langle f_i, x \rangle \leq \frac{1}{2} \sum_{i=1}^{k} \|\text{supp}(f_i)(x)\| \leq \|x\|.$$

Thus, we showed for $x \in c_{00}$ that $\|x\| \geq \sup_{f \in A} |\langle f, x \rangle|$.
The proof of $\leq$ will be shown by induction on the cardinality of $\text{supp}(x)$. For $k = \# \text{supp}(x) = 1$ the claim is trivial.

If our claim is true for $k$ and if $x \in c_{00}$ with $\text{supp}(x) = k + 1$ then either $\|x\| = \|x\|_{\infty}$, in which case we are done. Otherwise we find sets $E_1, E_2, \ldots, E_n$ in $[\mathbb{N}]^{<\omega}$ with $n \leq E_1 < E_2 \ldots E_n$ so that

$$
\|x\| = \frac{1}{2} \sum_{i=1}^{n} \|E_i(x)\| \\
\leq \frac{1}{2} \sum_{i=1}^{n} f_i(x)
$$

[By induction for appropriate $f_1, \ldots, f_n \in A$, and, since $(e_i)$ is 1-unconditional basis we can assume that $n \leq \text{supp}(f_1) < \ldots \text{supp}(f_2)$]

$$
\leq f(x) \\
[\text{with } f = \frac{1}{2} \sum_{i=1}^{n} f_i].
$$

Proposition 3.2.6. For any normalized block basis $(x_i)$ in $T$ it follows that

$$
\left\| \sum_{i=1}^{n} x_i \right\| \geq \frac{n}{4}.
$$

In particular, $T$ cannot contain any copy of $\ell_p$, with $1 < p < \infty$, or $c_0$.

Theorem 3.2.7. For every $\lambda < 2$ the space $T$ is $\lambda$-distortable.

Remark. It is not known whether or not $T$ is $\lambda$-distortable for some $\lambda > 2$.

Proof. For $k \in \mathbb{N}$ define

$$
\| \cdot \|_k : T \to [0, \infty) : \quad x \mapsto \sup_{(E_i)_{i=1}^{k} \text{ is } S_1\text{-admissible}} \frac{1}{2} \sum_{i=1}^{k} \|E_i(x)\|.
$$

It is clear that

$$
(3.4) \quad \frac{1}{2} \|x\| \leq \|x\|_k \leq \|x\| \quad \text{and}
$$

$$
(3.5) \quad \|x\| = \max_{k \in \mathbb{N}} \left(\|x\|_{\infty}, \sup_{k \in \mathbb{N}} \|x\|_k\right) \quad \text{for all } k \in \mathbb{N} \text{ and } x \in c_{00}.
$$

We will show that $\| \cdot \|_k$ is a $\lambda_k$-distortion of $T$, with $\lim_{k \to \infty} \lambda_k = 2$. 
CHAPTER 3. DISTORTION OF BANACH SPACES

Step 1: \( \ell_1 \) is finitely block represented in every blocks pace of \( T \). This means that for any normalized block \( (x_i) \), any \( \varepsilon > 0 \) and any \( n \in \mathbb{N} \) there is a normalized block \( (y_i)_{i=1}^n \) of \( (x_i) \) so that \( (y_i)_{i=1}^n \) is \((1 + \varepsilon)\)-equivalent to the \( \ell_1^n \)-unit vector basis.

This follows from the fact that any block sequence \( (y_i)_{i=1}^\ell \), with \( \ell \leq \text{supp}(y_1) \), is 2-equivalent to the \( \ell_1^n \)-unit vector basis and then we can choose \( k \) so that \( 2^{1/k} < 1 + \varepsilon \), \( \ell = n^k \) and apply exercise 3.1.12.

For \( \varepsilon > 0 \) and \( n \in \mathbb{N} \) we will call \( z \in T \) an \( \varepsilon \)-\( \ell_1^n \)-average if

\[
z = \frac{1}{n} \sum_{i=1}^n z_i
\]

where \( (z_i) \) is a normalized block in \( T \) which is \((1 + \varepsilon)\)-equivalent to the \( \ell_1^n \) unit vector basis.

Note that in particular it follows for an \( \varepsilon \)-\( \ell_1^n \)-average \( z \) that \( \|z\| \geq \frac{1}{1 + \varepsilon} \).

Step 2: For \( n, k \in \mathbb{N} \) and any normalized block \( (z_i)_{i=1}^n \)

\[
\| \frac{1}{n} \sum_{i=1}^n z_i \|_k \leq \frac{1}{2} \frac{n+k}{n}
\]

Indeed write \( z = \frac{1}{n} \sum_{i=1}^n z_i \), let \( k \leq E_1 < E_2 < \ldots E_k \) in \( \mathbb{N}^{<\omega} \) so that

\[
\|z\|_k = \frac{1}{2} \sum_{j=1}^k \|E_j(z)\|
\]

For \( j = 1, 2, \ldots k \) put \( \ell_j = \max \{1 \leq i \leq n : \min \text{supp}(z_i) \leq \max E_j\} \) (putting \( \max \emptyset = 0 \)) and put \( \ell_0 = 1 \). Since \( (z_i)_{i=1}^n \) is a block basis it follows

\[
\{ i \in \{1, 2 \ldots n \} : \text{supp}(z_i) \cap E_j \neq \emptyset \} \subset [\ell_{j-1}, \ell_j],
\]

since the unit vector basis in \( T \) is 1-unconditional we deduce that

\[
\sum_{j=1}^k \|E_j(z)\| \leq \frac{1}{n} \sum_{j=1}^k \left\| \sum_{i=\ell_{j-1}}^{\ell_j} z_i \right\| \leq \frac{1}{n} \sum_{j=1}^k (\ell_j - \ell_{j-1} + 1) = \frac{n+k}{n},
\]

which implies our claim.

Step 3: Let \( \varepsilon > 0 \) and \( \ell \). We call a block \( (y_i)_{i=1}^\ell \) a \( \varepsilon \)-rapidly increasing sequence of length \( \ell \) (RIS) if for each \( i = 1, \ldots, \ell \) \( y_i \) is a \( \varepsilon \)-\( \ell_1^n \) average with \( (n_i) \subset \mathbb{N} \) satisfying:

\[
\ell < \varepsilon n_1
\]

\[
\max \text{supp}(y_{i-1}) < n_i \text{ and } n_{i-1} < \varepsilon n_i \text{ for } i = 2, 3, \ldots n \ell
\]

Let \( \varepsilon > 0 \) and let \( \ell \in \mathbb{N}, \ell \geq 1 \). If \( (y_i)_{i=1}^\ell \) is \( \varepsilon \)-rapidly increasing sequence of length \( \ell \) then

\[
\left\| \frac{1}{\ell} \sum_{i=1}^\ell y_i \right\| \leq \frac{2}{\ell} + \frac{1}{2} (1 + \varepsilon)
\]
Indeed let $k \in \mathbb{N}$ for $y = \frac{1}{\ell} \sum_{i=1}^{\ell} y_i \|y\| = \|y\|_k$ (note that $\|y\| \neq \|y\|_\infty$).

If $k \leq \varepsilon n_1$ we deduce from (3.6) that

$$\|y\|_k \leq \frac{1}{\ell} \sum_{i=1}^{\ell} \|y_i\|_k \leq \frac{1}{\ell} \sum_{i=1}^{\ell} \frac{1}{2} \frac{n_i + k}{n_i} \leq \frac{1}{2} \frac{n_1 + k}{n_1} \frac{1}{2}(1 + \varepsilon) \tag{3.10}$$

Choose $k \leq E_1 < E_2 < \ldots E_k$ so that

$$\|y\| = \|y\|_k = \frac{1}{2} \sum_{j=1}^{k} \|E_i(y)\|,$$

Put $n_{\ell + 1} := \infty$ and $n_0 := \lfloor \varepsilon n_1 \rfloor$, $n_{\ell + 1} := +\infty$ and choose $i_0 \in \{0, 1, 2, \ldots\}$ so that $k \in (n_{i_0}, n_{i_0+1})$. Then for it follows for $i = 1, 2, \ldots, i_0 - 1$ and $j = 1, 2, \ldots k$ (note that by (3.8) max(supp($x_i$)) $\leq n_{i_0} < k \leq E_1$) $E_j(y_i) = 0$. For $i = i_0 + 2, i_0 + 3, \ldots \ell$ it follows from (3.6) and (3.8) that

$$\|y_i\|_k \leq \frac{1}{2} \frac{n_i + k}{n_i} < \frac{1}{2} \frac{n_i + n_{i_0+1}}{n_i} \leq \frac{1}{2}(1 + \varepsilon),$$

and thus we deduce in this case that

$$\|y\|_k \leq \frac{1}{\ell} \sum_{i=i_0}^{\ell} \|y_i\|_k \leq \frac{1}{\ell} \left[ 2 + \sum_{i=i_0+2}^{\ell} \|y_i\|_k \right] \leq \frac{2}{\ell} + \frac{1}{2}(1 + \varepsilon),$$

which implies our claim of Step 3.

We will show that for $k \in \mathbb{N}$ the equivalent norm $\| \cdot \|$ is an $\frac{2k}{k+20}$-distortion, which will finish the proof of our theorem.

Let $Y$ be an arbitrary block subspace. If $n > k^2$ and if $x_1$ is an $\frac{1}{n-l_1^a}$ average we deduce for $z_1 = x_1/\|x_1\|$ that

$$\|z_1\|_k = \frac{1}{\|x_1\|} \|x\|_k \leq \frac{1}{2} \left( 1 + \frac{1}{n} \right) \frac{n + k}{n} \leq \frac{1}{2} \left( 1 + \frac{1}{k} \right)^2$$

If we let and choose a

$x_2 = \frac{1}{k} \sum y_i$ and $(y_i)$ is a $\frac{1}{k}$-rapidly increasing sequence of length $k$ and we put

$$z_2 = x_2/\|x_2\|$$

Then, obviously,

$$\|z_2\|_k \geq \frac{1}{\|x_2\|} \frac{1}{2k} \sum_{i=1}^{k} \|y_i\| \geq \frac{1}{2} \frac{1}{\|x_2\|} \frac{1}{1 + \frac{1}{k}},$$

and (3.9) of Step 3 yields that

$$\|x_2\| \leq \frac{2}{k} + \frac{1}{2} \left( 1 + \frac{1}{k} \right) = \frac{1}{2} \left( 1 + \frac{5}{k} \right).$$
Thus
\[ \|z_2\|_k \geq \frac{1}{1 + \frac{1}{k}} \frac{1}{1 + \frac{5}{k}}, \]
and
\[ \frac{\|z_2\|_k}{\|z_1\|_k} \geq 2 \frac{1}{(1 + \frac{1}{k})^2} \frac{1}{1 + \frac{5}{k}} > \frac{2k}{k + 20}. \]
\[ \square \]

Since by James’ theorem 3.1.11 \( \ell_1 \) is not distortable deduce the following

**Corollary 3.2.8.** \( T \) does not contain a copy of \( \ell_1 \).

**Remark.** In the definition (see Definition 3.2.2) of the norms \( |\cdot|_n \) one can replace replace \( \frac{1}{2} \) by some \( \gamma \in (0,1) \), and put \( \|\cdot\|_\gamma = \max |\cdot|_n \) and denote the completion of \( c_{00} \) with respect to \( \|\cdot\|_\gamma \) by \( T_\gamma \). Using the same arguments as in the proof of Theorem 3.2.7 one shows that \( T_\gamma \) is \( \frac{1}{\gamma} \)-distortable. The next construction is a space which is arbitrarily distortable.

**Definition 3.2.9.** [S1] For \( s \in [1, \infty) \) define \( f(s) = \log_2(s + 1) \). For \( n \in \mathbb{N}_0 \) and \( x \in c_{00} \) define by induction as follows.

\[ |x|_0 = \|x\|_\infty \]

and, assuming that \( |x|_{n-1} \) has been defined for all \( x \in c_{00} \) we put

\[ |x|_n = \max \left( |x|_{n-1}, \max_{E \in E_1 < E_2 < \ldots < E_\ell} \frac{1}{f(\ell)} \sum_{i=1}^\ell |E_i(x)|_{n-1} \right). \]

Then put for \( x \in c_{00} \)

\[ \|x\| = \sup_{n \in \mathbb{N}_0} |x|_0. \]

\( \|\cdot\| \) is a norm on \( c_{00} \) and we denote the completion of \( c_{00} \) with respect to \( \|\cdot\| \) by \( S \).

The following observations can be shown easily.

**Proposition 3.2.10.**  

a) The unit vector basis is a 1-subsymmetric basis of \( S \)

b) The norm \( \|\cdot\| \) on \( S \) satisfies the following implicit equation:

\( (3.11) \quad \|x\| = \max \left( \|x\|_\infty, \sup_{E \geq E_1 < E_2 < \ldots < E_\ell} \frac{1}{f(\ell)} \sum_{i=1}^\ell \|E_i(x)\| \right) \)

whenever \( x \in S \).
c) For every normalized block \((y_i)_{i=1}^\ell\) in \(S\) and any \((a_i)_{i=1}^\ell \subset \mathbb{R}\)
\[
\left\| \sum_{i=1}^\ell a_i y_i \right\| \geq \frac{1}{\ell} \sum_{i=1}^\ell |a_i|
\]
d) \(\ell_1\) is finitely block represented in every block subspace of \(S\).
(Use (c) and the finite version of James’ Theorem, Exercise 3.1.12).
d) The norm on \(S\) can also be obtained by describing its normalizing functionals in the following way.
Define \(A_0 = \{ \pm e_i : i \in \mathbb{N} \}\) and assuming \(A_{n-1}\) has been defined for some \(n \in \mathbb{N}\) we put
\[
A_n = \left\{ \frac{1}{f(\ell)} \sum_{i=1}^\ell f_i : \ell \in \mathbb{N}, (f_i)_{i=1}^\ell \subset A_{n-1} \text{ is block sequence} \right\} \cup A_{n-1}.
\]
Let \(A = \bigcup_{n=1}^\infty A_n\) Then it follows for \(x \in S\) that
\[
\|x\| = \sup_{f \in A} \langle f, x \rangle.
\]
Similar to the space \(T\) we define the following equivalent norms \(\| \cdot \|_\ell\) for \(\ell \in \mathbb{N}\).
For \(x \in S\) define
\[
(3.12) \quad \|x\|_\ell = \frac{1}{f(\ell)} \sup_{E_1 < E_2 < \ldots < E_\ell} \sum_{i=1}^\ell \|E_i(x)\|.
\]
Clearly,
\[
(3.13) \quad \frac{1}{f(\ell)} \|x\| \leq \|x\|_\ell \leq \| \cdot \| \quad \text{and}
\]
\[
(3.14) \quad \|x\| = \max \left( \|x\|_\infty, \sup_{\ell \in \mathbb{N}, \ell \geq 2} \|x\|_\ell \right)
\]

**Theorem 3.2.11.** For every \(\ell \in \mathbb{N}\) the norm \(\| \cdot \|_\ell\) is an af(\(\ell\))-distortion of \(S\) for some fixed constant \(c\)

**Proof.** For \(\varepsilon > 0\) and \(n \in \mathbb{N}\) use definition for \(\varepsilon-\ell^n_1\) averages introduced in Step 1 of the proof of Theorem 3.2.7 and note that by Proposition 3.2.10 (d) every block subspace of \(S\) contains for any \(\varepsilon > 0\) and \(n \in \mathbb{N}\) an \(\varepsilon-\ell^n_1\) average.

Following similar arguments one can show that if \(z\) is an \(\varepsilon-\ell^n\)-average in \(S\), and thus \(\|z\| \geq \frac{1}{1+\varepsilon}\), and \(\ell \in \mathbb{N}\) it follows that
\[
(3.15) \quad \|z\|_\ell \leq \frac{1}{f(\ell)} \frac{\ell}{n + \ell},
\]
which is close to \(\frac{1}{f(\ell)}\) if \(n >> \ell\).
In order to show that for any $\ell \in \mathbb{N}$ there are vectors in each block subspace for which the $\| \cdot \|_\ell$ and $\| \cdot \|$ norm are approximately the same we have to redefine what we mean by a rapidly increasing sequence in $S$.

We call a sequence $(y_i)_{i=1}^k$ a *rapidly increasing sequence (RIS)* if there are $\varepsilon_i > 0$ and $n_i \in \mathbb{N}$ for $i = 1, \ldots, k$ so that

\begin{align*}
(3.16) \quad & \sum_{i=1}^{k} \varepsilon_i < 1 \\
(3.17) \quad & y_i \text{ is an } \varepsilon_i - \ell_1^{n_i} \text{ average} \\
(3.18) \quad & n_1 > 2k \\
(3.19) \quad & f(n_i) \geq \max \text{supp}(y_{i-1}) \text{ and } \frac{n_{i-1}}{n_i} < \varepsilon_i \text{ for } i = 2, 3, 4, \ldots, k
\end{align*}

and we will show that there is a constant $C \geq 1$ so that if $(y_i)_{i=1}^k$ is a rapidly increasing sequence it follows that

\begin{equation}
(3.20) \quad \left\| \sum_{i=1}^{k} y_i \right\| \leq \frac{Ck}{f(k)}.
\end{equation}

Since clearly (use for the second $\leq$ inequality (3.15))

\begin{equation*}
\left\| \sum_{i=1}^{k} y_i \right\| \geq \frac{1}{f(k)} \sum_{i=1}^{k} \| y_i \| \geq \frac{k}{f(k)}
\end{equation*}

it follows together with (3.15) that $\| \cdot \|_k$ is a $\frac{f(k)}{C}$-distortion.

In order to prove (3.20) we first need a little Calculus lemma

**Lemma 3.2.12.** For $\ell \in \mathbb{N}$ and $D > \ell$ it follows that

\begin{equation*}
\max \left\{ \sum_{i=1}^{\ell} \frac{\xi_i}{f(\xi_i)} : \xi_i \geq 1 \text{ for }, i = 1, \ldots, \ell \text{ and } \sum_{i=1}^{\ell} \xi_i = D \right\} = \frac{D}{f(D\ell)}.
\end{equation*}

(*i.e. maximum is achieved when $\xi_1 = \xi_2 = \ldots = \xi_\ell = D/\ell$*)

**Proof.** $[1, \infty) \ni \xi \mapsto \frac{\xi}{f(\xi)}$ is convex.

**Continuation of Proof of Theorem 3.2.11.** We are choosing $C > 1$ large enough to satisfy the following conditions

\begin{align*}
(3.21) \quad & C \geq 5 \text{ and } C \geq 2 \frac{f(k)}{f(k/2)} \text{ if } k \geq 2 \text{ (note that } \lim_{k \to \infty} \frac{f(k)}{f(k/2)} = 1 \text{ )} \\
(3.22) \quad & C \geq 2 \frac{f(k/\ell)f(\ell)}{f(\ell)f(k/\ell) - f(k)} \text{ whenever } \ell, k \in \mathbb{N} \text{ with } 2 \leq \ell \leq k/2
\end{align*}
For the fulfilment of (3.22) note that for \( k \in \mathbb{N}, k \geq 4 \) that the function
\[
g_k : [2, k/2] \to \mathbb{R}, \quad x \mapsto \frac{f(k)}{f(x)f(k/x)},
\]
achieves its maximal value at each of the endpoints and note that
\[
\sup_{k \in \mathbb{N}, 4 \geq 4} g_k(2) = \sup_{k \in \mathbb{N}, 2 \leq \ell \leq k/2} \frac{f(k)}{f(\ell)f(k/\ell)} = \frac{4}{f(2)f(2)} < 1,
\]

Thus
\[
\sup_{k \in \mathbb{N}, 2 \leq \ell \leq k/2} \frac{f(k/\ell)f(\ell)}{f(\ell)f(k/\ell)} = \sup_{k \in \mathbb{N}, 2 \leq \ell \leq k/2} \frac{1}{1 - \frac{f(k)}{f(\ell)f(k/\ell)}} < \infty.
\]

We will show by induction on \( k \in \mathbb{N} \) that (3.15) hold, or equivalently that for a rapidly increasing sequence \((y_i)_{i=1}^k\) of length \( k \) and for all \( \ell \geq 2 \) we have
\[
(3.23) \qquad \left\| \sum_{i=1}^k y_i \right\|_\ell \leq \frac{kC}{f(k)}.
\]

For \( k = 1 \) the claim is trivial. Assume that the claim is true for all \( 1 \leq k' < k \), let \((y_i)_{i=1}^k\) be rapidly increasing sequence of length \( k \), put \( y = \sum_{i=1}^k y_i \) and let \( \ell \in \mathbb{N} \).

We consider three different cases.
Case 1: \( \ell \in [2, \frac{k}{2}] \) (in particular \( k \geq 4 \)). Choose \( E_1 < E_2 < \ldots E_\ell \) so that
\[
\| y \|_\ell = \frac{1}{f(\ell)} \sum_{j=1}^\ell \| E_j(y) \|.
\]

Using the 1-unconditionality of the unit vector basis of \((e_i)\) in \( S \) we can assume that \((E_j)_{j=1}^\ell\) is a partition of the the interval \([\min \text{supp}(y_i), \max \text{supp}(y_k)]\) into \( \ell \) intervals and choose for \( j = 1, 2, \ldots, \ell \) choose \( i(j) \in \{1, 2, \ldots, k\} \) such that
\[
\max E_j \in (\max \text{supp}y_{i(j)-1}, \max \text{supp}y_{i(j)}).
\]
(with \( y_0 = 0 \) and \( \max \text{supp}(0) = 0 \)). We observe that (for the first inequality it is
CHAPTER 3. DISTORTION OF BANACH SPACES

recommended to draw a picture)

\[ \left\| \sum_{i=1}^{k} y_i \right\|_\ell \leq \frac{1}{f(\ell)} \left[ \sum_{j=1}^{\ell} \left\| \sum_{i=i(j-1)+1}^{i(j)-1} y_i \right\| + \sum_{j=1}^{\ell} \sum_{s=1}^{\ell} \| E_s(y_{i(j)}) \| \right] \]

[If \( i_{j-1} = i(j) - 1 \) the first \( \sum \) vanishes, and let \( i(0) = 0 \)]

\[ \leq \frac{1}{f(\ell)} \sum_{j=1}^{\ell} C \frac{i(j) - i(j-1) - 1}{f(i(j)) - i(j-1) - 1} + \frac{\ell}{f(\ell)} \frac{n_1 + \ell}{n_1} \]

[For first part note that subsequences of rapidly increasing sequence are rapidly increasing and use induction hypothesis, and let \( \frac{0}{0} := 0 \)]

[For second part use (3.15)]

\[ \leq C \frac{k}{f(\ell)f(k/\ell)} + \frac{2\ell}{f(\ell)} \]

[By Lemma 3.2.12 and by (3.18)]

\[ = C \left[ \frac{k}{f(\ell)f(k/\ell)} + \frac{\ell}{f(\ell)} \frac{f(k)/f(k/\ell) - f(k)}{f(k/\ell)f(\ell)} \right] \]

[Using (3.22)]

\[ = C \left[ \frac{k}{f(\ell)f(k/\ell)} + \frac{k}{f(k)} \frac{f(k)/f(k/\ell) - f(k)}{f(k/\ell)f(\ell)} \right] = C \frac{k}{f(k)}. \]

[The function \([1, \infty) \ni x \mapsto f(x)/x\) is increasing]

Case 2: \( k/2 \leq \ell \leq n_1 \).

