COEFFICIENT QUANTIZATION IN BANACH SPACES

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ABSTRACT. Let (e_i) be a dictionary for a separable Banach space X. We consider the problem of approximation by linear combinations of dictionary elements with quantized coefficients drawn usually from a 'finite alphabet'. We investigate several approximation properties of this type and connect them to the Banach space geometry of X. The existence of a total minimal system with one of these properties, namely the *coefficient quantization property*, is shown to be equivalent to X containing c_0.

CONTENTS

Date: April 20, 2006.

The research of the second and third authors was supported by the NSF. The first, second, and fourth authors were supported by the Linear Analysis Workshop at Texas A&M University in 2005. All authors were supported by BIRS..

1. Introduction

We begin with the problem which motivates this paper. Let $(X, \|\cdot\|)$ be a separable infinite-dimensional Banach space and let (e_i) be a semi-normalized dictionary for X (i.e. (e_i) has dense linear span in X). For a given choice of $N \in \mathbb{N}$, consider the problem of approximating an element $x \in X$ by an element of the 'lattice'

$$\mathcal{D}^N((e_i)) = \{\sum_{i\in E} \frac{k_i}{2^N} e_i \colon k_i \in \mathbb{Z}, E \subset \mathbb{N} \text{ finite}\}.$$

In many situations (e.g. when (e_i) is a Schauder basis for X) each coefficient $k_i/2^N$ of an approximant from $\mathcal{D}^N((e_i))$ will be bounded by a constant that depends only on (e_i) and $\|x\|$. In this case the approximant will be chosen from a collection of vectors in $\