In this case we use (3.15) and observe that

\[ \left\| \sum_{i=1}^{k} y_i \right\|_\ell \leq \sum_{i=1}^{k} \| y_i \|_\ell \]

\[ \leq \frac{1}{f(\ell)} \sum_{i=1}^{k} \frac{n_i + \ell}{n_i} \]

\[ \leq \frac{2k}{f(\ell)} \leq \frac{2k}{f(k/2)} \leq \frac{Ck}{f(k)} \text{[By (3.19) and (3.21)]} \]

Case 3: \( \ell > n_1 \) Choose \( i_0 \in \{1, 2 \ldots k\} \) so that \( \ell \in (n_{i_0}, n_{i_0+1}] \), with \( n_{k+1} := \infty \).
Then we deduce that
\[
\left\| \sum_{i=1}^{k} y_i \right\|_{\ell} \leq \left\| \sum_{i=1}^{i_0-1} y_i \right\|_{\ell} + 2 + \sum_{i=i_0+2}^{k} \|y_i\|_{\ell}
\]
\[
\leq \frac{\max \supp(y_{i_0} - 1)}{f(n_{i_0})} + 2 + \frac{2}{f(\ell)}
\]
[By (3.15), (3.16) and (3.19) (part 2)]
\[
\leq 5 \leq C k f(k) \left\{ \text{By (3.19) (part 1) and (3.21)} \right. 
\]
which handles the third case and finishes the proof of the induction step.

Remark. With more careful estimation one can actually proof that the constant \( C \) in the proof of Theorem 3.2.11 is 1. It was actually shown in [S1] that if \((y_i)\) is a sequence of increasing \(\ell_1\)-averages (i.e. for some \(\varepsilon > y_i\) is an \(\varepsilon-\ell_1^n\)-average for \(i \in \mathbb{N}\) and \(n_i \not\rightarrow \infty\) then \((y_i)\) has a subsequence \((\tilde{y}_i)\) whose spreading model is isometrically equivalent to the unit vector basis \((e_i)\) in \(S\).

This implies that
\[
\lim_{n_k > n_{k-1} > \ldots n_1 \rightarrow \infty} \left\| \sum_{i=1}^{k} \tilde{y}_{n_i} \right\| = \left\| \sum_{i=1}^{k} e_i \right\| = \frac{k}{f(k)},
\]
where the last equality follows easily from Lemma 3.2.12 by induction on \(k \in \mathbb{N}\). This implies, using the arguments of the proof of 3.2.11 that \(\| \cdot \|_{\ell}\) is a \(f(\ell)\)-distortion.

In [AS] a stronger result was shown: Using a strong enough definition of what rapidly increasing sequences are (but so that nevertheless every block subspace contains such sequences) it was shown that rapidly increasing sequences are equivalent to the unit vector basis. From that it was deduced that \(S\) is complementably minimal, every infinite dimensional subspace of \(S\) has a further subspace which is isomorphic to \(S\). It is unknown whether or not \(S\) is prime, i.e. whether or not every infinite dimensional complemented subspace of \(S\) is isomorphic to \(S\).

Remark. From our arguments in the proof of Theorem 3.2.11 we can deduce the following observations which will be important for the construction of the space of Gowers and Maurey.

Let \((y_i)_{i=1}^{k}\) be an RIS of length \(k\) and \(\ell \in \mathbb{N}\).

If \(\ell \leq k/2\) we have shown (third inequality of handling case 1) that
\[
\left\| \frac{f(k)}{k} \sum_{j=1}^{k} y_j \right\|_{\ell} \leq C \frac{f(k)}{f(\ell)f(k/\ell)} + 2 \frac{\ell}{f(k)} \frac{f(k)}{k f(\ell)}
\]
This implies for some constant \(C_1\) (using that \(f\) is a logarithmic function) that
\[
\left\| \frac{f(k)}{k} \sum_{j=1}^{k} y_j \right\|_{\ell} \leq \frac{C_1}{f(\ell)} \text{ if } \ell \leq \sqrt{k},
\]
and
\[ \left\| \frac{f(k)}{k} \sum_{j=1}^{k} y_j \right\|_\ell \leq C \leq C_1 \text{ if } \sqrt{k} \leq \ell \leq k^2. \]

If \( k^2 \leq \ell \leq n_1 \) we deduce from the computation done in Cases 2 that
\[ \left\| \frac{f(k)}{k} \sum_{j=1}^{k} y_j \right\|_\ell \leq C \frac{f(k)}{f(\ell)}, \]
and if \( \ell > n_1 \) we deduce from our inequalities in case Case 3 that
\[ \left\| \frac{f(k)}{k} \sum_{j=1}^{k} y_j \right\|_\ell \leq C \frac{f(k)}{k}. \]

So combining all these inequalities we finally get for some universal constant \( C' \) and all \( k, \ell \in \mathbb{N} \) that
\[ (3.24) \quad \left\| \frac{f(k)}{k} \sum_{j=1}^{k} y_j \right\|_\ell \leq C' \min \left( \frac{f(\ell)}{f(k)}, \max \left( \frac{f(k)}{f(\ell)}, \frac{f(k)}{k} \right) \right). \]

From Remark 3.2 we deduce the following:

**Corollary 3.2.13.** Let \( \varepsilon > 0 \) For \( m \in \mathbb{N} \) define:
\[
A_m = \{ \frac{f(m)}{m} \sum_{i=1}^{m} y_i : (y_i)_{i=1}^{m} \text{ is an RIS of length } m \} \\
A^*_m = \{ \frac{f(m)}{m} \sum_{i=1}^{m} y^*_i : (y^*_i)_{i=1}^{m} \text{ is a block in } S_{S^*} \}
\]

Note that \( A_m \subset CB_S \) and that \( A^*_m \subset B_{S^*} \) and that for and \( x \in A_m \) there is a \( x^* \in A^*_m \) so that \( x^*(x) \geq 1/2 \) (note that \( \|y_i\| \geq 1/2 \) for all elements of an RIS \( (y_i) \)). Also note that
\[ \|x\|_m = \sup_{x^* \in A_m^*} |x^*(x)| \text{ for all } x \in S. \]

Let \( \varepsilon_i \downarrow 0 \). Using Remark 3.2, we can find a lacunary enough sequence \( M = (m_i) \in [\mathbb{N}]^\omega \) So that
\[ (3.25) \quad \forall i, j \in \mathbb{N} \forall x \in A_{m_i}, x^* \in A_{m_j} \quad |\langle x^*, x \rangle| \leq \varepsilon_{\min(i,j)}. \]

**Definition 3.2.14.** A Banach space \( X \) is biorthogonally distortable If there is a sequence of subsets \( (B_n) \) of \( B_X \) and a sequence of subsets \( (B^*_n) \) of \( B_{X^*} \) so that

a) For all \( n \in \mathbb{N} \) the set \( B_n \) is asymptotic, i.e. for all infinite dimensional subspaces \( Y \subset X, \text{dist}(Y, B_n) = 0. \)
b) There is a $c > 0$ so that for all $n \in \mathbb{N}$ and all $x \in B_n$ there is an $x^* \in B_n^*$ so that $x^*(x) \geq c$.

c) There is a sequence $\varepsilon_n \nearrow 0$ so that for all $n \neq m$ in $\mathbb{N}$

$$\langle B_n^*, B_n \rangle = \sup_{x^* \in B_n^*, x \in B_n} |x^*(x)| \leq \varepsilon_{\min(m,n)}.$$ 

In this case we call $(B_n, B_n^*)$ nearly biorthogonal sequences in $X$.

**Exercise 3.2.15.** Assume $X$ separable and has a normalized 1-unconditional basis $(e_i)$ (so we assume it is the completion of $c_{00}$ under the norm $\| \cdot \|$) and assume that $X$ is biorthogonally distortable. Let $(B_n)$ and $(B_n^*)$ be given as in Definition 3.2.14 after passing to subsequences we can assume that $(\varepsilon_n)$, as given in c), is summable and $\sum_{n=1}^{\infty} \varepsilon_n \leq 1$. We can also assume that for $n \in \mathbb{N}$ $B_n \subset c_{00}$ and that $B_n^*$ is countable and $B_n^* \subset \text{span}(e_i^*) \equiv c_{00}$ (pass to a perturbation of a subset for which (b) is still satisfied, with $c/2$ instead of $c$). Let $B^* = \bigcup B_n^*$ and let $\sigma : B^{*<\omega} \to \mathbb{N}$ be injective. For $\ell \in \mathbb{N}$ define

$$\Gamma_\ell = \left\{ \sum_{i=1}^{\ell} f_i : f_i \in B^*, \text{ for } 1 \leq i \leq \ell \text{ and } f_i \in B_{\sigma(f_1,f_2,...,f_{i-1})}^* \right\},$$ 

and for $x \in c_{00}$

$$|||x|||_\ell = \|x\| + \sup_{f \in \Gamma_\ell} |f(x)|.$$ 

Show that $||| \cdot |||_\ell$ is an equivalent norm on $X$, and that there is a sequence $(\lambda_\ell)$ with $\lambda_\ell \not\nearrow \infty$, so that (Use the arguments in the construction of Maurey and Rosenthal in Example 2.2.11).

We can also construct a Banach space which does not contain any unconditional basic sequence

**Definition 3.2.16.** (The space of Gowers and Maurey, slightly modified)

We first consider a set $Q \subset c_{00} \cap [-1,1]^\omega$ with the following properties:

a) $c_{00} \cap (-1,1] \cap Q)^\omega \subset Q$,

b) If $(x_1, x_2, \ldots, x_\ell) \subset Q$ is a finite block sequence then

$$\frac{1}{f(\ell)} \sum_{i=1}^{\ell} x_i, \frac{1}{\sqrt{f(\ell)}} \sum_{i=1}^{\ell} x_i \in Q$$

c) $Q$ is countable.
Let \((\varepsilon_i) \subset (0, 1)\) summable such that

\[
\sum_{i=1}^{\infty} 2\varepsilon_i \leq \sqrt{f(2)} - 1
\]

Let \(M \subset \mathbb{N}\) be subset which is lacunary enough satisfying the following condition:

\[
\frac{k}{f(\ell)} < \varepsilon_k \text{ whenever } k, \ell \in M \text{ and } k < \ell.
\]

Let \(\sigma : Q^\omega = \{(z^*_i)_{i=1}^\ell : \ell \in \mathbb{N}, z^*_i \in Q \text{ for } i \leq \ell\} \rightarrow M\) be injective.

By induction on \(n \in \mathbb{N}_0\) we define a set \(F_n \subset c_00\) as follows:

For \(n = 0\) we put as usual \(F_0 = \{\pm e_i : i \in \mathbb{N}\}\) and assume that \(F_{n-1}\) has been defined for some \(n \in \mathbb{N}\). The define for \(k \in \mathbb{N}\),

\[
U(n, k) = \left\{ \frac{1}{f(k)} \sum_{i=1}^{k} z_i^* : k \in \mathbb{N}, (z_i^*)_{i=1}^k \subset F_{n-1} \text{ finite block} \right\}
\]

and put \(U_n = \bigcup_{k=1}^{\infty} U(n, k)\). Then define for \(\ell \in \mathbb{N}\),

\[
C(n, \ell) = \left\{ \frac{1}{\sqrt{f(\ell)}} \sum_{i=1}^{\ell} z_i^* : (z_i^*)_{i=1}^\ell \subset U_n \text{ block and } z_i^* \in U(n, \sigma(z_i^*, \ldots z_{i-1}^*), \text{ if } i > 1 \right\}
\]

Then let

\[
F_n = \{ [m, n](x^*) : x^* \in \bigcup_{\ell \in \mathbb{N}} C(n, \ell) \}.
\]

For \(x \in c_00\) we finally define

\[
\|x\|_{GM} = \sup_{x^* \in \bigcup F_n} |\langle f, x \rangle|.
\]

And we let \(GM\) be the completion of \(c_00\) under \(\| \cdot \|_{GM}\).

**Theorem 3.2.17.** [GM1] \(GM\) does not contain any unconditional basis sequence.

**Sketch of a proof.** An application of the finite version of James’ Theorem, Exercise 3.1.12) shows that every infinite dimensional blockspace space contains \(\ell_1\) finitely block represented, this implies every infinite dimensional block space contains Rapidly increasing sequences defined as in the space \(S\).

For \(k \in \mathbb{N}\) we put \(U(k) = \bigcup_{n \in \mathbb{N}} U(n, k)\) and \(C(k) = \bigcup_{n \in \mathbb{N}} C(n, k)\), and define as in \(S\):

\[
\|x\|_k = \sup_{x^* \in \bigcup U_k} |x^*(x)| \text{ for } x \in c_00.
\]
3.2. SPACES NOT CONTAINING $\ell_p$ OR $c_0$

Let $(x_i)_{i=1}^n \subset GM$ be a normalized block in $GM$ and $\varepsilon > 0$. We call $(x_i)_{i=1}^n \ell^+_1$ average if

$$\left\| \frac{1}{n} \sum_{i=1}^n x_i \right\| \geq \frac{1}{1+\varepsilon}.$$ 

We define RIS as in $S$, using $\ell^+_1$ averages.

We call a sequence $(y_i)_{i=1}^k$ a a rapidly increasing sequence (RIS) in $GM$ if there are $\varepsilon_i > 0$ and $n_i \in \mathbb{N}$ for for $i = 1, \ldots k$ so that

1. \[ \sum_{i=1}^k \varepsilon_i < 1 \] (3.28)
2. \[ y_i \text{ is an } \varepsilon_i - \ell^+_1 \text{ average} \] (3.29)
3. \[ n_1 > 2k \] (3.30)
4. \[ f(n_i) \geq \max \text{supp}(y_{i-1}) \text{ and } \frac{n_{i-1}}{n_i} < \varepsilon_i \text{ for } i = 2, 3, 4, \ldots k \] (3.31)

Step 1: There is a constant $C > 0$ so that for any $(y_i)$ be an RIS and any $\ell \in \mathbb{N}$

$$\left\| \frac{f(\ell)}{\ell} \sum_{i=1}^{\ell} y_i \right\|_k \leq C \min \left( \frac{f(\ell)}{f(k)}, \frac{f(k)}{f(\ell)} \right)$$ (3.32)

$$\left\| \frac{f(\ell)}{\ell} \sum_{i=1}^{\ell} y_i \right\| \leq C. \tag{3.33}$$

We show this claim by induction on $\ell$ for the $C$ chosen as in Theorem 3.2.11. For $\ell = 1$ this is trivial and assuming our claim is true for $\ell - 1$ and let

$$y = \frac{1}{f(\ell)} \sum_{i=1}^{\ell} y_i.$$ 

By following the same arguments as in the proof of Theorem 3.2.11 we deduce (3.32). In order to show (3.33) we still need to show that for any $\ell \geq 2$ and $x^* \in C(\ell)$ it follows that $|x^*(y)| \leq C$. But this follows from (3.32), the condition (3.27), and from the lacunarity condition (3.27) on $M$.

Note: If $m << k$ and if $(y_i)_{i=1}^{mk}$ is an RIS of length $mk$. Define for $i = 1, 2, \ldots m$

$$z_i = \frac{f(k)}{k} \sum_{j=1}^{k} y_{j+(i-1)m}.$$ 

Then $\|z_i\| \sim 1$, for $i = 1, \ldots, m$ and

$$\left\| \sum_{i=1}^{m} z_i \right\| = \frac{f(k)}{k} \left\| y_i \right\| \sim \frac{f(k)}{k} \frac{km}{f(km)} \sim m.$$
Thus \( \frac{k_m}{f(k_m)} \sum_{i=1}^{m} y_i \) is close to an \( \ell_1^m \) average.

Step 2: \( GM \) has no unconditional basic sequence.

Let \( Y \) be a block subspace let \( k \in \mathbb{N} \) We define by induction on \( \ell \in \{1, 2, \ldots, k\} \)
the following elements \( n_\ell \in \mathbb{N} \), \( x_\ell \in CB_Y \), with \( \text{supp}(x_{\ell-1}) < \text{supp}(x_\ell) \), if \( \ell > 1 \), and \( x^* \in U_{n_\ell} \). \( n_1 = 1 \), \( x_1 \in B_X \cap c_{00} \) arbitrary, and \( x^* \in U_1^* \) with \( x^*(x) \geq 1/2 \) (note that \( U_1 \) is normalizing).

Assume \( n_{\ell-1} \), \( x_{\ell-1} \) and \( x^*_{\ell-1} \), have been chosen, we put \( n_\ell = \sigma(x_1^*, \ldots x_{\ell-1}^*) \) and we choose \( x_\ell \) with \( \text{supp}(x_\ell) > \text{supp}(x_{\ell-1}) \) to be of the form

\[
x_\ell = \frac{1}{f(n_\ell)} \sum_{i=1}^{n_\ell} y_i,
\]

where \( (y_i) \) is an RIS with \( \text{supp}(y_1) > \text{supp}(x_{\ell-1}) \). Then we choose \( x^*_\ell \in U^*(n_\ell) \) with \( \text{supp}(x^*_\ell) > \text{supp}(x^*_{\ell-1}) \) with \( x^*_\ell(x_\ell) \geq 1/2 \).

Also note: At the moment we choose \( x^*_\ell \) we have infinitely many choices (i.e. arbitrary small perturbations of \( x^*_\ell \)) this means we can, assure that \( n_{\ell+1} = \sigma(x_1^*, x_2^*, \ldots, x^*_\ell) \) can be made arbitrarily large, depending on our previous choice. This implies together with the note after step 1, we can assure that \( (x_i)_{i=1}^k \) can be assumed to be an RIS itself.

From our construction it follows that for

\[
\frac{1}{\sqrt{f(\ell)}} \sum_{i=1}^{\ell} x^*_i \in C_\ell \subset B_{GM^*}.
\]

and, thus,

\[
\left\| \sum_{i=1}^{\ell} x_i \right\| \geq \frac{1}{2} \frac{\ell}{\sqrt{f(\ell)}}.
\]

Using the same arguments as in Example 2.2.11 (Maurey-Rosenthal) we can show that there is a constant \( C' \) so that

\[
\left\| \sum_{i=1}^{\ell} (-1)^i x_i \right\| \leq C' \frac{\ell}{\sqrt{f(\ell)}}.
\]

Since \( Y \) was an arbitrary block space this implies that \( GM \) cannot contain an unconditional basic sequence.
3.3 Hilbert space is arbitrarily distortable

The main goal of this section is to show the following

**Theorem 3.3.1.** ([OS1] $\ell_p$. $1 < p < \infty$ is biorthogonally distortable.

We will achieve the distortion of $\ell_p$ by “transporting” the nearly biorthogonal sequences of sets $(A_n)$ and $(A_n^*)$ from the space $S$ and $S^*$ to sequences of sets $(B_n)$ and $(B_n^*)$ in $\ell_p$ and $\ell_q^*$. This will be done via the generalized Mazur map which might be of independent interest.

**Exercise 3.3.2.** Let $1 < p < \infty$. Then the mapping

$$M_p : S_{\ell_1} \to S_{\ell_p}, \quad (\xi_i)_{i \in \mathbb{N}} \mapsto (\text{sign}(\xi_i)|\xi_i|^{1/p})_{i \in \mathbb{N}}.$$ 

Is a uniform homomorphism from $S_{\ell_p}$ onto $S_{\ell_1}$.

**Definition 3.3.3.** The generalizations is as follows. Let $X$ have a 1-unconditional normalized basis $(e_i)$ By the positive cone of $X$ we mean the set

$$X^+ = \left\{ \sum \xi_i e_i : \in X : \xi_i \geq 0, \text{ for } i \in \mathbb{N} \right\}.$$ 

We put $S_X^+ = S_X \cap X^+$ and $B_X^+ = B_X \cap X^+$.

If $\sigma = (\sigma_i) \in \{-1, 1\}^\omega$

$$X^\sigma = \left\{ \sum \xi_i e_i : \in X : \sigma_i \xi_i \geq 0, \text{ for } i \in \mathbb{N} \right\}.$$ 

is the cone for $\sigma$ in $X$.

The entropy function $E$ is defined by

$$E : \ell_1 \times X \to [0, \infty), \quad ((h_i), (\xi_i)) \mapsto \sum_{i=1}^{\infty} |h_i| \log |\xi_i|,$$

with the convention that $0 \log 0 = 0$

**Remark.** If $h = (h_i) \in \ell_1^+$ with $\|h\|_{\ell_1} = 1$ on can think of $(h)$ being a probability on $\mathbb{N}$, and for $x \in X^+ E(h, x)$ would then be the entropy of the random variable $x$.

**Proposition 3.3.4.** Let $h = (h_i) \in \ell_1 \cap c_{00}$ and put $B = \text{supp}(h)$. Then there exists a unique $x = (\xi_i) \in S_X$ so that

a) $E(h, x) = \max_{y \in S_X} E(h, y),$

b) $\text{supp} h = \text{supp} x = B$

c) $\text{sign}(\xi_i) = \text{sign}(h_i)$ for $i \in B$. 

This unique $x$ is then denoted by $F_X(h)$ and we call $F_X : S_{\ell_1} \to S_X$ the generalized Mazur map from $S_{\ell_1}$ to $S_X$. Moreover we put $E_X(h) = E(h, F_X(h)) = \max_{y \in S_X} E(h, y)$.

Proof. For $h \in \ell_1 \cap c_0$, $B = \text{supp}(h)$ we consider the restriction

$$E(h, \cdot) : \{ x \in S_X^+ : \text{supp}(x) \subset B \} \to [-\infty, 0].$$

Then $E(h, \cdot)$ is continuous taking real values for $x \in S_X^+$ with $\text{supp}(x) = B$ and taking the value $-\infty$ if $\text{supp}(x) \subset B$, and thus there is an $\tilde{x} = (\tilde{\xi}_i) \in S_X^+$, with $\text{supp}(\tilde{x}) = B$ for which

$$E(h, \tilde{x}) = \max \{ E(h, z) : z \in S_X^+ : \text{supp}(z) \subset B \}.$$

From the strict monotonicity and strict concavity of the $\log(\cdot)$ we deduce that for $z_1, z_2 \in S_X^+$

$$E(h, z_1 + z_2) \geq E(h, \frac{1}{2} (z_1 + z_2)) \geq \frac{E(h, z_1) + E(h, z_2)}{2},$$

and thus that the $\tilde{x}$ is unique. By letting $x = (\xi_i)$ with $\xi_i = \text{sign}(h_i) \tilde{\xi}_i$ we satisfy (a), (b) and (c).

Exercise 3.3.5. Justify the name "generalized Mazur map" and show that $F_X = M_p$ if $X = \ell_p$.

For the next result we need to define the following mapping $\Psi : (0, 1) \to (0, 1)$. First not that the function

$$g : (0, \infty) \to (0, \infty), \quad a \mapsto a + \frac{1}{a},$$

has a minimum at $a = 1$, is strictly decreasing on $(0, 1)$ and strictly increasing on $(1, \infty)$. Therefore we put

$$\eta : (0, 1) \to (0, 1), \quad \eta(\varepsilon) = \varepsilon \inf \left\{ \log \left( \frac{1}{2} \left( \sqrt{a} + \frac{1}{\sqrt{a}} \right) \right) : |a - 1| > \varepsilon \right\},$$

$$\Psi : (0, 1) \to (0, 1), \quad \Psi(\varepsilon) = \varepsilon \eta(\varepsilon)$$

and note that $\Psi(\varepsilon) > 0$ for $\varepsilon > 0$ and $\Psi(\varepsilon) \downarrow 0$ if $\varepsilon \downarrow 0$.

Proposition 3.3.6. Let $X$ have a 1-unconditional basis $(e_i)$.

a) Let $h \in S_{\ell_1} \cap c_0$, $v \in B_X^+$, and $\varepsilon > 0$ so that

$$E(h, v) \geq E_X(h) - \Psi(\varepsilon),$$

i.e. $E(h, v)$ is close to its maximum $E(h, \tilde{v})$ of over all $v \in B_X^+$, and let $u = F_X(v)$. Then, there is an $A \subset \text{supp}(h)$ so that
3.3. HILBERT SPACE IS ARBITRARILY DISTORTABLE

\(\|Ah\| \geq 1 - \varepsilon\) and \((1 - \varepsilon)u_i \leq v_i \leq (1 + \varepsilon)u_i\) for \(i \in A\).

b) Let \(h_1, h_2 \in S^+_0 c_{00}\) with \(\|h_1 - h_2\| \leq \varepsilon\) and let \(u_i = F_X(h_i), i = 1, 2\). Then

\[
\left\| \frac{1}{2}(u_1 + u_2) \right\| \geq 1 - \sqrt{\|h_1 - h_2\|}
\]

(Recall that if \(X\) is uniform convex this means that the closer the value of \(\left\| \frac{1}{2}(u_1 + u_2) \right\|\) is to 1 the closer \(u_1\) and \(u_2\) are to each other).

Proof. We have \(B := \supp(h) = \supp(u)\) and if \(E(h, v) \geq F_X(h) - \Psi(\varepsilon)\) we must have \(\supp(v) \subset B\). After passing to \(\tilde{v} = B(v)\) we might assume that \(\supp(v) = B\).

We observe that

\[
\Psi(\varepsilon) \geq E_X(h) - E(h, v)
\]

\[
\geq E\left(\frac{1}{2}(u + v)\right) - E(h, v)
\]

[Maximality of \(E_X(h)\)]

\[
= \sum_{i \in B} h_i \left[ \log \left( \frac{1}{2}(u_i + v_i) \right) - \log v_i \right]
\]

\[
= \sum_{i \in B} h_i \left[ \frac{1}{2} \log u_i + \frac{1}{2} \log v_i + \log \left( \frac{1}{2}(u_i + v_i) \right) - \log \sqrt{u_i v_i} - \log v_i \right]
\]

\[
= \frac{1}{2} \sum_{i \in B} h_i (\log u_i - \log v_i) + \sum_{i \in B} \log \frac{1}{2} \left( \sqrt{\frac{v_i}{u_i}} + \sqrt{\frac{u_i}{v_i}} \right) \geq \sum_{i \in B} \log \frac{1}{2} \left( \sqrt{\frac{v_i}{u_i}} + \sqrt{\frac{u_i}{v_i}} \right)
\]

Define

\[
A = \left\{ i \in B : \frac{v_i}{u_i} - 1 \leq \varepsilon \right\}.
\]

First note that \(i \in A \iff (1 - \varepsilon)u_i \leq v_i \leq (1 + \varepsilon)u_i\), and by (3.34) if \(i \notin A\) then

\[
\eta(\varepsilon) \leq \log \frac{1}{2} \left( \sqrt{\frac{u_i}{v_i}} + \sqrt{\frac{v_i}{u_i}} \right)
\]

Thus we conclude that

\[
\sum_{i \in B \setminus A} h_i \leq \sum_{i \in B \setminus A} h_i \frac{1}{\eta(\varepsilon)} \log \frac{1}{2} \left( \sqrt{\frac{u_i}{v_i}} + \sqrt{\frac{v_i}{u_i}} \right) \leq \frac{\Psi(\varepsilon)}{\eta(\varepsilon)} = \varepsilon.
\]

In order to show (b) let \(h_1, h_2 \in S^+_0 c_{00}\) and \(u_i = F_X(h_i)\) and define \(\varepsilon \geq 0\) by \(\|\frac{1}{2}(u_1 + u_2)\| = 1 - 2\varepsilon\). W.l.o.g we assume that \(\varepsilon > 0\). Let \(\tilde{u}_1 = u_1 + \varepsilon u_2\) and \(\tilde{u}_2 = u_2 + \varepsilon u_1\). Thus \(\supp(\tilde{u}_1) = \supp(\tilde{u}_2) = \supp h_1 \cup \supp h_2, \|\frac{1}{2}(\tilde{u}_1 + \tilde{u}_2)\| \leq 1 - \varepsilon\)
and, thus,
\[ E(h_1, \tilde{u}_1) \geq E(h_1, u_1) \]
\[ \geq E\left(h_1, \frac{\tilde{u}_1 + \tilde{u}_2}{2(1 - \varepsilon)}\right). \]
[Maximality of \(E(h_1, u_1)\)]
\[ = E\left(h_1, \frac{\tilde{u}_1 + \tilde{u}_2}{2} + |\log(1 - \varepsilon)| \right) \geq \frac{1}{2} E(h_1, \tilde{u}_1) + \frac{1}{2} E(h_1, \tilde{u}_2) + |\log(1 - \varepsilon)| \]
and, thus,
\[ |\log(1 - \varepsilon)| \leq \frac{1}{2} E(h_1, \tilde{u}_1) - \frac{1}{2} E(h_1, \tilde{u}_2) \]
and similarly we can show
\[ |\log(1 - \varepsilon)| \leq \frac{1}{2} E(h_2, \tilde{u}_2) - \frac{1}{2} E(h_2, \tilde{u}_1). \]
Since for \(j \in \text{supp}(\tilde{x}_1)\)
\[ |\log \tilde{x}_{1,j} - \log \tilde{x}_{2,j}| = |\log \left(\frac{x_{1,j} + \varepsilon x_{2,j}}{x_{2,j} + \varepsilon x_{1,j}}\right)| \leq |\log \varepsilon|, \]
we obtain
\[ \varepsilon \leq |\log(1 - \varepsilon)| \leq \frac{1}{4} (E(h_1, \tilde{u}_1) - E(h_1, \tilde{u}_2) - E(h_2, \tilde{u}_1) + E(h_2, \tilde{x}_2)) \]
\[ \leq \frac{1}{4} \sum_{j \in \text{supp}(\tilde{x}_1)} (h_{1,j} - h_{2,j})(\log \tilde{x}_{1,j} - \log \tilde{x}_{2,j}) \]
\[ \leq \frac{1}{4} \|h_1 - h_2\|_1 |\log \varepsilon| \leq \frac{1}{4} \|h_1 - h_2\|_1 \varepsilon^{-1}, \]
and, thus, \( \varepsilon \leq \frac{1}{2} \|h_1 - h_2\|^{1/2} \), hence
\[ \left\| \frac{u_1 + u_2}{2} \right\| = 1 - 2\varepsilon \geq \|h_1 - h_2\|^{1/2}. \]
\[ \square \]

**Remark.** Proposition 3.3.6 implies that if \(X\) is uniform convex the map \(F_X\) is uniformly continuous on \(S_X^+\) and thus on any cone of \(X\). Moreover the modulus of uniform continuity only depends on the modulus of uniform convexity.

Indeed, recall that the modulus of uniform convexity:
\[ \delta_X(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : \|x - y\| \geq \varepsilon \right\}. \]
So, if \(h_1, h_2 \in S_{c_0}^+\) with \(\|h_1 - h_2\| < \delta^2(\varepsilon)\) then it follows from Proposition 3.3.6 that \(1 - \left\| \frac{F_X(h_1) + F_X(h_2)}{2} \right\| < \delta(\varepsilon)\) which implies that \(\|F_X(h_1) - F_X(h_2)\| < \varepsilon.\)
In our next step we extend that remark to all of $S_X$

**Proposition 3.3.7.** Let $X$ be a uniform convex Banach space with a 1-unconditional basis. The map $F_X : S_{\ell_1} \cap c_00 \rightarrow S(X)$ is uniformly continuous and the modulus of uniform continuity solely depends on the modulus of uniform convexity of $X$.

**Proof.** Let $g(\varepsilon)$ be the modulus of uniform continuity of $F_X|_{S_{\ell_1}^+}$. We first note that if $h \in S_{\ell_1}^+$ and $u = F_X(h)$ and $I \subset \supp(h)$ so that $\sum_{i \in I} h_i \leq \varepsilon$, then $\|\sum_{i \in I} u_i e_i\| \leq g(2\varepsilon)$. Indeed, 

$$\left\| h - \left( \sum_{i \not\in I} h_i e_i \right) / \left\| \sum_{i \not\in I} h_i e_i \right\| \right\|_{\ell_1} \leq \left\| \sum_{i \in I} h_i e_i \right\|_{\ell_1} + \left\| \sum_{i \not\in I} h_i e_i - \left( \sum_{i \not\in I} h_i e_i \right) / \left\| \sum_{i \not\in I} h_i e_i \right\|_{\ell_1} \right\|_{\ell_1} \leq 2\varepsilon$$

Since the coordinates in $I$ of the vector $u = F_X(h)$ and the vector 

$$F_X(h) - F_X \left( \left( \sum_{i \not\in I} h_i e_i \right) / \left\| \sum_{i \not\in I} h_i e_i \right\|_{\ell_1} \right)$$

coincide it follows that 

$$\left\| \sum_{i \in I} u_i e_i \right\| = \left\| F_X(h) - F_X \left( \left( \sum_{i \not\in I} h_i e_i \right) / \left\| \sum_{i \not\in I} h_i e_i \right\|_{\ell_1} \right) \right\|_{\ell_1} \leq g(2\varepsilon).$$

Let $h_1, h_2 \in S_{\ell_1}$ and $u_i = F_X(h_i)$, for $i = 1, 2$ and put $\varepsilon = \|h_1 - h_2\|$ Then $\text{sign}(u_{i,j}) = \text{sign}(h_{i,j})$ for $i = 1, 2$ and $j \in \supp(h_i)$. Letting 

$$I = \{ j \in \supp(h_1) \cup \supp(h_1) : \text{sign}(x_{1,j}) \neq \text{sign}(x_{2,j}) \}$$

and $h_i = (h_{i,j})_{j \in \mathbb{N}} \in \ell_1^+$, and, similarly, $|u_i| = F_X(|h_i|) = (u_{i,j})_{j \in \mathbb{N}} \in X^+$, for $i = 1, 2$. 

$$\|u_1 - u_2\| \leq \|u_1| - |u_2\| + \left\| \sum_{j \in \mathbb{I}} (|u_{1,j}| + |u_{2,j}|) e_j \right\|$$

$$\leq g(\|h_1| - |h_2\|_{\ell_{-1}}) + \left\| \sum_{j \in \mathbb{I}} |u_{1,j}| e_j \right\| + \left\| \sum_{j \in \mathbb{I}} |u_{2,j}| e_j \right\| \leq g(\varepsilon) + 2g(2\varepsilon).$$

[Note that $\sum_{j \in \mathbb{I}} |h_{i,j}| \leq \|h_1 - h_2\| = \varepsilon$, for $i = 1, 2$]

\[\square\]

**Definition 3.3.8.** A Banach space $X$ is uniformly smooth if for each $x \in S_X$ there is a unique $x^* \in S_{X^*}$ so that $x^*(x) = 1$ and if the mapping $\Phi : S_X \rightarrow S_{X^*}$, $x \rightarrow x^*$ is uniformly continuous.

The map $\Phi$ is called support map.
Proposition 3.3.9. Let $X$ be uniform smooth. Extend the support homogeneously map $(\cdot)^* : S_X \to S_{X^*}$ to a map $(\cdot)^* X \to X^*$, i.e for $x \in X \setminus \{0\}$, $x^* = \|x\| \cdot \left(\frac{x^*}{\|x\|}\right)$.

Then the norm function is differentiable on $X \setminus \{0\}$ with
\[
\frac{\partial}{\partial x} \|x\| = \nabla \|x\| = x^*,
\]

Proof. We will show that for $x, y \in S_X$, and $\lambda > 0$
\[
x^*(y) \leq \frac{\|x + \lambda y\| - \|x\|}{\lambda} \leq \frac{(x + \lambda y)^*(y)}{\|x + \lambda y\|},
\]
which implies our claim after letting $\lambda \searrow 0$.
\[
x^*(y) = \frac{x^*(\lambda y)}{\lambda} = \frac{x^*(x + \lambda y) - 1}{\lambda}
\leq \frac{\|x + \lambda y\| - \|x\|}{\lambda}
= \frac{\|x + \lambda y\|^2 - \|x\| \cdot \|x + \lambda y\|}{\lambda \|x + \lambda y\|}
\leq \frac{(x + \lambda y)^*(x + \lambda y) - (x + \lambda y)^*(x)}{\lambda \|x + \lambda y\|}
= \frac{(x + \lambda y)^*(\lambda y)}{\lambda \|x + \lambda y\|} = \frac{(x + \lambda y)^*(y)}{\|x + \lambda y\|}
\]

\[ \square \]

Theorem 3.3.10. (Pisier) Every uniformly convex Banach space admits an equivalent norm which is also uniformly smooth and every uniformly smooth Banach space admits an equivalent norm which is also uniform convex.

Proposition 3.3.11. Let $X$ be a uniform smooth and uniform convex Banach space with a 1-unconditional basis $(e_i)$. Then $F_X : S_{\ell_1} \to S_X$ is invertible and $F^{-1}$ is also uniformly continuous, with the modulus of continuity solely on the modulus of uniform smoothness of $X$.

Moreover
\[
F_X^{-1} : S_X \to \ell_1, x = \sum_{i=1}^{\infty} x_i e_i \mapsto \sum_{i=1}^{\infty} \operatorname{sign}(x_i) x_i x^*_i e_i,
\]
where $x^* = \sum_{i=1}^{\infty} x^*_i e_i$ normalizes $x$.

Proof. The biorthogonal functionals $(e^*_i)$ are a 1-unconditional basis for $X^*$ and thus we can express for $x \in S_X$, $x^*$ as
\[
x^* = \sum_{i \in \operatorname{supp}(x)} x^*_i e_i <
\]
with \( \text{sign}(x^*_i) = \text{sign}(x_i) \) and \( x^* \circ x := (x^*_i \cdot x_i) \in S^+_\ell_i \).

Let \( G(x) = |x^*| \circ x \). We claim that \( G : S_X \rightarrow S_{\ell_i} \) is uniform continuous. Indeed, for \( x, y \in S_X \) and \( u = G(x) \) and \( v = G(y) \) we deduce that

\[
\|u - v\|_{\ell_1} = \| |x^*| \circ x - |y^*| \circ y \| \\
\leq \| |x^*| \circ (x - y) \| + \| |x^* - y^*| \circ (y) \| \\
\leq \|x - y\| + \|x^* - y^*\|,
\]

which proves that \( G \) is uniform continuous and that the modulus of uniform continuity solely depends on the modulus of uniform continuity of the map \( S_X \ni x \rightarrow x^* \in S_{X^*} \).

It remains to show that \( G = F^{-1}X \). Since \( G(x) = \text{sign}(x) \circ G(|x|) \), and \( F_X(x) = \text{sign}(x) \circ F(|x|) \) it is enough to show that for \( h \in S^+_{\ell_i} \cap c_{00} \) and \( x \in S^+_X \cap c_{00} \) we have \( G(F(h)) = h \) and \( F(G(x)) = x \).

If \( h \in S^+_{\ell_i} \cap c_{00} \) and \( x = F(h) \) then, by the Lagrange multiplier method, for some \( \lambda > 0 \)

\[
(h_i/x_i)_{i \in \text{supp}(h)} = \nabla_x E(h, x) = \lambda \nabla \| \cdot (x) = x^*,
\]

Thus \( h = \lambda x \circ x^* = \lambda G(x) \). It follows (take norm on both sides) that \( \lambda = 1 \) and that \( G(F(h)) = G(x) = h \).

In order to show that \( F(G(x)) = x \), it is enough to show that \( G \) injective. So assume that \( x, y \in S^+_X \) and \( h = x^* \circ |x| = y^* \circ |y| \). Let \( U = \{ u \in X^+ : \text{supp}(u) = \text{supp}(h) \} \) and for \( u \in U \) put \( f(u) := \| u \| - E(h, u) \). \( f \) is strictly convex on \( U \), and therefore there is at most one point \( z \in U \) for which \( \nabla f(z) = 0 \). But

\[
\nabla f(z) = 0 \iff z^* - \sum_{i \in \text{supp}(h)} \frac{h_i}{z_i} = 0 \iff h = z \circ z^*.
\]

Now we will sketch how to show that \( \ell_2 \) is biorthogonally distortable. We consider the nearly biorthogonal sets \( (A_m) \) and \( (A^*_m) \) in \( S \) and \( S^* \), respectively, as constructed in Corollary 3.2.13

\[
A_m = \left\{ \frac{f(m)}{m} \sum_{i=1}^m y_i : (y_i)_{i=1}^m \text{ is an RIS of length } m \right\}
\]

\[
A^*_m = \left\{ \frac{f(m)}{m} \sum_{i=1}^m y_i^* : (y_i)_{i=1}^m \text{ is a block in } S_{S^*} \right\}
\]

Let \( \varepsilon_k \searrow 0 \). As we have shown in Corollary 3.2.13 we can pass to subsequence \( (B_m, B^*_m) \) of \( (A_m, A^*_m) \) so that for some constant \( C \):

\[
(3.37) \quad \forall m \in \mathbb{N} \forall x \in B_m \exists x^* \in B^*_m \quad |x^*(x)| \geq (1 - \varepsilon_m), 1 \leq \| x \| \leq C,
\]

\[
(3.38) \quad \forall m \neq k, m, k \in \mathbb{N} \forall x \in B_m, x^* \in B^*_m \quad |x^*(x)| \leq \varepsilon_{\min(m,k)}
\]

\[
(3.39) \quad \forall m \in \mathbb{N} \forall Y \hookrightarrow X, \text{dim}(Y) = \infty \quad \text{dist}(B_m, Y) = 0.
\]
Since \((e_i)\) is an unconditional and subsymmetric basis of \(S\), it follows also that

\[(3.40) \quad \forall m \in \mathbb{N} \ B_m \text{ and } B_m^* \text{ are unconditional and spreading}\]

\((A \subset X)\) unconditionall and spreading means that if \(x = \sum a_i e_i \in A\) then \(\sum \sigma_i a_i e_{n_i} \in A\), for and \((\sigma_i) \subset \{1, -1\}\) and increasing \((n_i) \subset \mathbb{N}\).

**Theorem 3.3.12.** For \(m \in \mathbb{N}\) define

\[C_m = \{ x^* \circ |x|, \text{ with } x^* \in B_m, \ x \in B_m \text{ and } x^*(x) \geq 1/2 \} \]

and for \(1 < p, q < \infty\) with \(1/p + 1/q = 1\) we put

\[C_m^{(p)} = M_p(C_m) = \{(\text{sign}|x_i|^{1/p}) : x \in C_m\} \text{ and } C_m^{*(p)} = M_q(C_m) = \{(\text{sign}|x_i|^{1/q}) : x \in C_m\}.\]

Then \((C_m^{(p)})\)and \((C_m^{*(p)})\) are nearly biorthogonal in \(\ell_p\).

**Lemma 3.3.13.** [OS1][Lemma 3.5] Let \(Y\) be a block subspace of \(\ell_1\) and \(\varepsilon > 0\) and \(m \in \mathbb{N}\). Then there is a vector \(u \in S\) which is an \((1 + \varepsilon)-\ell_m^m\) average and a \(u^* \in S_{S^*}\), so that \(u^*(u) > 1 - \varepsilon\) and \(\text{dist}(u^* \circ |u|, Y) < \varepsilon\).

We skip the quite technical proof of Lemma 3.3.13 which makes use of Proposition 3.3.6 (a).

**Proof of Theorem 3.3.12.** We will show our claim for \(p = 2\), and put \(D_m = C_m^{(2)}\) (note \(C_m^{*(2)} = C_m^{(2)}\)).

**Step 1:** For \(m \in N\) \(C_m\) is asymptotic in \(\ell_1\). Indeed, let \(Y\) be a block subspace of \(\ell_1\). Then we can use Lemma 3.3.13 to find an RIS \((x_i)^m_{i=1}\) in \(S\) and \((x_i^*)^m_{i=1} \subset S_{S^*}\), so that \(x_i^*(x_i) > 1 - \varepsilon\) and \(\text{dist}(|x_i| \circ x_i^*, Y)\) then choose

\[y = \frac{f(m)}{m} \sum_{i=1}^m x_i \in A_m \text{ and } y^* = \frac{1}{f(m)} \sum_{i=1}^m x_i^* \in A^*m,\]

and

\[\text{dist}(y^* \circ |y|, Y) \leq \frac{1}{m} \sum_{i=1}^m \text{dist}(x_i^* \circ |x_i|) < \varepsilon.\]

**Step 2:** If \(A \subset \ell_1\) is asymptotic then \(M_2(A)\) is asymptotic in \(\ell_2\). This follows from the fact that if \(Y \subset \ell_1\) is a block subspace then \(M_2(Y \cap S_Y)\) is also a sphere of a block subspace of \(\ell_2\).

**Step 3:** \((C_m)\) is nearly biorthogonal (to itself). Let \(k \neq \ell\), and \(v \in C_k\), and \(w \in C_\ell\). We write

\[v = (v_i), \text{ with } v_i = \text{sign}(x_i)\sqrt{|x_i| \cdot |x_i^*|} \text{ and } x \in B_k, x^* \in B_k^*\]
$$ w = (w_i), \text{ with } w_i = \text{sign}(y_i) \sqrt{|y_i| \cdot |y_i^*|} \text{ and } y \in B_k, y^* \in B_k^* $$

Then

$$ \langle v, w \rangle \leq \sum_{i \in \mathbb{N}} |v_i| \cdot |w_i| $$

$$ = \sum \left( |x_i| \cdot |x_i^*| \cdot |y_i| \cdot |y_i^*| \right)^{1/2} $$

$$ \leq \left( \sum |x_i^*| \cdot |y_i| \right)^{1/2} \left( \sum |y_i^*| \cdot |x_i| \right)^{1/2} \leq \varepsilon_{\min\{k, \ell\}}. $$

(Since $B_\ell$ and $B_k$ are unconditional)
Chapter 4

Versions of Ramsey’s theorem in Banach spaces

4.1 Gowers’ game on blocks and the dichotomy theorem

In this section we present a special case of Gowers’ Ramsey theorem on block spaces of a Banach space with a basis. Throughout this section let $X$ be a separable Banach space with a normalized and bimonotone basis $(e_i)$. The assumed bimonotonicity of $(e_i)$ is not really necessary but will simplify the arguments.

As usual we think of $X$ being the completion of $c_{00}$ under some norm. For $x, y \in c_{00}$ we write $x < y$ if $\max \text{supp}(x) < \min \text{supp}(y)$. We adopt the convention that $0 < x$ and $0 > x$ for any $x \in c_{00}$.

If $Y, Z$ are block subspaces of $X$ we write $Y \preceq Z$ or $Z \succeq Y$ if $Y$ is a block subspace of $Z$. For $x \in c_{00}$ and $Z \preceq X$ we write $x < Y$ if $x < y$ for all $y \in Y$. of $Z$.

We also assume that our space satisfies the following technical condition:

\[(*)\] For all $x \in c_{00}$ there is an $\varepsilon = \varepsilon(\text{supp}(x))$ so that for all $y \in c_{00}$, with $x > y$ it follows that

$$\|x + y\| \geq \|x\| + \varepsilon \|y\|.$$ 

**Exercise 4.1.1.** In order to show that condition $(*)$ holds for an arbitrary small perturbation of a norm we define the following norm on $c_{00}$. Let $c > 1$. For $x = (x_i) \in c_{00}$ put

$$\|x\| = \sup \left\{ \sum_{i=0}^{\infty} c^{-i}x_{n_i} : n_0 < n_1 < n_2 \right\}.$$ 

Show that $\|\cdot\|$ is an equivalent norm on $c_0$ which satisfies $(*)$.

Let $B^\infty$ be the set of all infinite normalized block sequences in $X$ and $B^f$ the set of all finite normalized block sequences in $X$. We also consider $\emptyset$ to be an element
of $\mathcal{B}^f$ (block of length 0). For $Y \preceq X$ we denote the set of all normalized infinite or finite block sequences in $Y$ (these are exactly the normalized block of the block basis generating $Y$) by $\mathcal{B}^\infty(Y)$ and $\mathcal{B}^f(Y)$.

If $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in \mathcal{B}^f$ and $\mathbf{y} = (y_1, x_2, \ldots, y_m) \in \mathcal{B}^f$ or $\mathbf{y} = (y_i) \in \mathcal{B}^\infty$ we write $\mathbf{x} < \mathbf{y}$ if $x_m < y_1$ and in this case we denote the concatenation of $x$ and $y$ by $(x, y)$, i.e.

\[(x, y) = \begin{cases} (x_1, x_2, \ldots, x_m, y_1, y_2, \ldots y_n) & \text{if } \mathbf{y} \in \mathcal{B}^f \\ (x_1, x_2, \ldots, x_m, y_1, y_2, \ldots) & \text{if } \mathbf{y} \in \mathcal{B}^\infty. \end{cases} \]

We denote the length of $\mathbf{x}$ by $|\mathbf{x}|$.

**Definition 4.1.2.** We consider on $\mathcal{B}^\infty$ the product topology of the discrete topology on $S_X \cap c_{00}$, i.e. if $\mathbf{y}(n) = (y_i(n))_{i=1}^\infty \in \mathcal{B}^\infty$ for $n \in \mathbb{N}$ and $\mathbf{y} = (y_i)_{i=1}^\infty \in \mathcal{B}^\infty$ then $\mathbf{y}(n)$ converges to $\mathbf{y}$ if and only if for any $m \in \mathbb{N}$ there is an $n_0 \in \mathbb{N}$ so that for all $n \geq n_0$ $y_i(n) = y_i$ whenever $n \geq n_0$ and $i = 1, 2, \ldots m$.

For $\mathcal{A} \subset \mathcal{B}^\infty$ we denote the closure in the product topology of the discrete topology on $S_X \cap c_{00}$ by $\overline{\mathcal{A}}$.

For $\mathbf{e} = (\varepsilon_i) \subset [0, \infty)$ and $\mathcal{A} \subset \mathcal{B}^\infty$ we define the $\mathbf{e}$ fattening of $\mathcal{A} \subset \mathcal{B}^\infty$ by

\[(4.1) \quad \mathcal{A}_{\mathbf{e}} = \{(z_i) \in B^\infty : \exists (x_i) \in \mathcal{A} \quad \|x_i - z_i\| \leq \varepsilon_i \text{ for } i = 1, 2, \ldots \}. \]

If $\mathbf{e} = (\varepsilon_i)_{i=1}^n$ is a finite sequence we understand by $\mathcal{A}_{\mathbf{e}}$ the set $\mathcal{A}_{\eta}$ with $\eta = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, 0, 0, 0, \ldots)$.

For $\mathcal{A} \subset \mathcal{B}^\infty$ and $Y \preceq X$ we consider now the following infinite game between two players

- Player I: chooses $Y_1 \preceq Y$
- Player II: chooses $y_1 \in S_{Y_1} \cap c_{00}$
- Player I: chooses $Y_2 \preceq Y$
- Player II: chooses $y_2 \in S_{Y_2} \cap c_{00}$

: Player I wins if the resulting sequence $(y_1, y_2, \ldots)$ lies in $\mathcal{A}$.

It follows from Theorem 1.2.4 that the game is determined if $\mathcal{A}$ is Borel with respect to the product of the discrete topology on $S_X$, i.e. that one of the Players has a winning strategy.

Write $W_I(\mathcal{A}, Y)$ if Player I has a winning strategy for the $(\mathcal{A}, Y)$-game and we write $W_{II}(\mathcal{A}, Y)$ if Player II has a winning strategy.

The main result of this section is the following.

**Theorem 4.1.3.** [Go3]. Assume $\mathcal{A} \subset \mathcal{B}^\infty$ is closed. Then the following are equivalent.

\[ \text{Proof:} \]
4.1. GOWERS’ GAME ON BLOCKS AND THE DICHOTOMY THEOREM

1. For all \( \varepsilon = (\varepsilon_i) \subset (0, 1] \) and for all \( Y \preceq X \) it follows \( W_1(\overline{X}, Y) \).

2. For all \( \varepsilon = (\varepsilon_i) \subset (0, 1] \) and for all \( Y \preceq X \) there exists \( Z \preceq Y \) so that every normalized block sequence \( (z_n) \) in \( Z \) is in \( \overline{X} \).

Remark. In [Go3] above result was proved for coanalytic sets \( A \subset B^\infty \).

Proof. Needs to be rewritten.

Theorem 4.1.3 was a central part for Gowers to prove his dichotomy result which lead to the solution of the homogeneous Banach space problem.

The Homogeneous Banach space problem.
Let \( X \) be an infinite dimensional separable Banach space which is isomorphic to all of its infinite dimensional closed subspace (we call such a space homogenous).

Does it follow that \( X \) is isomorphic to Hilbert space?

In order to outline the solution we need Gowers’ dichotomy Theorem.

Definition 4.1.4. Hereditary Indecomposable Spaces.
An infinite dimensional Banach space \( X \) is called indecomposable if it is not isomorphic the complemented sum of two infinite dimensional Banach spaces \( Z_1 \) and \( Z_2 \).

\( X \) is called hereditary indecomposable (HI) if no infinite dimensional closed subspace is decomposable.

Remark. Note that an (HI) space cannot contain an unconditional basic sequence. The existence of (HI) spaces was shown in [GM1] and [GM2].

Assume that \( Y \) and \( Z \) are two closed subspaces of a Banach space \( X \) in order for

\[ Y + Z = \{ y + z : y \in Y \text{ and } z \in Z \} \]

to be the complemented sum of \( Y \) and \( Z \) it is necessary and sufficient that the map

\[ Y \times Z \rightarrow y + Z, \quad (y, z) \mapsto (y + z) \]

is an isomorphism. Here \( Y \times Z \) is the topological product of \( Y \) and \( Z \) which can be endowed for example with the norm \( \|(y, z)\| = \|y\| + \|z\| \).

Thus, for \( Y + Z \) not being the complemented sum of \( Y \) and \( Z \), it is necessary and equivalent that there are sequences \( (y_n) \subset S_Y \) and \( (z_n) \subset S_Z \) for which \( \lim_{n \to \infty} \|y_n - z_n\| = 0 \). Therefore we observed the following proposition.

Proposition 4.1.5. For space \( X \) the following are equivalent

a) \( X \) is (HI)

b) \( \text{dist}(S_Y, S_Z) = \inf_{y \in S_Y, z \in S_Z} \|y - z\| = 0 \) for any two infinite dimensional subspaces \( Y, Z \preceq X \).
**Theorem 4.1.6. Gowers’ Dichotomy** [Go3]

If $X$ is an infinite dimensional Banach space. Then either it contains an unconditional basic sequence or it has a subspace which is (HI).

**Proof.** W.l.o.g. $X$ has a normalized basis $(e_i)$ and let $C > 1$.

Note that a normalized block sequence $(x_n)$ is $C$-unconditional if and only if for all normalized blocks $(y_n)$, all $n \in \mathbb{N}$, and all $(\lambda_i) \in [0, 1]^n$

$$\left\| \sum_{i=1}^{n} (-1)^i \lambda_i x_i \right\| \leq C \left\| \sum_{i=1}^{n} \lambda_i x_i \right\|.$$

Let $X$ be a Banach space with normalized basis $(e_i)$. For a fixed $C > 1$ we define

$$A^{(C)} = \left\{ (x_n) : \forall n \in \mathbb{N} \forall (\lambda_i)_{i=1}^{n} \subset [0, 1], \, \left\| \sum_{i=1}^{n} (-1)^i \lambda_i x_i \right\| \leq C \left\| \sum_{i=1}^{n} \lambda_i x_i \right\| \right\}$$

$$= \bigcap_{n \in \mathbb{N}, (\lambda_i)_{i=1}^{n} \subset [0, 1]} \left\{ (x_n) \subset S_X \text{ block}: \left\| \sum_{i=1}^{n} (-1)^i \lambda_i x_i \right\| \leq C \left\| \sum_{i=1}^{n} \lambda_i x_i \right\| \right\}.$$

Note that $A^{(C)}$ is closed in the product topology of the discrete topology, and that therefore the $A^{(C)}$-game is determined and that Theorem 4.1.3 applies.

Let $\varepsilon > 0$. Now, either Player I has a winning strategy for $(A^{(C)})_{(2^{-n} \varepsilon)} \subset A^{(C+\varepsilon)}$ on every closed $Y \hookrightarrow X$. Then we deduce from Theorem 4.1.3 that there is a normalized block sequence $(y_n)$ all of its normalized blocks are in $A^{(C+\varepsilon)}$ and it follows that $(y_n)$ is $(C + \varepsilon)$-unconditional. Or Player II has a winning strategy on some closed block subspace $Y$ of $X$. He in particular has a winning strategy if Player I chooses $Z_1, Z_2, Z_1, Z_2, \ldots$ for any given block subspaces $Z_1$ and $Z_2$ of $Y$. Therefore Player II can choose a normalized block $(z_i)$ in $Y$ so that $z_{2i-1} \in Z_1$ and $z_{2i} \in Z_2$ for which there is an $n \in \mathbb{N}$ and a sequence $(\lambda_i)_{i=1}^{n} \in [-1, 1]$ So that

$$\left\| \sum_{i=1}^{2n} (-1)^i \lambda_i z_i \right\| > C \left\| \sum_{i=1}^{2n} \lambda_i z_i \right\|.$$

We choose

$$y_1 = \sum_{i=1}^{n} \lambda_{2i-1} z_{2i-1}$$
$$y_2 = \sum_{i=1}^{n} \lambda_{2i} z_{2i}$$

and assume w.l.o.g that $\|y_1\| = 1$ (otherwise divide $y_1$ and $y_2$ by $\|y_1\|$) and that $\|y_2\| \leq \|y_1\|$ (otherwise swap the roles of $y_1$ and $y_2$).
It follows that
\[ 1 \geq \|y_2\| \geq 1 - \|y_1 + y_2\| \geq 1 - \frac{1}{C} \|y_1 - y_2\| \geq 1 - \frac{2}{C} = \frac{C - 2}{C} \]
and thus
\[ \left\| \frac{y_1 + y_2}{\|y_2\|} \right\| \leq \left\| y_1 + y_2 \right\| + \|y_2\| \left( 1 - \frac{C - 2}{C} \right) \leq \frac{4}{C}. \]

It follows that for any two infinite dimensional subspaces \( Z_1 \) and \( Z_2 \) of \( Y \) \( \text{dist}(S_{Z_1} S_{Z_2}) \leq \frac{5}{C} \).

It follows that
\[ \|y_1 + y_2\| \leq \frac{1}{C} \|y_1 - y_2\| \leq \frac{2}{C} \]
which yields \( \text{dist}(Z_1, Z_2) \leq \frac{2}{C} \).

Now we consider the a sequence \( C_n \uparrow \infty \) and apply the argument successively for each \( C_n \). We will get block spaces \( X = Y_0 \succ Y_1 \succ Y_2 \) so that either for some \( n \) \( Y_n \) has a \( C_n + 1 \)-unconditional subsequence or for all \( n \) and all closed subspaces \( Z_1, Z_2 \subset Y_n \) it follows that \( \text{dist}(S_{Z_1} S_{Z_2}) \leq \frac{5}{C_n} \).

In the second case we take \( Z \) to be a diagonal space of the \( Y_n \)'s (i.e. \( Z = \text{span}(z_i) \) with \( z_1 < z_2 < \ldots \) and \( z_n \in Y_n \) for \( n \in \mathbb{N} \)) and deduce that \( Z \) is (HI).

\textbf{Theorem 4.1.7.} [GM2] A hereditary indecomposable space is not isomorphic to any proper subspaces.

In particular Theorems 4.1.6 and 4.1.7 yield that if \( X \) is an infinite dimensional Banach space which is isomorphic to all of its subspaces, it must contain an unconditional basic sequence and, thus, have an unconditional basis and all its infinite dimensional subspaces have unconditional bases.

But this only happens in Hilbert space as the following result by Komorowski and Tomczak-Jaegermann shows.

\textbf{Theorem 4.1.8.} [KT] Let \( X \) be a homogenous Banach space not containing \( \ell_2 \). Then \( X \) has an infinite dimensional closed subspace without unconditional basis.
4.2 Trees and Branches in Banach spaces and embedding theorems

In this section we present a second version of Ramsey like theorems in Banach space and consider a similar game to the one introduced in Section 4.1. But this time Player I will only be able to choose cofinite dimensional subspaces of a given space.

We first introduce some notation.

**Definition 4.2.1.** Let $Z$ be a Banach space and $E = (E_n)$ be a sequence of finite dimensional subspaces of $Z$. We call $(E_n)$ a finite dimensional decomposition of $Z$, and we abbreviate it by FDD, if for every $x \in X$ there is a unique sequence $(x_n) \subset X$, with $x_n \in E_n$, for $n \in \mathbb{N}$, so that $x = \sum_{n=1}^{\infty} x_n$.

Note that FDD’s can be thought of as a generalization of bases. Indeed, if $\dim(E_n) = 1$, for all $n \in \mathbb{N}$ for an FDD $(E_n)$ then $(x_n)$, where $x_n \in E_n \setminus \{0\}$ for $n \in \mathbb{N}$, is a basis. Many basic observations on bases can be extended to FDDs.

Let $Z$ be a Banach space with an FDD $E = (E_n)$. For $n \in \mathbb{N}$ we denote the $n$-th coordinate projection by $P^E_n$, i.e. $P^E_n : Z \to E_n$, $\sum z_i \mapsto z_n$. For finite $A \subset \mathbb{N}$ we put $P^E_A = \sum_{n \in A} P^E_n$. As in the case of bases one can show that the projection constant of $(E_n)$ (in $Z$)

$$K = K(E, Z) = \sup_{m \leq n} \| P^E_{[m,n]} \|$$

is finite.

As in the case of bases we call $(E_i)$ bimonotone (in $Z$) if $K = 1$. By passing to the equivalent norm

$$\| \cdot \| : Z \to \mathbb{R}, \quad z \mapsto \sup_{m \leq n} \| P^E_{[m,n]}(z) \|,$$

we can always renorm $Z$ so that $K = 1$.

$(E_i)$ is called a $C$-unconditional FDD of $Z$ if for all $(x_i)_{i=1}^{n} \subset Z$, with $x_i \in E_i$, for $i \in \mathbb{N}$, it follows that

$$\left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| \leq \left\| \sum_{i=1}^{n} x_i \right\| \text{ for all } (\varepsilon_i) \subset \{+1, -1\},$$

and it is called a suppression $C$-unconditional FDD of $Z$ if

$$\left\| \sum_{i \in A} x_i \right\| \leq \left\| \sum_{i=1}^{n} x_i \right\| \text{ for all } A \subset \{1, 2, \ldots, n\}.$$

As in the case of bases, $C$-unconditionality implies suppression $C$-unconditionality, and suppression $C$-unconditionality implies $C$-unconditionality. We call $(E_i)$ unconditional or suppression unconditional if there is a $C \geq 1$ for which $(E_i)$ unconditional or suppression unconditional, respectively.
4.2. TREES AND BRANCHES IN BANACH SPACES

In an important example are the $\ell_p$-sums of finite dimensional spaces $(E_i)$.

$$(\oplus_{i=1}^{\infty} E_i)_{\ell_p} = \{(x_i) : x_i \in E_i, \text{ for } i \in \mathbb{N}, \text{ and } \sum_i \|x_i\| < \infty\},$$

$$(\oplus_{i=1}^{\infty} E_i)_{c_0} = \{(x_i) : x_i \in E_i, \text{ for } i \in \mathbb{N}, \text{ and } \lim_{i \to \infty} \|x_i\| = 0\}.$$  

For a sequence $(E_i)$ of finite dimensional spaces we define the vector space

$$c_00(\oplus_{i=1}^{\infty} E_i) = \left\{ (z_i) : z_i \in E_i, \text{ for } i \in \mathbb{N}, \text{ and } \{i \in \mathbb{N} : z_i \neq 0\} \text{ is finite} \right\},$$

which is dense in each Banach space for which $(E_n)$ is an FDD. For $A \subset \mathbb{N}$ we denote by $\oplus_{i \in A} E_i$ the linear subspace of $c_00(\oplus E_i)$ generated by the elements of $(E_i)_{i \in A}$ and we denote its closure in $Z$ by $(\oplus E_i)_Z$. As usual we denote the vector space of sequences in $\mathbb{R}$ which are eventually zero by $c_00$ and its unit vector basis by $(e_i)$.

The vector space $c_00(\oplus_{i=1}^{\infty} E_i^*)$, where $E_i^*$ is the dual space of $E_i$, for $i \in \mathbb{N}$, is a $w^*$-dense subspace of $Z^*$. (More precisely $E_i^*$ is the subspace of $Z^*$ generated by all elements $z^*$ for which $z^*|E_n = 0$ if $n \neq i$. $E_i^*$ is uniformly isomorphic to the dual space of $E_i$ and is isometric to it if $K(E,Z) = 1$.) We denote the norm closure of $c_00(\oplus_{i=1}^{\infty} E_i^*)$ in $Z^*$ by $Z^{(*)}$. $Z^{(*)}$ is $w^*$-dense in $Z^*$, the unit ball $B_{Z^{(*)}}$ norms $Z$ and $(E_i^*)$ is an FDD of $Z^{(*)}$ having a projection constant not exceeding $K(E,Z)$. If $K(E,Z) = 1$ then $B_{Z^{(*)}}$ is 1-norming and $Z^{(*)} = Z$.

For $z \in c_00(\oplus E_i)$ we define the $E$-support of $z$ by

$$\text{supp}_E(z) = \left\{ i \in \mathbb{N} : P_i E(z) \neq 0 \right\}.$$  

A non-zero sequence (finite or infinite) $(z_j) \subset c_00(\oplus E_i)$ is called a block sequence of $(E_i)$ if

$$\text{max supp}_E(z_n) < \text{min supp}_E(z_{n+1}), \text{ whenever } n \in \mathbb{N} \text{ (or } n < \text{length}(z_j)),$$

and it is called a skipped block sequence of $(E_i)$ if $1 < \text{min supp}_E(z_1)$ and

$$\text{max supp}_E(z_n) < \text{min supp}_E(z_{n+1}) - 1, \text{ whenever } n \in \mathbb{N} \text{ (or } n < \text{length}(z_i)).$$

Let $\delta = (\delta_i) \subset (0, 1]$. A (finite or infinite) sequence $(z_j) \subset S_Z = \{z \in Z : \|z\| = 1\}$ is called a $\delta$-block sequence of $(E_n)$ or a $\delta$-skipped block sequence of $(E_n)$ if there are $1 \leq k_1 < \ell_1 < k_2 < \ell_2 < \cdots$ in $\mathbb{N}$ so that

$$\|z_n - P^{E}_{[k_n, \ell_n]}(z_n)\| < \delta_n, \text{ or } \|z_n - P^{E}_{[k_n, \ell_n]}(z_n)\| < \delta_n, \text{ respectively},$$

for all $n \in \mathbb{N}$ (or $n < \text{length}(z_j)$). Of course one could generalize the notion of $\delta$-block and $\delta$-skipped block sequences to more general sequences, but we prefer to introduce this notion only for normalized sequences. It is important to note that in the definition of $\delta$-skipped block sequences $k_1 \geq 1$, and that therefore the $E_1$-coordinate of $z_1$ is small (depending on $\delta_1$).

A sequence of finite-dimensional spaces $(G_n)$ is called a blocking of $(E_n)$ if there are $0 = k_0 < k_1 < k_2 < \cdots$ in $\mathbb{N}$ so that $G_n = \oplus_{i=k_{n-1}+1}^{k_n} E_i$, for $n = 1, 2, \ldots$. 

**Definition 4.2.2.** An FDD $(E_i)$ of a Banach space is called *shrinking* if the sequence of its coordinate functionals $(E_i^*)$ is an FDD of $X^*$, and $(E_i)$ is called *boundedly complete* if the series $\sum x_i, x_i \in E_i$, for $i \in \mathbb{N}$ converges whenever $\sup_{n \in \mathbb{N}} \| \sum_{i=1}^n x_i \|$. 

$(F_i)$ is called an *unconditional finite dimensional decomposition* (UFDD), if for all $x \in X$, the representation as $x = \sum_{i=1}^\infty x_i$, with $x_i \in E_i$, for $i \in \mathbb{N}$, converges unconditionally.

Let $A \subset S^Z_\omega$ and $B = \prod_{i=1}^\infty B_i$, where $B_n \subset S^Z_i$ for $n \in \mathbb{N}$.

We consider the following $(A, B)$-game between two players: Assume that $E = (E_i)$ is an FDD for $Z$.

- Player I chooses $n_1 \in \mathbb{N}$,
- Player II chooses $z_1 \in c_00\left( \bigoplus_{i=n_1+1}^\infty E_i \right) \cap B_1$,
- Player I chooses $n_2 \in \mathbb{N}$,
- Player II chooses $z_2 \in c_00\left( \bigoplus_{i=n_2+1}^\infty E_i \right) \cap B_2$,

\[ \vdots \]

Player I wins the $(A, B)$-game if the resulting sequence $(z_n)$ lies in $A$. If Player I has a winning strategy (forcing the sequence $(z_i)$ to be in $A$) we will write $W_I(A, B)$ and if Player II has a winning strategy (being able to choose $(z_i)$ outside of $A$) we write $W_{II}(A, B)$. If $A$ is a Borel set with respect to the product of the discrete topology on $S^Z_\omega$ (note that $B$ is always closed in the product of the discrete topology on $S^Z_\omega$), it follows from Theorem 1.2.4 that the game is determined, i.e., either $W_I(A, B)$ or $W_{II}(A, B)$.

Let us define $W_{II}(A, B)$ slightly differently from the definition provided in Section 1.1 and use *trees in Banach spaces*.

We define

$$ T_\infty = \bigcup_{\ell \in \mathbb{N}} \{ (n_1, n_2, \ldots, n_\ell) : n_1 < n_2 < \cdots n_\ell \text{ are in } \mathbb{N} \}. $$

If $\alpha = (m_1, m_2, \ldots, m_\ell) \in T_\infty$, we call $\ell$ the *length of $\alpha$* and denote it by $|\alpha|$, and $\beta = (n_1, n_2, \ldots, n_k) \in T_\infty$ is called an *extension of $\alpha$*, or $\alpha$ is called a *restriction of $\beta$*, if $k \geq \ell$ and $n_i = m_i$, for $i = 1, 2, \ldots, \ell$. We then write $\alpha \leq \beta$ and with this order $(T_\infty, \leq)$ is a tree.

In this section *trees* in a Banach space $X$ are families in $X$ indexed by $T_\infty$, thus they are countable infinitely branching trees of countably infinite length.

For a tree $(x_\alpha)_{\alpha \in T_\infty}$ in a Banach space $X$, and $\alpha = (n_1, n_2, \ldots, n_\ell) \in T_\infty \cup \{ \emptyset \}$ we call the sequences of the form $(x_{(\alpha, n)})_{n>n_i}$ *nodes of $(x_\alpha)_{\alpha \in T_\infty}$*. The sequences $(y_n)$, with $y_i = x_{(n_1, n_2, \ldots, n_i)}$, for $i \in \mathbb{N}$, for some strictly increasing sequence $(n_i) \subset \mathbb{N}$, are called *branches of $(x_\alpha)_{\alpha \in T_\infty}$*. Thus, branches of a tree $(x_\alpha)_{\alpha \in T_\infty}$ are sequences of the form $(x_{(\alpha, n)})$ where $(\alpha_n)$ is a maximal linearly ordered (with respect to extension) subset of $T_\infty$. 
4.2. TREES AND BRANCHES IN BANACH SPACES

If \((x_\alpha)_{\alpha \in T_\infty}\) is a tree in \(X\) and if \(T' \subset T_\infty\) is closed under taking restrictions so that for each \(\alpha \in T' \cup \{\emptyset\}\) infinitely many direct successors of \(\alpha\) are also in \(T'\) then we call \((x_\alpha)_{\alpha \in T'}\) a full subtree of \((x_\alpha)_{\alpha \in T_\infty}\). Note that \((x_\alpha)_{\alpha \in T'}\) could then be relabeled to a family indexed by \(T_\infty\) and note that the branches of \((x_\alpha)_{\alpha \in T'}\) are branches of \((x_\alpha)_{\alpha \in T_\infty}\) and that the nodes of \((x_\alpha)_{\alpha \in T'}\) are subsequences of certain nodes of \((x_\alpha)_{\alpha \in T_\infty}\).

We call a tree \((x_\alpha)_{\alpha \in T_\infty}\) in a Banach space \(X\) normalized if \(\|x_\alpha\| = 1\), for all \(\alpha \in T_\infty\) and weakly null if every node is weakly null. More generally if \(T\) is a topology on \(X\) and a tree \((x_\alpha)_{\alpha \in T_\infty}\) in a Banach space \(X\) is called \(T\)-null if every node converges to 0 with respect to \(T\).

If \((x_\alpha)_{\alpha \in T_\infty}\) is a tree in a Banach space \(Z\) which has an FDD \((E_n)\) we call it a block tree of \((E_n)\) if every node is a block sequence of \((E_n)\).

We will also need to consider trees of finite length. For \(\ell \in \mathbb{N}\) we call a family \((x_\alpha)_{\alpha \in T_\infty, |\alpha| \leq \ell}\) in \(X\) a tree of length \(\ell\). Note that the notions nodes, branches, \(T\)-null and block trees can be defined analogously for trees of finite length.

**Remark.** Assume that \((x_n)\) is a sequence in \(X\). We can define the associated tree as follows: For \((n_1, n_2, \ldots, n_k) \in T_\infty\) we put \(x_{(n_1, n_2, \ldots, n_k)} := x_{n_k}\).

Note that then every subsequence of \((x_n)\) is a branch of \((x_\alpha)_{\alpha \in T_\infty}\) and vice versa, and that \((x_\alpha)_{\alpha \in T_\infty}\) is normalized, weakly null or a block tree, if \((x_n)\) is a normalized, weakly null or a block sequence.

Using the formal definition of winning strategies as introduced in chapter 1 we can easily derive the following Proposition.

**Proposition 4.2.3.** Assume that \(Z\) is a Banach space with an FDD \((E_i)\), \(A \subset S_Z^c\) and \(B = \prod_{i=1}^\infty B_i\), with \(B_i \subset S_Z\) for \(i \in \mathbb{N}\).

Then Player II has a winning strategy for the \((A, B)\)-game if and only if

\[(W_{II}(A, B)) \quad \text{There exists a block tree } (x_\alpha)_{\alpha \in T_\infty} \text{ of } (E_i) \text{ in } S_Z \text{ all of whose branches are in } B \text{ but none of its branches are in } A.\]

In case that the \((A, B)\)-game is determined \(W_I(A, B)\) can be therefore stated as follows.

\[(W_I(A, B)) \quad \text{Every block tree } (x_\alpha)_{\alpha \in T_\infty} \text{ of } (E_i) \text{ in } S_X, \text{ all of whose branches are in } B, \text{ has a branch in } A.\]

The proof of the following Proposition is easy.

**Proposition 4.2.4.** Let \(A, \tilde{A} \subset S_Z^c\), \(B = \prod_{i=1}^\infty B_i\), with \(B_i \subset S_Z\) for \(i \in \mathbb{N}\). Assume that the \((A, B)\)-game and the \((\tilde{A}, B)\)-game are determined.

a) If \(A \subset \tilde{A}\), then

\[W_I(A, B) \Rightarrow W_I(\tilde{A}, B) \text{ and } W_{II}(\tilde{A}, B) \Rightarrow W_{II}(A, B).\]

b) \(W_I(A, B) \iff \exists n \in \mathbb{N} \ \forall x \in (\oplus_{i=n+1}^\infty E_i) \cap B_1 \ \ W_I(A(x), \prod_{i=2}^\infty B_i)\)
c) If $\ell \in \mathbb{N}$, $\varepsilon = (\varepsilon_i)_{i=1}^\ell \subset [0, \infty)$ and $x_i, y_i \in B_i$ with $\|x_i - y_i\| \leq \varepsilon_i$ for $i = 1, 2, \ldots, \ell$ then

$$W_1\left(A(x_1, x_2, \ldots, x_\ell), \prod_{i=\ell+1}^\infty B_i\right) \Rightarrow W_1\left(A_\varepsilon(y_1, y_2, \ldots, y_\ell), \prod_{i=\ell+1}^\infty B_i\right).$$

Here $A(x_1, x_2, \ldots, x_n)$, for $x_1, \ldots, x_n \in S_Z$, and $A_\varepsilon$, the $\varepsilon$-fattening of $A$, was defined in (4.1) of Section 4.1

Now we can state one of our main combinatorial principles.

**Theorem 4.2.5.** Let $Z$ have an FDD $(E_i)$ and let $B_i \subset S_Z$, for $i = 1, 2, \ldots$. Put $B = \prod_{i=1}^\infty B_i$ and let $A \subset S_Z^\varepsilon$.

Assume that for all $\varepsilon = (\varepsilon_i) \subset (0, 1]$ we have $W_1(A_\varepsilon(B), B)$.

Then for all $\varepsilon = (\varepsilon_i) \subset (0, 1]$ there exists a blocking $(G_i)$ of $(E_i)$ so that every skipped block sequence $(z_i)$ of $(G_i)$, with $z_i \in B_i$, for $i \in \mathbb{N}$, is in $A_\varepsilon$.

**Proof.** Let $\varepsilon = (\varepsilon_i) \subset (0, 1]$ be given. W.l.o.g assume that $W_1(A, B)$ (otherwise replace $A$ by $A_{\varepsilon/2}$ and $B$ by $\varepsilon/2$).

For $k = 0, 1, 2, \ldots$ put $\varepsilon^{(k)}(i) = (\varepsilon_i^{(k)})$ with $\varepsilon_i^{(k)} = \varepsilon_i(1 - 2^{1-k})$ for $i \in \mathbb{N}$.

By induction we choose for $k \in \mathbb{N}$ numbers $n_k \in \mathbb{N}$ so that $0 = n_0 < n_1 < n_2 < \cdots$, and so that for any $k \in \mathbb{N}$, if $G_k = \oplus_{i=n_{k-1}+1}^{n_k} E_i$,

\begin{equation}
(4.2) \quad W_1\left(A_\varepsilon^{(k)}(\sigma, x), B^{(k+1)}\right)
\end{equation}

for any $0 \leq \ell < k$ and any normalized skipped block

$$\sigma = (x_1, x_2, \ldots, x_\ell) \in \prod_{i=1}^{\ell} B_i \text{ of } (G_i)_{i=1}^{k-1} \quad (\sigma = \emptyset \text{ if } \ell = 0)$$

and any $x \in S_{\oplus_{i=n_{k-1}+1}^{n_k} E_i} \cap B_{\ell+1}$

\begin{equation}
(4.3) \quad W_1\left(A_\varepsilon^{(k)}(\sigma), B^{(k)}\right)
\end{equation}

for any $0 \leq \ell < k$ and any normalized skipped block

$$\sigma = (x_1, x_2, \ldots, x_\ell) \in \prod_{i=1}^{\ell} B_i \text{ of } (G_i)_{i=1}^{k}$$

For $k = 1$ we deduce from Proposition 4.2.4 (b), the fact that and the hypothesis that there is an $n_1 \in \mathbb{N}$ so that $W_1\left(A_\varepsilon^{(1)}(x), B^{(1)}\right)$ for any $x \in S_{\oplus_{i=n_{1}+1}^{n_1} E_i} \cap B_1$. This implies (4.2) and (4.3) (note that for $k = 1$ $\sigma$ can only be chosen to be $\emptyset$ in (4.2) and (4.3)).

Assume $n_1 < n_2 < \cdots n_k$ have been chosen for some $k \in \mathbb{N}$. We will first choose $n_{k+1}$ so that (4.2) is satisfied. In the case that $k = 1$ we simply choose $n_2 = n_1 + 1$ and note that (4.2) for $k = 2$ follows from (4.2) for $k = 1$ since in both cases $\sigma = \emptyset$ is the only choice. If $k > 1$ we can use the compactness of the sphere of a finite dimensional space and choose a finite set $F$ of normalized skipped blocks
Assume that \( \sigma'(x_1', x_2', \ldots, x_\ell') \in \mathcal{F} \) with \( |x_i - x_i'| < \varepsilon_i 2^{-k-2} \), for \( i = 1, 2, \ldots, \ell \). Then, using the induction hypothesis (4.3) for \( k \), and Proposition 4.2.4 (b), we choose \( n_{k+1} \in \mathbb{N} \) large enough so that \( W_I(\mathcal{A}_{\varepsilon(i)}(\sigma, x), \mathcal{B}(\ell+1)) \) for any \( \sigma \in \mathcal{F} \) and \( x \in S_{\oplus_{i=n_{k+1}+1}^{\infty} E_i} \cap B_{\ell+1} \).

From Proposition 4.2.4 (c) and our choice of \( \mathcal{F} \) we deduce \( W_I(\mathcal{A}_{\varepsilon(k+1)}(\sigma, x), \mathcal{B}(\ell+1)) \) for any \( 0 \leq \ell < k \), any normalized skipped block \( \sigma \) of \( (G_i)_{i=1}^k \) of length \( \ell \in \prod_{i=1}^{\ell} B_i \) and any \( x \in S_{\oplus_{i=n_{k+1}+1}^{\infty} E_i} \cap B_{\ell+1} \), and, thus, (using the induction hypothesis for \( \sigma = \emptyset \)) we deduce (4.2) for \( \ell + 1 \).

In order to verify (4.3) let \( \sigma = (x_1, x_2, \ldots, x_\ell) \in \prod_{i=1}^{\ell} B_i \) be a normalized skipped block of \( (G_i)_{i=1}^k \) (the case \( \sigma = \emptyset \) follows from the induction hypothesis). Then \( \sigma'(x_1, x_2, \ldots, x_{\ell-1}) \) is empty or a normalized skipped block sequence of \( (G_i)_{i=1}^{k-1} \) in \( \prod_{i=1}^{\ell-1} B_i \). In the second case \( W_I(\mathcal{A}_{\varepsilon(k+1)}(\sigma), \mathcal{B}(\ell)) = W_I(\mathcal{A}_{\varepsilon(k+1)}(\sigma', x_\ell), \mathcal{B}(\ell)) \) follows from (4.2) for \( k \) and from Proposition 4.2.4 (a). This finishes the recursive definition of the \( n_k \)'s and \( G_k \)'s.

Let \( (z_n) \) any normalized skipped block sequence of \( (G_i) \) which lies in \( \mathcal{B} \). For any \( n \in \mathbb{N} \) it follows from (4.3) for \( \sigma = (z_n)_{i=1}^n \) that \( W_I(\mathcal{A}_{\varepsilon(2)}(\sigma), \mathcal{B}) \), and, thus, \( \mathcal{A}_{\varepsilon}(\sigma) \neq \emptyset \), which means that \( \sigma \) is extendable to a sequence in \( \mathcal{A}_{\varepsilon} \) (note that \( \lim_{n \to \infty} \varepsilon_i(n) = \varepsilon_i \)). Thus, any normalized skipped block sequence which is element of \( \mathcal{B} \) lies in \( \mathcal{A}_{\varepsilon} \).

Now let \( X \) be a closed subspace of \( Z \) having an FDD \((E_i)\) and \( \mathcal{A} \subset S^\infty_X \). We consider the following game.

- Player I chooses \( n_1 \in \mathbb{N} \),
- Player II chooses \( x_1 \in (\oplus_{i=n_1+1}^{\infty} E_i)_Z \cap X \), \( \|x_1\| = 1 \),
- Player I chooses \( n_2 \in \mathbb{N} \),
- Player II chooses \( x_2 \in (\oplus_{i=n_2+1}^{\infty} E_i)_Z \cap X \), \( \|x_2\| = 1 \),
- \( \vdots \)

As before, Player I wins if \( (x_i) \in \mathcal{A} \). Since the game does not only depend on \( \mathcal{A} \) but on the superspace \( Z \) in which \( X \) is embedded and its FDD \((E_i)\) we call this the \((\mathcal{A}, Z)\)-game.

**Definition 4.2.6.** Assume that \( X \) is a subspace of a space \( Z \) which has an FDD \((E_i)\) and that \( \mathcal{A} \subset S^\infty_X \). Define for \( n \in \mathbb{N} \)

\[
X_n = X \cap (\oplus_{i=n+1}^{\infty} E_i)_Z = \{ x \in X : \forall z^* \in \oplus_{i=1}^{n} E^*_i \quad z^*(x) = 0 \} ,
\]

a closed subspace of finite codimension in \( X \).

We say that **Player II has a winning strategy in the \((\mathcal{A}, Z)\)-game** if

\[
W_{II}(\mathcal{A}, Z) \quad \text{there is a tree} \quad (x_{(\alpha)})_{\alpha \in T_{\infty}} \in S_X \quad \text{so that for any} \quad \alpha = (n_1, \ldots, n_\ell) \in T_{\infty} \cup \emptyset \quad \text{whenever} \quad n > n_\ell, \quad \text{and so that no branch lies in} \quad \mathcal{A}.
\]
In the case that the \((\mathcal{A}, Z)\)-game is determined, Player I has a winning strategy in the \((\mathcal{A}, Z)\)-game if the negation of \(W_I(\mathcal{A}, Z)\) is true and thus 

\[ W_I(\mathcal{A}, Z) \]

for any tree \((x_n)_{n\in\mathbb{T}_\infty}\) in \(S_X\) so that for any \(\alpha = (n_1, \ldots, n_\ell) \in T_\infty \cup \emptyset\) \(x_{(n,n)} \in X_n\) whenever \(n > n_\ell\), there is branch in \(\mathcal{A}\).

For \(\mathcal{A} \subset S_X \subset S_Z^\omega\) and a sequence \(\varepsilon = (\varepsilon_i)\) in \([0, \infty)\) we understand by \(\mathcal{A}_\varepsilon\) the \(\varepsilon\)-fattening of \(\mathcal{A}\) as a subset of \(S_Z^\omega\). In case we want to restrict ourselves to \(S_X\) we write \(\mathcal{A}_\varepsilon^X\), i.e.

\[ \mathcal{A}_\varepsilon^X = \mathcal{A}_\varepsilon \cap S_X^\omega = \{(x_i) \in S_X^\omega : \exists (y_i) \in \mathcal{A} \mid \|x_i - y_i\| \leq \varepsilon_i\text{ for all } i \in \mathbb{N}\}. \]

Since \(S_X^\omega\) is closed in \(S_Z^\omega\) with respect to the product of the discrete topology, we deduce that \(\mathcal{A}_\varepsilon^X = \mathcal{A}_\varepsilon^X\) for \(\mathcal{A} \subset S_X\).

The following Proposition reduces the \((\mathcal{A}, Z)\)-game to a game we treated before. In order to be able to do so we need some technical assumption on the embedding of \(X\) into \(Z\) (see condition (4.4) below).

**Proposition 4.2.7.** Let \(X \subset Z\), a space with an FDD \((E_i)\). Assume the following condition on \(X, Z\) and the embedding of \(X\) into \(Z\) is satisfied:

\[ \text{(4.4) There is a } C > 0 \text{ so that for all } m \in \mathbb{N} \text{ and } \varepsilon > 0 \text{ there is an } n = n(\varepsilon, m) \geq m \]

\[ \|x\|_{X/X_m} \leq C[\|P_{[1,n]}^E(x)\| + \varepsilon] \text{ whenever } x \in S_X. \]

Assume that \(\mathcal{A} \subset S_X\) and that for all null sequences \(\varepsilon = (\varepsilon_i) \subset (0, 1]\) we have \(W_I(\mathcal{A}_\varepsilon^X, Z)\).

Then it follows for all null sequences \(\varepsilon = (\varepsilon_i) \subset (0, 1]\) that \(W_I(\mathcal{A}_\varepsilon, (S_X^\omega)_\delta)\) holds, where \(\delta = (\delta_i)\) with \(\delta_i = \varepsilon_i/28CK\) for \(i \in \mathbb{N}\), with \(C\) satisfying (4.4) and \(K\) being the projection constant of \((E_i)\) in \(Z\).

**Question.** (open) Is the technical condition (4.4) necessary to derive the conclusion of Proposition 4.2.7

**Proof.** Let \(\mathcal{A} \subset S_X\) and assume that \(W_I(\mathcal{A}_\varepsilon^X, Z)\) is satisfied for all null sequences \(\varepsilon = (\varepsilon_i) \subset (0, 1]\). For a null sequence \(\varepsilon = (\varepsilon_i) \subset (0, 1]\) we need to verify \(W_I(\mathcal{A}_\varepsilon, (S_X^\omega)_\delta)\) (with \(\delta_i = \varepsilon_i/28K\) for \(i \in \mathbb{N}\)) and so we let \((z_\alpha)_{\alpha \in \mathbb{T}_\infty}\) be a block tree of \((E_i)\) in \(S_Z\) all of whose branches lie in \((S_X^\omega)_\delta = \{(z_i) \in S_Z^\omega : \text{dist}(z_i, S_X) \leq \delta_i\text{ for } i = 1, 2, \ldots\}\).

After passing to a full subtree of \((z_\alpha)\) we can assume that for any \(\alpha = (m_1, \ldots, m_\ell)\) in \(T_\infty\)

\[ z_\alpha \in \oplus_{j=1+n(\delta_\ell,m_\ell)}^\infty E_j \]

(\(n(\varepsilon, m)\) is chosen as in (4.4)).

For \(\alpha = (m_1, m_2, \ldots, m_\ell) \in T_\infty\) we choose \(y_\alpha \in S_X\) with \(\|y_\alpha - z_\alpha\| < 2\delta_\ell\) and, thus, by (4.5)

\[ \|P_{E_{[1,n(\delta_\ell,m_\ell)]}}(y_\alpha)\| = \|P_{E_{[1,n(\delta_\ell,m_\ell)]}}(y_\alpha - z_\alpha)\| \leq 2K\delta_\ell. \]
4.2. TREES AND BRANCHES IN BANACH SPACES

Using (4.4) we can therefore choose an \( x'_\alpha \in X_{m_\ell} \) so that
\[
\| x'_\alpha - y_\alpha \| \leq C(2K\delta_\ell + \delta_\ell) \leq 3CK\delta_\ell,
\]
and thus
\[
1 - 3CK\delta_\ell \leq \| x'_\alpha \| \leq 1 + 3CK\delta_\ell.
\]
Letting \( x_\alpha = x'_\alpha/\| x'_\alpha \| \) we deduce that
\[
\| y_\alpha - x_\alpha \| \leq \| y_\alpha - x'_\alpha \| + \| x'_\alpha - x_\alpha \|
\leq 3CK\delta_\ell + (1 + 3CK\delta_\ell)3CK\delta_\ell/(1 - 3CK\delta_\ell) \leq 12CK\delta_\ell
\]
(the last inequality follows from the fact that \((1 + 3CK\delta_\ell)/(1 - 3CK\delta_\ell) \leq 3\)) and, thus,
\[
\| z_\alpha - x_\alpha \| \leq 14CK\delta_\ell = \varepsilon_\ell/2.
\]
Using \( W_I(\overline{A^X_{\varepsilon/2}}, Z) \) and noting that \( x_\alpha \in X_{m_\ell} \), for \( \alpha = (m_1, m_2, \ldots, m_\ell) \in T_\infty \) we can choose a branch of \((x_\alpha)\) which is in \( \overline{A^X_{\varepsilon/2}} \). Thus, the corresponding branch of \((z_\alpha)\) lies in \( \overline{A^X_{\varepsilon/2}} \).

From [OS1, Lemma 3.1] it follows that every separable Banach space \( X \) is a subspace of a space \( Z \) with an FDD satisfying the condition (4.4) (with \( n(m) = m \)). The following Proposition exhibits two general situations in which (4.4) is automatically satisfied.

**Proposition 4.2.8.** Assume \( X \) is a subspace of a space \( Z \) having an FDD \((E_i)\). In the following two cases (4.4) holds:

a) If \((E_i)\) is a shrinking FDD for \( Z \). In that case \( C \) in (4.4) can be chosen arbitrarily close to 1.

b) If \((E_i)\) is boundedly complete for \( Z \) (i.e., \( Z \) is the dual of \( Z^{(\ast)} \)) and the ball of \( X \) is a \( w^\ast \)-closed subset of \( Z \). In that case \( C \) can be chosen to be the projection constant \( K \) of \((E_i)\) in \( Z \).

**Proof.** In order to prove (a) we will show that for any \( m \in \mathbb{N} \) and any \( 0 < \varepsilon < 1 \) there is an \( n = n(\varepsilon, m) \) so that
\[
\| x \|_{X/X_m} \leq (1 + \varepsilon)[\| P_{[1,n]}^E(x) \| + \varepsilon], \text{ whenever } x \in S_X
\]
(i.e., \( C \) in (4.4) can be chosen arbitrarily close to 1).

Since \( X/X_m \) is finite dimensional and
\[
(X/X_m)^* = X^\perp_m = \{ x^* \in X^*: x^*|_{X_m} \equiv 0 \},
\]
we can choose a finite set \( A_m \subset S_{X^\perp_m} \subset S_X \) for which
\[
\| x \|_{X/X_m} \leq (1 + \varepsilon) \max_{f \in A_m} |f(x)| \text{ whenever } x \in X.
\]
By the Theorem of Hahn Banach we can extend each \( f \in A_m \) to an element \( g \in S_{Z^*} \). Let us denote the set of all of these extensions \( B_m \). Since \( B_m \) is finite and since \((E_i)\) is an FDD of \( Z^* \) we can choose an \( n = n(\varepsilon,m) \) so that \( \| P_{[1,n(m)]}^E(g) - g \| < \varepsilon \) for all \( g \in B_m \). Since \( P_{[1,n(m)]}^E \) is the adjoint operator of \( P_{[1,n(m)]}^E \) \( \varepsilon \) is the dual of \( Z \) which is the dual of \( Z^* \). For fixed \( m \) and \( n \in N \) we have that \( \| P_{[1,m(n)]}^E(g) - g \| \leq \varepsilon \) and \( \| P_{[1,n(m)]}^E(g) \| + \varepsilon \leq \| P_{[1,n(m)]}^E(g) \| + \varepsilon \), which proves our claim and finishes the proof of part (a).

In order to show (b) we assume that \( X \) is a subspace of a space \( Z \) which has a boundedly complete FDD \((E_i)\) and the unit ball of \( X \) is a \( w^* \)-closed subset of \( Z \), which is the dual of \( Z^* \).

For \( m \in N \) and \( \varepsilon > 0 \) we will show that the inequality in (4.4) holds for some \( n \) and \( C = K \). Assuming that this was not true we could choose a sequence \( (x_n) \subset S_X \) so that for any \( n \in N \)

\[
\| x_n \|_{X/X_m} > K \left[ \| P_{[1,n]}^E(x_n) \| + \varepsilon \right].
\]

By the compactness of \( B_X \) in the \( w^* \) topology we can choose a subsequence \( x_{n_k} \) which converges \( w^* \) to some \( x \in B_X \). For fixed \( \ell \) it follows that \( (P_{[1,\ell]}^E(x_{n_k})) \) converges in norm to \( P_{[1,\ell]}^E(x) \). Secondly, since \( X/X_m \) is finite dimensional it follows that \( \lim_{k \to \infty} \| x_{n_k} \|_{X/X_m} = \| x \|_{X/X_m} \), and, thus, it follows that

\[
\| x \| = \lim_{\ell \to \infty} \| P_{[1,\ell]}^E(x) \|
= \lim_{\ell \to \infty} \lim_{k \to \infty} \| P_{[1,\ell]}^E(x_{n_k}) \|
\leq K \limsup_{k \to \infty} \| P_{[1,n_k]}^E(x_{n_k}) \|
\leq \limsup_{k \to \infty} \| x_{n_k} \|_{X/X_m} - K\varepsilon = \| x \|_{X/X_m} - K\varepsilon,
\]

which is a contradiction since \( \| x \| \geq \| x \|_{X/X_m} \). \qed

By combining Theorem 4.2.5 and Proposition 4.2.7 we deduce

**Corollary 4.2.9.** Let \( X \) be a subspace of a space \( Z \) with an FDD \((E_i)\) and assume that this embedding satisfies condition (4.4). Let \( K \geq 1 \) be the projection constant of \((E_i)\) in \( Z \) and let \( C \geq 1 \) be chosen so that (4.4) holds.

For \( A \subset S_X \) the following conditions are equivalent

- \( A \subset S_X \) is a boundedly complete FDD of \( Z \)
- \( A \subset S_X \) is a boundedly complete FDD of \( Z \)
- \( A \subset S_X \) is a boundedly complete FDD of \( Z \)
- \( A \subset S_X \) is a boundedly complete FDD of \( Z \)
- \( A \subset S_X \) is a boundedly complete FDD of \( Z \)
4.2. TREES AND BRANCHES IN BANACH SPACES

a) For all null sequences \( \bar{\epsilon} = (\epsilon_n) \subset (0,1] \), \( W_I(\overline{\mathcal{A}_\epsilon^X}, X) \) holds.

b) For all null sequences \( \bar{\epsilon} = (\epsilon_n) \subset (0,1] \) there exists a blocking \((G_n)\) of \((F_n)\) so that every \( \bar{\epsilon}/420CK\)-skipped block sequence \((z_n) \subset X\) of \((G_n)\) is in \( \overline{\mathcal{A}_\epsilon} \).

In the case that \( X \) has a separable dual \((a)\) and \((b)\) are equivalent to the following condition

c) For all null sequences \( \bar{\epsilon} = (\epsilon_n) \subset (0,1] \) every weakly null tree in \( S_X \) has a branch in \( \overline{\mathcal{A}_\epsilon} \).

In the case that \((E_i)\) is a boundedly complete FDD of \( Z \) and \( B_X \) is \( w^* \)-closed in \( Z = (Z^*)^* \) the conditions \((a)\) and \((b)\) are equivalent to

d) For all null sequences \( \bar{\epsilon} = (\epsilon_n) \subset (0,1] \) every \( w^*\)-null tree in \( S_X \) has a branch in \( \overline{\mathcal{A}_\epsilon} \).

Proof. \((a) \Rightarrow (b)\) Let \( \bar{\epsilon} = (\epsilon_i) \subset (0,1] \) be a null sequence, choose \( \bar{\eta} = (\eta_i) \) with \( \eta_i = \epsilon_i/3 \), for \( i \in \mathbb{N} \), and \( \bar{\delta} = (\delta_i) \) with \( \delta_i = \eta_i/140CK = \epsilon_i/420CK \).

We deduce from Proposition 4.2.7 that \( W_I(\overline{\mathcal{A}_{\bar{\eta}}} (S^\omega_X)_{\bar{\delta}}) \) holds. Using Theorem 4.2.5 we can block \((E_i)\) into \((G_i)\) so that every skipped block of \((G_i)\) in \((S^\omega_X)_{\bar{\delta}}\) (as a subset of \( S_Z \)) is in \( \overline{\mathcal{A}_{\bar{\eta}}} \) (actually we are using the quantified result given by the proof of Theorem 4.2.5).

Assume \((x_i) \subset S_X\) is a \( \bar{\delta} \)-skipped block sequence of \((G_i)\) and let \( 1 \leq k_1 < \ell_1 < k_2 < \ell_2 < \cdots \) in \( \mathbb{N} \) so that

\[
\|x_n - P^{E}_{(k_n, \ell_n]}(x_n)\| < \delta_n, \text{ for all } n \in \mathbb{N}.
\]

The sequence \((z_n)\) with \( z_n = P^{E}_{(k_n, \ell_n]}(x_n) / \|P^{E}_{(k_n, \ell_n]}(x_n)\|\), for \( n \in \mathbb{N} \), is a skipped block sequence of \( S_Z \) and we deduce that

\[
\|x_n - z_n\| \leq \|x_n - P^{E}_{(k_n, \ell_n]}(x_n)\| + \|P^{E}_{(k_n, \ell_n]}(x_n)\| \left| 1 - \frac{1}{\|P^{E}_{(k_n, \ell_n]}(x_n)\|} \right| \\
\leq \delta_n + (1 + \delta_n) \frac{\delta_n}{1 - \delta_n} \leq 5\delta_n.
\]

This implies that \((z_n) \in \overline{\mathcal{A}_{\bar{\eta}}} \) and thus by our choice of \( \bar{\eta} \),

\[
(x_i) \in (\overline{\mathcal{A}_{\bar{\eta}}})_{\bar{\eta}} \subset \overline{\mathcal{A}_\epsilon}
\]

which finishes the verification of \((b)\).

\((b) \Rightarrow (a)\) is clear since for any blocking \((G_i)\) of \((E_i)\) and any null sequence \( \bar{\delta} = (\delta_i) \subset (0,1] \) every tree \((x_\alpha)_{\alpha \in T_\infty} \in S_X\) with the property that \( x_{(\alpha, n)} \in X_n \), whenever \( n > n_\ell \) and \( \alpha = (n_1, \ldots, n_\ell) \in T_\infty \cup \emptyset \) has a full subtree all of whose branches are \( \bar{\delta} \)-skipped block sequences of \((G_i)\).
Now assume that $X$ has a separable dual, or $(E_i)$ is a boundedly complete FDD of $Z$ and $B_X$ in $Z$ $w^*$-closed.

It is clear that (c) or (d), respectively, imply (a). Secondly, since for any null sequence $\delta = (\delta_i) \subset (0,1]$ and any blocking $(G_i)$ every weakly null tree in $S_X$ (in the case that $X$, has a separable dual) or every $w^*$ null tree (in the boundedly complete case) has a full subtree all of whose branches are $\delta$-skipped block sequences of $(G_i)$ we deduce that (b) implies (c) or (d) respectively.

Motivated by the asymptotic structure of a Banach space we introduce the following "coordinate-free" variant of our games. Again let $X$ be a separable Banach space and for $A \subset S^\omega_X$ we consider the following coordinate-free $A$-game.

Player I chooses $X_1 \in \text{cof}(X)$,
Player II chooses $x_1 \in X_1$, $\|x_1\| = 1$,
Player I chooses $X_2 \in \text{cof}(X)$,
Player II chooses $x_2 \in X_2$, $\|x_2\| = 1$,

As before, Player I wins if $(x_i) \in A$. We will show that $X$ can be embedded into a space $Z$ with an FDD so that for all $\varepsilon = (\varepsilon_i) \subset (0,1]$ Player I has a winning strategy in the coordinate-free $A_{\varepsilon}$-game, which we will denote by $W_I(A_{\varepsilon}, \text{cof}(X))$, if and only if for all $\varepsilon \subset (0,1]$ he has a winning strategy for the $(A_{\varepsilon},Z)$-game.

First note that since we only considering fattened sets and their closures, Player II has a winning strategy if and only if he has a winning strategy choosing his vectors out of a dense and countable subset of $S_X$ determined before the game starts. But this implies that there is countable set of cofinite dimensional subspaces, say $\{Y_n : n \in \mathbb{N}\}$ from which player I can choose if he has a winning strategy. Moreover if we consider a countable set $B$ of coordinate free games, there is a countable set $\{Y_n : n \in \mathbb{N}\}$ so that for all $A \in B$

$$\forall \varepsilon \subset (0,1] \ W_I(A_{\varepsilon}, \text{cof}(X)) \iff \forall \varepsilon \subset (0,1] \ W_I(A_{\varepsilon}, \{Y_n : n \in \mathbb{N}\})$$

where we write $W_I(A_{\varepsilon}, \{Y_n : n \in \mathbb{N}\})$, if player I has a winning strategy for the coordinate-free $A$-game, even if he can only choose his spaces out of the set $\{Y_n : n \in \mathbb{N}\}$. Note that by passing to $\bigcap_{i=1}^n Y_i$ we can always assume that the $Y_n$’s are decreasing in $n \in \mathbb{N}$. In case that $X$ has a separable dual and we let $(x_i^*)$ be a dense subset of $X^*$, we can put for $n \in \mathbb{N}$

$$Y_n = \{x_1^*, x_2^*, \ldots, x_n^*\}^\perp = \{x \in X : \forall i \leq n x_i^*(x) = 0\},$$

and observe that (4.6). holds for all $A \subset S^\omega_Z$.

The following result was shown in [OS1, Lemma 3.1] and its proof was based on techniques and results of W.B. Johnson, H. Rosenthal and M. Zippin [JRZ].
Lemma 4.2.10. Let \((Y_n)\) be a decreasing sequence of closed subspaces of \(X\), each having finite codimension. Then \(X\) is isometrically embeddable into a space \(Z\) having an FDD \((E_i)\) so that (we identify \(X\) with its isometric image in \(Z\))

a) \(c_0(\bigoplus_{i=n+1}^{\infty} E_i) \cap X\) is dense in \(X\).

b) For every \(n \in \mathbb{N}\) the finite codimensional subspace \(X_n = \bigoplus_{i=n+1}^{\infty} E_i \cap X\) is contained in \(Y_n\).

c) There is a \(c > 0\) so that for every \(n \in \mathbb{N}\) there is a finite set \(D_n \subset S_{\bigoplus_{i=1}^{n} E_i^*}\) such that whenever \(x \in X\)

\[
\|x\|_{X/Y_n} = \inf_{y \in Y_n} \|x - y\| \leq c \max_{w^* \in D_n} w^*(x).
\]

From (a) it follows that \(c_0(\bigoplus_{i=n+1}^{\infty} E_i) \cap X\) is a dense linear subspace of \(X_n\).

Moreover if \(X\) has a separable dual \((E_i)\) can be chosen to be shrinking (every normalized block sequence in \(Z\) with respect to \((E_i)\) converges weakly to 0, or, equivalently, \(Z^* = \bigoplus_{i=1}^{\infty} E_i^*\)), and if \(X\) is reflexive \(Z\) can also be chosen to be reflexive.

So assume that for a countable set \(B\) of games that \((Y_n)\) is a sequence of decreasing finite codimensional closed spaces satisfying the equivalences of (4.6). We then use Lemma 4.2.10 to embed \(X\) into a space \(Z\) with an FDD \((E_i)\).

Note that b) of Lemma 4.2.10 implies that for all \(A \subset S_{\omega X}\) such that we have \(W_1(A, \text{cof}(X)) \iff \forall \varepsilon \subset (0, 1) W_1(A, Z)\).

Using the embedding of \(X\) given by Lemma 4.2.10 a result similar to Proposition 4.2.7 can be shown. The proof is very similar, therefore we will only present a sketch.

Proposition 4.2.11. Assume that \(X\) is a Banach space and \(\{Y_n : n \in \mathbb{N}\}\) a decrasing sequence of cofinite dimensional subspaces. Let \(Z\) be a space with an FDD \((E_i)\) which satisfies the conclusion of Lemma 4.2.10.

Assume that \(A \subset S_{X}^\omega\) such that we have \(W_1(A, (Y_n : n \in \mathbb{N}))\) for all null sequences \(\varepsilon \subset (0, 1]\).

Then for all null sequences \(\varepsilon = (\varepsilon_i) \subset (0, 1], W_1(A, (S_{X}^\omega)\bar{\delta})\) holds, where \(\bar{\delta} = (\delta_i) = (\varepsilon_i/28cK)\), with \(c\) as in Lemma 4.2.10, \(K\) is the projection constant of \((E_i)\) in \(Z\), and where the fattenings \(A\) and \((S_{X}^\omega)\bar{\delta}\) are taken in \(Z\).

Sketch of proof. Note that instead of condition (4.4) the following condition is satisfied.

\[
\|x\|_{X/Y_m} \leq C\|P_{[1,m]}^E(x)\| \text{ whenever } x \in S_X.
\]
Also note that $W_I(\mathcal{A}_\varepsilon, \{Y_n : n \in \mathbb{N}\})$ means that every tree $(x_\alpha) \subset S_X$, with the property that for $\alpha = (m_1, m_2, \ldots, m_\ell) \in T_\infty$ we have that $x_\alpha \in Y_{m_\ell}$, has a branch in $\mathcal{A}_\varepsilon$.

We follow the proof of Proposition 4.2.7 until choosing the $x_\alpha$'s which we will not choose in $X_{m_\ell}$ but in $Y_{m_\ell}$ instead. Then the proof continues as the proof of Proposition 4.2.7.

Using Proposition 4.2.11 and Theorem 4.2.5 we deduce the following.

**Corollary 4.2.12.** Let $\mathcal{A} \subset S_X^{\omega}$ and assume that $Z$ is a space with an FDD $(E_i)$ which contains $X$ and satisfies the conclusion of Lemma 4.2.10.

Then the following conditions are equivalent:

a) For all null sequences $\varepsilon = (\varepsilon_n) \subset (0, 1]$, $W_I(\mathcal{A}_\varepsilon^{X}, \text{cof})$ holds.

b) For all null sequences $\varepsilon = (\varepsilon_n) \subset (0, 1]$, $W_I(\mathcal{A}_\varepsilon^{X}, Z)$ holds.

c) For all null sequences $\varepsilon = (\varepsilon_n) \subset (0, 1]$, there exists a blocking $(G_n)$ of $(E_n)$ so that every $\varepsilon/420CK$-skipped block sequence $(z_n) \subseteq X$ of $(G_n)$ is in $\mathcal{A}_\varepsilon$.

In the case that $X$ has a separable dual (a), (b) and (c) are equivalent to the following condition (which is independent of the choice of $Z$).

d) For all null sequences $\varepsilon = (\varepsilon_n) \subset (0, 1]$ every weakly null tree in $S_X$ has a branch in $\mathcal{A}_\varepsilon$.

Moreover, in the case that $X$ has a separable dual we deduce from the remarks after the equivalence (4.6), Corollary 4.2.9 and Proposition 4.2.8 that above equivalences hold for any embedding of $X$ into a space $Z$ having a shrinking FDD.
4.3 Embedding and Universality Theorems

We will use the main Theorems of section 4.2 to derive results of the following type:

a) Given a Banach space of a certain class, can it be embedded into a Banach space of the same class, or some closely related class, which has a basis or an FDD?

b) Given a class $\mathcal{C}$ of spaces. Does there exist a Banach space $X$ of this class $\mathcal{C}$, or of a closely related class $\tilde{\mathcal{C}}$, which is universal for $\mathcal{C}$, i.e. contains an isomorphic copy of every element of $\mathcal{C}$?

The fact that every separable infinite dimensional Banach space $X$ embeds into $C[0,1]$, and, thus that $C[0,1]$ is universal for all separable Banach spaces, dates back to the early days of Banach space theory [Ba, Théorème 9, page 185]. Pełczyński [Pe] showed that there is a Banach space having a basis/unconditional basis, which is complementably universal for all Banach spaces having basis/unconditional bases.

An example for the second type of question was answered by Zippin.

**Theorem 4.3.1.** [Z2] Every Banach space with a separable dual is embeddable into a Banach space with shrinking basis.

Every reflexive Banach space is embeddable into a reflexive space with basis.

In this section we interested in characterizing the property of a Banach space (which or may not have an FDD) into spaces with an FDD satisfying $C$-$(p,q)$-estimates.

**Definition 4.3.2.** Let $1 \leq q \leq p \leq \infty$ and $C < \infty$. A (finite or infinite) FDD $(E_i)$ for a Banach space $Z$ is said to satisfy $C$-$(p,q)$-estimates if for all $n \in \mathbb{N}$ and block sequences $(x_i)_{i=1}^n$ w.r.t. $(E_i)$,

$$C^{-1} \left( \sum_{i=1}^n \|x_i\|_p \right)^{1/p} \leq \left\| \sum_{i=1}^n x_i \right\| \leq C \left( \sum_{i=1}^n \|x_i\|_q \right)^{1/q}. \quad (4.9)$$

The *coordinate free version* of the $C$-$(p,q)$-estimates is the following

**Definition 4.3.3.** Let $1 \leq q \leq p \leq \infty$ and $C < \infty$. A space $X$ satisfies $C$-$(p,q)$-tree estimates if for all weakly null trees in $S_X$ there exist branches $(x_i)_{i=1}^\infty$ and $(y_i)_{i=1}^\infty$ satisfying for all $(a_i) \in c_0$,

$$C^{-1} \left( \sum_{i=1} |a_i|^p \right)^{1/p} \leq \left\| \sum_{i=1} a_i x_i \right\| \quad \text{and} \quad \left\| \sum_{i=1} a_i y_i \right\| \leq C \left( \sum_{i=1} |a_i|^q \right)^{1/q}. \quad (4.9)$$

If $X \subseteq Y^*$, a separable dual space, we say that $X$ satisfies $C$-$(p,q)$-w$^*$-tree estimates if each w$^*$ null tree in $S_X$ admits branches $(x_i)$ and $(y_i)$ satisfying (4.9).

We will say that $X$ satisfies $(p,q)$-tree estimates if it satisfies $C$-$(p,q)$-tree estimates for some $C < \infty$ and similarly for $(p,q)$-w$^*$ tree estimates.
In the definition of $q$-upper and $p$-lower tree estimates it is actually not necessary to assume that $C$ exists uniformly for all trees as the following proposition shows.

**Proposition 4.3.4.** [OSZ, Proposition 1.2] Let $1 \leq q \leq p \leq \infty$. Assume that $X$ is a Banach space with the property that every normalized, weakly null tree in $X$ has a branch which dominates the $\ell_p$-unit vector basis and a branch which is dominated by the $\ell_q$-unit vector basis. Then $X$ satisfies $(p,q)$-tree estimates.

**Proof.** For $C \geq 1$ define

$$A^{(C)} = \left\{ (x_n) \subset S_X : (x_n) \text{is } C\text{-basic and for all } (a_i) \in c_0 \text{ } \left( \sum |a_i|^p \right)^{1/p} \leq \| \sum_{i=1}^{\infty} x_i \| \leq \left( \sum |a_i|^q \right)^{1/q} \right\}.$$

We consider the $(A^{(C)},\text{cof})$-game on $X$. Assuming that for every weakly null tree there is a $C$ and a branch which is in $A^{(C)}$, we need to show that there is a uniform $C$ working for all trees.

Assume that this is not true.

By Corollary 4.2.12 we conclude that for any $C > 1$ Player II has a winning strategy for the $(A^{(C)},\text{cof})$-game (since $A^{(C)}$ is closed under the product of the discrete topology on $S_X$, this game is determined by Theorem 1.2.4, and also note that for $\varepsilon > 0$ there are nullsequences $\delta = (\delta_i) \subset (0,1)$ and $\eta = (\eta_i) \subset (0,1)$ so that $A^{(C)}_{\delta} \subset A^{(C+\varepsilon)}_{\eta} \subset \overline{A^{(C)}_{\eta}}$).

Player II could choose a sequence $(C_n)$ in $\mathbb{R}^+$ which increases to $\infty$ and could play the following strategy: first he follows his strategy for achieving a sequence $(x_n)$ outside of $A^{(2C_1)}$ and after finitely many steps $s_1$ he must have chosen a sequence $x_1, x_2, \ldots, x_{s_1}$ which is either not $C_1$-basic or does satisfy one of the two required inequalities for some $a = (a_i)_{i=1}^{s_1} \in \mathbb{R}^{s_1}$. Then Player II follows his strategy for getting a sequence outside of $A^{(2C_3)}$, and continues that way using $C_3, C_4$ etc. It follows that the infinite sequence $(x_n)$, which is obtained by Player II cannot be in any $A^{(C)}$. Therefore Player II has a winning strategy for choosing a sequence outside of $\bigcup_{C \geq 1} A^{(C)}$ which means that there are weakly null tree $(z_n)$ none of whose branches is in $\bigcup_{C \geq 1} A^{(C)}$. □

The following result is a typical *Embedding Theorem* on can show using the results of Section 4.2

**Theorem 4.3.5.** [OS3] Let $X$ be a reflexive and separable Banach space and let $1 < p < \infty$. Assume that every weakly null tree in $S_X$ has branch which is equivalent to the unit vector basis of $\ell_p$.

Then there is a sequence of finite dimensional spaces $(G_n)$ so that $X$ can be isomorphically embedded into $(\oplus G_n)_{\ell_p}$. 

**Remark.** In [J01] it was shown that a subspace $X$ of $L_p$ can be embedded into $(\oplus G_n)_{\ell_p}$ if and only if there is a $C \geq 1$ every normalized weakly null sequence has a subsequence which is $C$-equivalent the $\ell_p$-unit vector basis.
This latter condition is in general (i.e. for reflexive spaces which are not subspaces of \( L_p \)) not sufficient to imply embedability into \((\oplus G_n)_{\ell_p}\). For an example see [OS3].

We will show the following generalization of Theorem 4.3.5

**Theorem 4.3.6.** Let \( X \) and \( Y \) be Banach spaces, assume that \( X \) is reflexive, let \( V \) be a Banach space with a sub-symmetric and normalized basis \((v_i)\) and let \( T : X \to Y \) be linear and bounded.

Assume that for some \( C \geq 1 \) every normalized weakly null tree of \( X \) has a branch \((x_n)\) so that

\[
\| \sum_{i=1}^{\infty} a_n x_n \|_X \sim_C \| \sum_{i=1}^{\infty} a_n v_n \|_V \vee \| T \left( \sum_{i=1}^{\infty} a_n x_n \right) \|_Y \quad \text{whenever} \quad (a_i) \in c_{00}.
\]

Then there is a sequence of finite dimensional spaces \((G_i)\) so that \( X \) is isomorphic to a subspace of \((\oplus_{i=1}^{\infty} G_i) \oplus Y\).

More precisely, if \( Z \) is any reflexive space with an FDD \((E_i)\) which contains a copy of \( X \) (such a space \( Z \) always exists \([Z2]\)) and if \( S : X \to Z \) is an isomorphic embedding, then there is a blocking \((G_i)\) so that \( S \) is a bounded linear operator from \( X \) to \((\oplus_{i=1}^{\infty} G_i) \oplus Y\) and the operator

\[
(S, T) : X \to (\oplus_{i=1}^{\infty} G_i) \oplus Y, \quad x \mapsto (S(x), T(x)),
\]

is an isomorphic embedding.

We will need the following Lemma which uses a blocking trick of Johnson \([Jo1]\).

**Lemma 4.3.7.** Let \( X \) be a subspace of a space \( Z \) having a boundedly complete FDD \((E_i)\) with projection constant \( K \) with \( B_X \) being a \( w^* \)-closed subset of \( Z \). Let \( \delta_i \downarrow 0 \). Then there exists a blocking \((F_i)\) of \((E_i)\) given by \( F_i = \oplus_{j=N_{i-1}+1}^{N_i} E_j \) for some \( 0 = N_0 < N_1 < \cdots \) with the following properties. For all \( x \in S_X \) there exists \((x_i)_{i=1}^{\infty} \subseteq X \) and for all \( i \in \mathbb{N} \) there exists \( t_i \in (N_{i-1}, N_i) \) satisfying \( (t_0 = 1 \text{ and } t_1 > 1) \)

\[
\begin{align*}
\text{a)} \quad & x = \sum_{j=1}^{\infty} x_j, \\
\text{b)} \quad & \| x_i \| < \delta_i \text{ or } \| P_{(t_{i-1}, t_i)}^E x_i - x_i \| < \delta_i \| x_i \|, \\
\text{c)} \quad & \| P_{(t_{i-1}, t_i)}^E x - x_i \| < \delta_i, \\
\text{d)} \quad & \| x_i \| < K + 1, \\
\text{e)} \quad & \| P_{t_i}^E x \| < \delta_i.
\end{align*}
\]

Moreover, the above hold for any blocking of \((F_i)\) (which would redefine the \( N_i \)'s).
Proof. We observe that for all $\varepsilon > 0$ and $N \in \mathbb{N}$ there exists $n > N$ such that if $x \in B_X$, $x = \sum y_i$ with $y_i \in E_i$ for all $i$, then there exists $t \in (N, n)$ with
\[
\|y_t\| < \varepsilon \quad \text{and} \quad \text{dist}\left(\sum_{i=1}^{t-1} y_i, X\right) < \varepsilon .
\]

Indeed, if this was not true for any $n > N$ we can find $y^{(n)} \in B_X$ failing the conclusion for $t \in (N, n)$. Choose a subsequence of $(y^{(n)})$ converging $w^*$ to $y \in X$ and choose $t > N$ so that $\|P_{[t,\infty)}^E y\| < \varepsilon /2K$. Then choose $y^{(n)}$ from the subsequence so that $t < n$ and $\|P_{[1,n)}^E (y - y^{(n)})\| < \varepsilon /2K$. Thus
\[
\|P_{[1,t)}^E y^{(n)} - y\| \leq \|P_{[1,t)}^E (y^{(n)} - y)\| + \|P_{[t,\infty)}^E y\| < \frac{\varepsilon}{2K} + \frac{\varepsilon}{2K} < \varepsilon .
\]

Also
\[
\|P_t^E y^{(n)}\| \leq \|P_t^E (y^{(n)} - y)\| + \|P_t^E y\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .
\]

This contradicts our choice of $y^{(n)}$.

Let $\varepsilon_i \downarrow 0$ and by the observation choose $0 = N_0 < N_1 < \cdots$ so that for all $x \in S_X$ there exists $t_i \in (N_i-1, N_i)$ and $z_i \in X$ with $\|P_t^E x\| < \varepsilon_i$ and $\|P_{[t_i,t_i-1)}^E x - z_i\| < \varepsilon_i$ for all $i \in \mathbb{N}$. Set $x_1 = z_1$ and $x_i = z_i - z_{i-1}$ for $i > 1$. Thus $\sum_{i=1}^n x_i = z_n \rightarrow x$ so a) holds. Also
\[
\|P_{(t_i-1,t_i)}^E x - x_i\| \leq \|P_{[1,t_i)}^E x - z_i\| + \|P_{[1,t_i-1]}^E x - z_{i-1}\| < \varepsilon_i + 2\varepsilon_{i-1} ,
\]

and
\[
\|P_{(t_i-1,t_i)}^E x_i - x_i\| = \|(I - P_{(t_i-1,t_i)}^E)(x_i - P_{(t_i-1,t_i)}^E x)\| < (K + 1)(\varepsilon_i + 2\varepsilon_{i-1}) .
\]

From these inequalities b), c) and d) follow if we take $(\varepsilon_i)$ so that $(K + 1)(\varepsilon_i + 2\varepsilon_{i-1}) < \delta_i^2$. 

**Proof of Theorem 4.3.6.** By Zippin’s theorem [Z2] we can assume that $X$ is the subspace of a reflexive space $Z$ with an FDD $E = (E_i)$. After renorming we can assume that the projection constant $K = \sup_{m \leq n} \|P_{[m,n]}^E\| = 1$. We also assume without loss of generality that $\|T\| \leq 1$.

For a sequence $\pi = (x_i) \in S_X$ and $a = \sum a_i e_i \in c_{00}$ we define
\[
\|\sum a_i e_i\|_{(V,T,\pi)} = \|\sum a_i v_i\|_V \vee \left\|T \left(\sum a_i x_i\right)\right\|_Y .
\]

Then $\|\cdot\|_{(V,T,\pi)}$ is a norm on $c_{00}$ and we denote the completion of $c_{00}$ with respect to $\|\cdot\|_{(V,T,\pi)}$ by $X(V,T,\pi)$.

Define
\[
\mathcal{A} = \left\{\pi = (x_n) \subset S_X : \pi \text{ is } \frac{3}{2}\text{-basic and } \frac{3}{2}C\text{-equivalent to } (e_i) \text{ in } X(V,T,\pi)\right\} .
\]
By Corollary 4.2.12 applied to an appropriate sequence \( \varepsilon = (\varepsilon_i) \subset (0, 1) \) we can find a blocking \( F = (F_i) \) of \( (E_i) \) and a sequence \( (\delta) \subset (0, 1) \), so that every \( \delta \)-skipped block \( (x_i) \subset S_X \) of \( (F_i) \) is 2-basic and \( 2C \)-equivalent to \( (e_i) \) in \( X(V, T, \varepsilon) \). Now we apply Lemma 4.3.7 to get a further blocking \( (G_i) \), \( G_i = \bigoplus_{j=N_i}^{N_i+1} F_j \), for \( i \in \mathbb{N} \) and some sequence \( 0 = N_0 < N_1 < N_2 \ldots \), so that for every \( x \in S_X \) there is a sequence \( (t_i) \subset \mathbb{N} \), with \( t_i \in (N_{i-1}, N_i] \) for \( i \in \mathbb{N} \), and a sequence \( (x_i) \) satisfying (a)-(e).

We also may assume that \( \sum_{i=1}^{\infty} \delta_i < 1/36C \) and will show that for every \( x \in X \)

\[
\|x\|_X \sim_{36C} \left( \left\| \sum_{i=1}^{\infty} \| P_i^G(x) \|_V \right\|_V \right) \vee \|T(x)\|_Y.
\]

This implies that the map \( X \to (\bigoplus G_i)_V \oplus Y, \quad x \mapsto ((P_i^G(x)), T(x)) \), is an isomorphic embedding.

Let \( x \in S_X \) and choose \( (t_i) \subset \mathbb{N} \) and \( (x_i) \subset X \) as prescript in Lemma 4.3.7. Letting \( B = \{ i \geq 2 : \| P_{(t_{i-1}, t_i)} \|_V (x_i - x_i) \leq \delta_i \| x_i \| \} \) it follows that \( (x_i)_{i \in B} \) is a \( \delta \)-skipped block and therefore

\[
\left\| \sum_{i \in B} x_i \right\|_X \sim_{2C} \left( \left\| \sum_{i \in B} \| x_i \|_V \right\|_V \right) \vee \left\| T \left( \sum_{i \in B} x_i \right) \right\|_Y.
\]

If \( \|x_1\| \geq 1/8C \) then we deduce that

\[
\frac{1}{8C} \leq \|x_1\| \leq \left( \left\| \sum_{i=1}^{\infty} \| x_i \|_V \right\|_V \right) \vee \|T(x)\|_V \leq \left( \left\| \sum_{i \in B} \| x_i \|_V \right\|_V + \|x_1\| + \sum_{i \notin B} \delta_i \right) \vee \|T(x)\|_Y \leq 2C \left\| \sum_{i \in B} x_i \right\| + 2 + \sum_{i \notin B} \delta_i \quad \text{[By (4.12) and (d) of Lemma 4.3.7]} \leq 2C \|x\| + 2C \|x_1\| + 3C \sum \delta_i + 2C \leq 9C.
\]
If \( \|x_1\| < 1/8C \) then

\[
\frac{1}{4C} \leq \left\| \sum_{i \in B} \|x_i\| v_i \right\|_V \vee \left( \left\| T\left( \sum_{i \in B} x_i \right) \right\|_Y - \frac{1}{4C} \right) \quad \text{[By (4.12)]}
\]

\[
\leq \left\| \sum_{i \in B} \|x_i\| v_i \right\|_V \vee \left\| T(x) \right\|_Y \quad \text{[Since \( \|T\| \leq 1 \)}
\]

\[
\leq \left\| \sum_{i=1}^{\infty} \|x_i\| v_i \right\|_V \vee \left\| T\left( \sum_{i \in B} x_i \right) \right\|_Y + \frac{1}{4C}
\]

\[
\leq 2C \left\| \sum_{i \in B} x_i \right\| + \frac{1}{4C} \quad \text{[By (4.12)]}
\]

\[
\leq 2C \|x\| + 2C \|x_1\| + 2C \sum_{i} \delta_i + \frac{1}{4C} \leq 4C.
\]

(4.13) and (4.14) imply that

\[
(4.15) \quad 1 \sim_{9C} \left\| \sum_{i=1}^{\infty} \|x_i\| v_i \right\|_V \vee \left\| T(x) \right\|.
\]

For \( n \in \mathbb{N} \) define \( y_n = P_{(t_{n-1}, t_n]}^F(x) \). From Lemma 4.3.7 (c) and (e) it follows that

\[
\|y_n - x_n\| \leq \|P_{(t_{n-1}, t_n]}^F(x) - x_n\| + \|P_{t_n}^F(x)\| \leq 2\delta_n \text{ and thus } \sum \|y_n - x_n\| \leq 1/18C
\]

which implies by (4.15) that

\[
(4.16) \quad 1 \sim_{18C} \left\| \sum_{i=1}^{\infty} \|y_i\| v_i \right\|_V \vee \left\| T(x) \right\|.
\]

Since for \( n \in \mathbb{N} \) we have \( (N_{n-1}, N_n] \subset (t_{n-1}, t_{n+1}) \) and \( (t_{n-1}, t_n] \subset (N_{n-2}, N_n) \) (put \( N_1 = N_0 = 0 \) and \( P_0^H = 0 \)) it follows from the assumed sub-symmetry of \((v_n)\) and the assumed bi-monotonicity of \((E_i)\) in \( Z \) that

\[
\frac{1}{2} \left\| \sum_{n \in \mathbb{N}} \|y_n\| v_n \right\|_V \leq \frac{1}{2} \left\| \sum_{n \in \mathbb{N}} (\|P_{n-1}^G(x)\| + \|P_n^G(x)\|) v_n \right\|_V
\]

\[
\leq \left\| \sum_{n \in \mathbb{N}} \|P_n^G(x)\| v_n \right\|_V
\]

\[
\leq \left\| \sum_{n \in \mathbb{N}} \|P_{(t_{n-1}, t_n]}^F(x)\| v_n \right\|_V
\]

\[
\leq \left\| \sum_{n \in \mathbb{N}} (\|y_n\| + \|y_{n+1}\|) v_n \right\|_V \leq 2 \left\| \sum_{n \in \mathbb{N}} \|y_n\| v_n \right\|_V.
\]
which implies with (4.16) that

\[ 1 \sim_{36C} \left\| \sum_{i=1}^{\infty} \|x_i\|v_i\right\|_Y \vee \|T(x)\|. \]

and finishes the proof of our claim. \qed

**Example 4.3.8.** Let \( 1 < p < \infty \). There exists a reflexive space \( X \) with an unconditional basis so that \( X \) satisfies: for all \( \varepsilon > 0 \) every normalized weakly null sequence in \( X \) admits a subsequence \( 1 + \varepsilon \)-equivalent to the unit vector basis of \( \ell_p \). Yet \( X \) is not a subspace of an \( \ell_p \)-sum of finite dimensional spaces.

**Proof.** Fix \( 1 < q < p \). We define \( X = (\sum X_n)_p \) where each \( X_n \) is given as follows. \( X_n \) will be the completion of \( c_{00}(\mathbb{N}^{\leq n}) \) under the norm

\[ \|x\|_n = \sup \left\{ \left( \sum_{i=1}^{m} \|x|_{\beta_i}^q\right)^{1/p} : (\beta_i)_{1}^{m} \text{ are disjoint segments in } [\mathbb{N}]^{\leq n} \right\}. \]

By a segment we mean a sequence \( (A_i)_{i=1}^{k} \in [\mathbb{N}]^{\leq n} \) with \( A_1 = \{n_1,n_2,\ldots n_{\ell}\} \), \( A_2 = \{n_1,n_2,\ldots n_{\ell},n_{\ell+1}\} \) \ldots \( A_k = \{n_1,n_2,\ldots n_{\ell},n_{\ell+1}\ldots n_{\ell+k-1}\} \), for some \( n_1 < n_2 < \ldots n_{\ell+k-1} \). Thus a segment can be seen as an interval of a branch (with respect to the usual partial order in \([\mathbb{N}]^{\leq n}\)), while a branch is a maximal segment.

Clearly the node basis \( (e^{(n)}_{A})_{A\in[\mathbb{N}]^{\leq n}} \) given by \( e^{(n)}_{A}(B) = \delta_{(A,B)} \) is a 1-unconditional basis for \( X_n \). Furthermore the unit vector basis of \( \ell_q^n \) is 1-equivalent to \( (e^{(n)}_{A_i})_{i=1}^{n} \), if \( (A_i)_{i=1}^{n} \) is any branch of \([\mathbb{N}]^{\leq n}\).

Thus no extension of the tree \( (e^{(n)}_{A})_{A\in[\mathbb{N}]^{\leq n}} \) to a weakly null tree of infinite length in \( S_X \) has a branch whose basis distance to the \( \ell_p \)-unit vector basis is closer than \( \text{dist}_b(\ell_{p,n},\ell_{q,n}) = n^{\frac{1}{2}}-\frac{1}{2} \to \infty \) for \( n \to \infty \). Since it is clear that in every subspace \( Y \) of an \( \ell_p \)-sum of finite dimensional spaces every weakly null tree in \( S_Y \) must have a branch equivalent (for a fixed constant) to the unit vector basis of \( \ell_p \) it follows that \( X \) cannot be embedded into a subspace of an \( \ell_p \)-sum of finite dimensional spaces.

Also each \( X_n \) is isomorphic to \( \ell_p \) and thus \( X \) is reflexive.

It remains to show that if \( (x_j) \) is a normalized weakly null sequence in \( X \) and \( \varepsilon > 0 \) then a subsequence is \( (1 + \varepsilon) \)-equivalent to the unit vector basis of \( \ell_p \). This will essentially follow from the following lemma.

We say that a Banach space has property \( (\ast) \) is

\[ \forall (x_n) \subset S_X\varepsilon > 0 \exists (x'_n) \subset (x_n) \ (x_n) \text{ is } (1 + \varepsilon) \text{-equiv. to } \ell_p \text{-u.v.b.} \]

**Lemma 4.3.9.** If \( Y_n \) has property \( (\ast) \) for all \( n \in \mathbb{N} \) then \( (\oplus Y_n)_{\ell_p} \) has property \( (\ast) \).

Firstly we use Lemma 4.3.9 to reduce our claim that \( X \) has property \( (\ast) \) to the claim that each \( X_n \) has property \( (\ast) \). Then we show by induction for each \( n \in \mathbb{N} \) that \( X_n \) has property \( (\ast) \). For \( n = 1 \) this is trivial. Assuming we showed that \( X_n \)
satisfies (*), we observe that \( X_{n+1} \) is the \( \ell_p \)-sum of spaces \( Y_n \) where \( Y_n \) isometrically isomorphic to a 1-dimensional extension of \( X_n \), say \( Y_n \equiv \mathbb{R} \oplus X_n \). Since every weakly null sequence \((z_k)\) in \( Y_n \) is up to passing to subsequence and small perturbation in \( X_n \), it follows that each \( Y_n \) has property (*), we apply Lemma 4.3.9 again to conclude that \( X_{n+1} \) has property (*).

\[ \square \]

**Proof of Lemma 4.3.9.** Let \( y_n \) be a normalized sequence in \( Y = \oplus_k Y_k \). For \( k \in \mathbb{N} \) denote by \( P_k \) the canonical projection of \( Y \) onto \( Y_k \) and put \( y(n, k) = P_k(y_n) \). For \( A \subset \mathbb{N} \) and \( z \in Y \) define \( P_A(z) = \sum_{k \in A} P_k(z) \).

After passing to a subsequence and an arbitrary perturbation we may assume that for each \( k \in \mathbb{N} \) \( a_k = \lim_{n \to \infty} \|y(n, k)\| \) exists and that for some sequence \((k_i)\) in \( \mathbb{N} \) \( y_n = P_{[1,k_i]}(y_n) \) for all \( n \in \mathbb{N} \).

Note that \( \alpha = \sum_{i=1}^{\infty} |a_k|^p \leq 1 \). If \( \alpha = 0 \), we can find a subsequence \((y'_n)\) which is an arbitrary perturbation of a sequence \((z_n)\) for which the sets \( A_n = \{k : P_k(z_n) \neq 0\} \) are disjoint. If \( \alpha > 0 \) we proceed as follows:

Let \( \varepsilon_i \to 0 \) fast enough (to be determined later). First choose \( \ell_1 \in \mathbb{N} \) so that \( \sum_{k > \ell_1} a_k^p < \varepsilon_1 \) and a subsequence \( N_1 \subset \mathbb{N} \) so that

\[ \left( \frac{P_{[1,\ell_1]}(y_n)}{\|P_{[1,\ell_1]}(y_n)\|} : n \in N_1 \right) \]

is \((1 + \varepsilon_1)\)-equivalent to the unit vector basis in \( \ell_p \) and choose some \( n_1 \in N_1 \).

Then choose \( \ell_2 \geq k_{n_1} \) so that \( \sum_{k > \ell_2} a_k^p < \varepsilon_2 \) and choose subsequence \( N_2 \subset N_1 \) so that

\[ \left( \frac{P_{[1,\ell_2]}(y_n)}{\|P_{[1,\ell_2]}(y_n)\|} : n \in N_2 \right), \]

is \((1 + \varepsilon_2)\)-equivalent to the unit vector basis in \( \ell_p \) and so that \( \|P_{(\ell_2,k_{n_1})}(y_n)\| \leq \varepsilon_1 \) for \( n \in N_2 \) and choose some \( n_2 \in N_2 \).

Continuing this way we deduce that \((y_{n_i})\) is \((1 - \varepsilon)\)-equivalent to the unit vector basis in \( \ell_p \) with \( \varepsilon \) depending on \( \sum_{i=1}^{\infty} \varepsilon_i \).

\[ \square \]

**Theorem 4.3.10.** Let \( X \) be a reflexive Banach space and let \( 1 \leq q \leq p \leq \infty \). The following are equivalent.

a) \( X \) satisfies \((p, q)\)-tree estimates.

b) \( X \) is isomorphic to a subspace of a reflexive space \( Z \) having an FDD which satisfies \((p, q)\)-estimates.

c) \( X \) is isomorphic to a quotient of a reflexive space \( Z \) having an FDD which satisfies \((p, q)\)-estimates.

**Sketch of a proof of (a)⇒(b).**

Step 1: Embed \( X \) in a reflexive space \( Z \) with FDD \((E_i)\). Using the same proof as in Theorem 4.3.6 we can prove:
4.3. EMBEDDING AND UNIVERSALITY THEOREMS

There is a blocking \( F = (F_i) \) of \( (E_i) \) so that \( X \) embeds in \( Z(F, p) \), which is the completion of \( c(\bigoplus_{i=1}^{\infty} F_i) \) under the following norm:

\[
\|x\|_{(z,p)} = \sup_{0 = n_0 < n_1 < \ldots < n_k, k \in \mathbb{N}} \left( \sum_{i=n_0}^{n_k} \|P_{(n_{i-1},n_i)}^F(x)\|_Z^p \right)^{1/p}.
\]

Step 2: The condition \( X \) satisfies upper \( q \)-tree estimates dualizes, i.e. \( X^* \) satisfies lower \( q' \)-tree estimates.

Step 3: Consider quotient map \( Q : Z^*_p \setminus X^* \).

For \( i \in \mathbb{N} \) let \( \tilde{F}_i \) be the quotient space of \( F^*_i \) determined by \( Q \). Thus if \( z \in F^*_i \), the norm on \( \tilde{z} \) (the equivalence class of \( z \) in \( F_i \)) is \( \|\tilde{z}\| = \|Qz\| \). We may assume \( \tilde{F}_i \neq \{0\} \) for all \( i \). More generally for \( \tilde{z} = \sum \tilde{z}_i \in c_00(\bigoplus_{i=1}^{\infty} \tilde{F}_i) \) with \( \tilde{z}_i \in \tilde{F}_i \) for every \( i \), we set

\[
\|\tilde{z}\| = \sup_{m \leq n} \left| \sum_{i=m}^{n} Qz_i \right| = \sup_{m \leq n} \|QP_{[m,n]}^F z\|.
\]

We let \( \tilde{Z} \) be the completion of \( (c_00(\bigoplus_{i=1}^{\infty} \tilde{F}_i)), \|\cdot\| \). Note that if \( \tilde{z} = \sum \tilde{z}_i \in c_00(\bigoplus_{i=1}^{\infty} \tilde{F}_i) \) then setting \( \tilde{Q} \tilde{z} = \sum Q \tilde{z}_i = \sum Qz_i \), we have \( \|\tilde{Q} \tilde{z}\| \leq \|\tilde{z}\| \). Thus \( \tilde{Q} \) extends to a norm one map from \( \tilde{Z} \) into \( X \).

a) \((\tilde{E}_i)\) is a bimonotone shrinking FDD for \( \tilde{Z} \).

b) \( \tilde{Q} \) is a quotient map from \( \tilde{Z} \) onto \( X \). More precisely if \( x \in X \) and \( z \in Z \) with \( Qz = x, \|z\| = \|x\| \), and \( z = \sum z_i \) with \( z_i \in F^*_i \), then \( \tilde{z} = \sum \tilde{z}_i \in \tilde{Z}, \|\tilde{z}\| = \|z\| \) and \( \tilde{Q} \tilde{z} = x \).

c) Let \( (\tilde{z}_i) \) be a block sequence of \( (\tilde{F}_i) \) in \( B_{\tilde{Z}} \) and assume that \( (\tilde{Q} \tilde{z}_i) \) is a basic sequence with projection constant \( K \) and \( a = \inf_i \|\tilde{Q} \tilde{z}_i\| > 0 \). Then for all scalars \( (a_i) \) we have

\[
\| \sum a_i \tilde{Q} (\tilde{z}_i) \| \leq \|\sum a_i \tilde{z}_i\| \leq \frac{3K}{a} \|\sum a_i \tilde{Q} \tilde{z}_i\|.
\]

Step 4: There is a blocking \( \tilde{H}_i \) of \( \tilde{F}_i \) so that \( \tilde{Q} \) is still a quotient map from \( \tilde{Z}(q', \tilde{H}) \) onto \( X^* \).

Step 5: \( X \) embeds into \( Z^*(q', \tilde{H}) \) and \( H^* = (\tilde{H}_i^*) \) is a an FDD of \( Z^*(q', H) \) which satisfies upper \( q \)-estimates.

Step 6: apply Step 1 again to \( Z^*(q', H) \).

\[ \square \]

In [Bo] Bourgain asked whether or not there is a reflexive space universal for all super reflexive spaces.
**Definition 4.3.11.** Let $E$ and $X$ be two Banach spaces. We say that $E$ is finitely represented in $X$ if there is a $C \geq 1$ for every finite dimensional sub space $F$ of $E$ there is a finite dimensional subspace $F'$ in $X$, so that the Banach mazur distance between $F$ and $F'$ is not larger than $C$.

$X$ is called super reflexive if every space $E$ which is finitely represented in $X$ is reflexive.

**Remark.** Since every Banach space $X$ is finitely represented in it self, every super reflexive space must be reflexive. On the other hand Tsireson space is reflexive but not super reflexive.

In order to solve Bourgain’s question we also need the following two results.

**Theorem 4.3.12.** [Pr] (Solution of Bourgain’s problem for spaces with FDD)

There exists a reflexive Banach space $X$ which is universal for all spaces with a finite dimensional decomposition (FDD) which satisfy $(p,q)$-estimates for some $1 < q \leq p < \infty$.

**Theorem 4.3.13.** [Ja3]. Let $c \geq 1$. For every super reflexive Banach space $X$ there are $1 < q \leq p < \infty$ and $C \geq 1$, so that every normalized basic sequence $(x_n)$, whose basis constant does not exceed $c$, satisfies $(p,q)$ esitmates.

Since every weakly null sequence has a subsequence whose basis constant is not larger than, say, 2. It follows that for every super reflexive Banach space $X$ there are $1 < q \leq p < \infty$ so that $X$ satisfies $(p,q)$-tree estimates.

**Remark.** From the arguments in [Ja3] it follows that separable Banach space which satisfies $(p,q)$-tree estimates for some $1 < p \leq q$ is reflexive.

But such a space does not need to be super reflexive as the example $(\oplus_{n=1}^{\infty} \ell_1^n)_{\ell_2}$ shows. There fore the following question is still open

**Question.** Does a separable super reflexive space embed in a super reflexive space with a basis, or with an FDD?

We will deduce Theorem 4.3.10 from Corollary 4.2.9 only in the spacial case that $p = q$. In this case it was already shown and answerd a problem of Johnson, asking for an intrinsic characterization of the property of a Banach space being a subspace of an $\ell_p$-sum of finite dimensional Banach spaces.
Chapter 5

Ordinal numbers

5.1 Definition of ordinal numbers

Definition 5.1.1. A well order on a set $S$ is a relation $<$ on $S$ which has the following properties.

(WO1) $(A, <)$ is a linear order, i.e. for any $a, b \in S$, one and only one of the following cases occur: Either $a < b$ or $b < a$ or $a = b$.

(WO2) Every nonempty subset $A$ of $S$ has a minimum, i.e. there is an $a_0 \in A$ so that for all $a \in A$ either $a_0 < a$ or $a_0 = a$.

In that case the pair $(S, <)$ is called a well ordering and we introduce the following notations:

For $a, b \in S$ we write $a \leq b$ if $a < b$ or $a = b$.

For $a, b \in S$, with $a \leq b$ we introduce the following intervals:

$$[a, b] = \{x \in S : a \leq x \leq b\}$$
$$[a, b) = \{x \in S : a \leq x < b\}$$
$$(a, b] = \{x \in S : a < x \leq b\}$$
$$(a, b) = \{x \in S : a < x < b\}$$

For a none empty $A \subset S$ we denote the (by (OW2) existing) minimal element of $A$ by $\min(A)$, and write $0_S = \min S$. If $a \in S$ is not a maximal element then the set of successors of $a$, namely the set

$$\text{Succ}(a) = \{x \in S : x > a\},$$

must have a minimal element which we call the direct successor of $a$ and denote it by $a^+$.

The following theorem is an easy consequence of the Hausdorff Maximal Principle.
Theorem 5.1.2. Every set can be well ordered.

Proof. Let $S$ be a set. Define

$$\mathcal{W} = \{(A, <_A) : A \subset S \text{ and } <_A \text{ well order on } A\}.$$ 

For $(A, <_A)$ and $(\tilde{A}, <_{\tilde{A}})$ in $\mathcal{W}$ we write $(A, <_A) < (\tilde{A}, <_{\tilde{A}})$ if $A \subset \tilde{A}$ and the restriction of $<_{\tilde{A}}$ to $A$ is $<_A$. Then it is easy to see that for every linear ordered subset in $\mathcal{W}$ has an upper bound. By the Hausdorff Maximal Principle there is a maximal element $(A, <_A)$ in $\mathcal{W}$, and since one could easily extend $<_A$ to one more element if $A \neq S$, it follows that $A = S$. □

Definition 5.1.3. Two $(S, <)$ and $(S', <)$ are called order isomorphic if there exists a bijective (1−1 and onto) map: $\Phi : S \to S'$ which is order preserving i.e.

$$\forall a, b \in S \quad (a < b \iff \phi(a) < \phi(b)).$$

In that case we call $\phi$ an order isomorphism between $S$ and $S'$.

Proposition 5.1.4. If $(S, <)$ and $(S', <)$ are order isomorphic well orderings, then the order isomorphism between them is unique.

Proof. W.l.o.g. $S, S' \neq \emptyset$. Let $\Psi, \Phi : S \to S'$ be two order isomorphism, and assuming that $\Psi \neq \Phi$ we can define:

$$a = \min\{x \in [0, S, a] : \Psi(x) \neq \Phi(x)\}.$$

Since $\Psi(a) \neq \Phi(a)$ we may w.l.o.g. assume that $\Psi(a) < \Phi(a)$. But now it follows that

$$\Phi(S) \subset [0, S', \Psi(a)) \cup \{x' \in S' : x' \succ \Phi(a)\},$$

(Indeed: $x < a \Rightarrow \Psi(x) = \Phi(x) \prec \Phi(a)$, and $x \geq a \Rightarrow \Psi(x) \geq \Psi(a) \succ \Phi(a)$), i.e. $\Phi(a) \not\in \Psi(S)$, which contradicts that $\Psi$ is surjective. □

Proposition 5.1.5. If $(S, <)$ is a well ordering and $a, b \in S$. Then $[0, S, a]$ and $[0, S, b]$ are order isomorph if and only if $a = b$.

Proof. Let $\Phi : [0, S, a] \to [0, S, b]$ be an order isomorphism and consider

$$A = \{x \in [0, S, a] : \Phi(x) \neq x\}.$$ 

If this set was not empty we could choose $x_0 = \min(A)$, and, using a similar argumentation as in the proof of Proposition 5.1.4 we would get contradiction to the assumed surjectivity of $\Phi$. If $A$ is empty, it follows that $\Phi$ is the identity on $[0, S, a]$, and, since $\Phi$ is supposed to be surjective it follows that $a = b$. □

Corollary 5.1.6. If $(S, <)$ and $(S', <)$ are two well orderings and there is an $a' \in S$ so that $(S, <)$ is order isomorph to $[0, S', a')$. Then such an $a' \in S'$ is unique.
5.1. DEFINITION OF ORDINAL NUMBERS

Theorem 5.1.7. Let \((S, <)\) and \((S', <')\) be two well orderings. Then one and only one of the following cases occurs.

Case 1. \((S, <)\) and \((S', <')\) are order isomorphic.

Case 2. There is an \(a \in S\) so that \([0_S, a)\) and \(S'\) are order isomorphic.

Case 3. There is an \(a' \in S'\) so that \([0_{S'}, a')\) and \(S\) are order isomorphic.

Proof. Define:

\[ A = \{ a \in S : \exists a' \in S' \quad [0_S, a] \text{ and } [0_{S'}, a'] \text{ are order isomorphic} \}. \]

Let us first assume that \(A = S\). If \(S\) has a maximal element \(a\) it follows that there exists an order isomorphism between \(S = [0_S, a]\) and some interval \([0_{S'}, b']\) (which might or might not be all of \(S'\)) which is order isomorphic to \(S\). This means that we are either in Case 1 or in Case 3 (choose \(a' = b^+\)). If \(S\) has not a maximal element we choose for \(a \in S\), an \(a' \in S'\) and an order isomorphism \(\Phi_a : [0_S, a] \to [0_{S'}, a']\) (note that by Proposition 5.1.4 and Corollary 5.1.6 \(a'\) and \(\Phi_a\) are unique for every \(a \in S\)) then choose

\[ \Phi : S \to S', a \mapsto \Phi_a(a). \]

From the uniqueness of the \(\Phi_a\)'s it follows for \(a < b \in S\) that \(\Phi_a(a) = \Phi_b(a)\). Moreover, if \(\Phi(S) = S'\) it follows that \(\Phi\) is an order isomorphism between \(S\) and \(S'\), and if \(\Phi(S) \neq S'\) it follows that \(\Phi\) is an order isomorphism between \(S\) and \(a'\), where \(a' = \min\{x' \in S' : x' \notin \Phi(S)\}\).

If \(A \neq S\) and we put \(a_0 = \min S \setminus A\). For \(a < a_0\) we choose (the uniquely existing) \(a' \in S'\) and \(\Phi_a : [0_S, a] \to [0_{S'}, a']\), where \(\Phi_a\) is an order isomorphism and claim that \(S' = \{a' : a \in S\}\). Indeed if this where not so, we could put \(a'_0 = \min S' \setminus \{a' : a \in S\}\) and define

\[ \Phi_{a_0} : [0, a_0] \to [0, a'_0], a \mapsto \begin{cases} \Phi_a(a) & \text{if } a < a_0 \\ a'_0 & \text{if } a = a_0 \end{cases} \]

and deduce from the uniqueness of the \(a'\) and \(\Phi_a\) for \(a \in S\) that \(\Phi_{a_0}\) is an order isomorphism between \([0, a_0]\) and \([0, a'_0]\), which contradicts our definition of \(a_0\).

Therefore it follows that

\[ A' = \{ a' \in S' : \exists a \in S \quad [0_S, a] \text{ and } [0_{S'}, a'] \text{ are order isomorphic} \} = S', \]

and using our previous arguments we can show that we are either in Case 1 or Case 2.

Moreover, only one of the three cases can happen. Indeed, assume Cases 1 and 2 hold. This would imply that for some \(a \in S\) the sets \(S\) and \([0_S, a)\) where order isomorphic. Thus, let \(\Phi : S \to [0_S, a)\) be an order isomorphism. But this yields that \(\Phi_{[0_S, a)} : [0_S, a] \to [0, S, \Phi(a)]\) is and order isomorphism, which implies that \(\Phi(a) = a\) which is a contradiction. Similarly we can show that Cases 1 and 3, as well as Cases 2 and 3 cannot hold at the same time. \(\square\)
Our next step will be, roughly speaking, the following: Call two well orderings equivalent if they are order isomorphic. We choose out of each equivalence class of well orderings one representant which we will call ordinal number. Since these classes are not sets we will not be able to use the Axiom of Choice to do so.

We can can proceed as follows. Note that “∈” can be thought of a (non symmetric, since $A \in A$ never holds) relation between sets.

**Definition 5.1.8.** An ordinal number is a set $\alpha$ which has the following two properties:

(Or1) $\alpha$ is well ordered by the relation $\in$, i.e. the relation $<$ on $\alpha$ defined by

\[ \beta < \gamma \iff \beta \in \gamma \text{ for } \beta, \gamma \in \alpha \]

is a well order on $\alpha$.

(Or2) Every element of $\alpha$ is also a subset of it.

If $\alpha$ is an ordinal we write $\alpha \in \text{Ord}$ (although Ord is actually not a set but a class).

**Example 5.1.9.** Let us write down the first “couple of ordinals”:

0 := $\emptyset$ (Note that the empty set is always well ordered, no matter how you define $<$)
1 := $\{\emptyset\}$
2 := $\{\emptyset, \{\emptyset\}\}$
3 := $\{\{\emptyset, \emptyset\}, \{\emptyset, \{\emptyset\}\}\}$
$\omega := \{0, 1, 2, 3, 4, \ldots\} = \bigcup_{n=0}^{\infty} n$
$\omega + 1 := \omega \bigcup \{\omega\}$
$\omega + 2 := \omega \bigcup \{\omega\}$

\[ \vdots \]
$\omega \cdot 2 := \bigcup_{n=1}^{\infty} \omega + n$
$\omega \cdot 2 + 1 := \omega \cdot 2 \bigcup \{\omega \cdot 2\}$

\[ \vdots \]

**Proposition 5.1.10.** Let $\alpha \in \text{Ord}$.

a) For all $\gamma \in \alpha$ it follows that $\gamma \in \text{Ord}$ and $\gamma = [0, \gamma)$. 
b) If \( \alpha \neq \emptyset \) then \( \emptyset \in \alpha \) and \( \emptyset = 0_{\alpha} = \min \alpha \). Therefore we will write instead of \( 0_{\alpha} \) from now on simply \( 0 \).

Proof. For \( \gamma \in \alpha \) it follows from (Or2) that:

\[
[0, \gamma) = \{ \beta \in \alpha : \beta \in \gamma \} = \{ \beta : \beta \in \gamma \} = \gamma.
\]

Since intervals of the form \([0, a)\), for a well ordering \((A, <)\) and \( a \in A \) is also a well ordering it follows that \((\gamma, \in) = [0, \gamma), \in)\) is a well ordering. For \( \beta \in \gamma = [0, \gamma) \subset \alpha \) it follows that

\[
\beta = [0, \beta) \subset [0, \gamma) = \gamma,
\]

thus \( \gamma \) also satisfies (Or2), which finishes the proof of (a). Since

\[
\emptyset = [0, 0_{\alpha}) = 0_{\alpha},
\]

we deduce (b).

In order to prove that the class of ordinal numbers is well ordered itself, we will need the following Well Foundedness Principle which is needed to build up Set theory.

**The Well Foundedness Principle.** Every nonempty set \( A \) contains an element \( a \) for which \( A \cap \{a\} = \emptyset \).

The next Proposition that

**Theorem 5.1.11.** The class of ordinals itself is well ordered by \( \in \).

Proof. Let \( \alpha, \beta \in \text{Ord} \). We need to show that either \( \alpha \in \beta \) or \( \beta \in \alpha \) or \( \alpha = \beta \). By Theorem 5.1.7 we can w.l.o.g assume that there is an injective and order preserving embedding map \( \Phi : \alpha \to \beta \) whose image is \([0, \beta)\) for some \( \beta \in \text{beta} \) or \( \beta = \beta \) (in the later case we define \([0, \beta) = \beta\)). We need to show that \( \alpha = \beta \) and \( \Phi(\gamma) = \gamma \) for all \( \gamma \in \alpha \). Assuming that this is not true we could choose

\[
\gamma_0 = \min \{ \gamma \in \alpha : \Phi(\gamma) \neq \gamma \},
\]

and deduce from Proposition

\[
\Phi(\gamma_0) = [0, \Phi(\gamma_0)) = [0, \gamma_0) = \gamma_0,
\]

which is a contradiction. So we deduce (WO1).

In order to show (WO2), let \( A \subset \text{Ord} \) a non empty subset of the (class) Ord. By the well foundedness principle there is an \( \alpha \in A \) which is disjoint from \( A \) (as sets). Therefore it must follow that \( \alpha \in \beta \) for all \( \beta \in A \setminus \{\alpha\} \). Indeed, otherwise it would follow from (WO1) that there is a \( \beta \in \alpha \cap A \). Thus, we showed (WO2).

To show our last claim, we let \((S, <)\) be a well ordering and assume that there is no \( \alpha \in \text{Ord} \) which is order isomorphic to \( S \). We can there for define \( 0 \).
Corollary 5.1.12. If $\alpha \in \text{Ord}$ then $\alpha^+ = \alpha \cup \alpha$ is also an ordinal.

If $A$ is a set of ordinals then,

$$\sup(A) = \bigcup \alpha^+$$

is also an ordinal and is the smallest ordinal which contains all elements of $A$.

Proof. The first claim is clear. The claim that $\sup(A)$ is an ordinal follows immediately from Theorem 5.1.11.

Theorem 5.1.13. For any well ordering $(S, <)$ there is a (unique) ordinal $\alpha$ which is order isomorphic to $S$. 

\[ \square \]
5.2 Arithmetic of ordinals

5.3 Classification of countable compacts by the Cantor Bendixson index

**Proposition 5.3.1.** Every countable compact space is metrizable.

**Lemma 5.3.2.** In a complete separable metric space there is no strictly descending chain of closed subsets indexed by $\omega_1$.

**Lemma 5.3.3.** Every non empty, countable and complete metric space has isolated points

*Proof.* Baire category theorem. \qed

**Lemma 5.3.4.** If $X$ is a compact space and has infinitely many points then it has accumulation points.

Lemma above allows us to define the Cantor Bendixson index for compact
Chapter 6

Banach indices

6.1 General definition of indices

6.2 Szlenk’s index

6.3 Classification of $C(K)$, $K$ countable compact
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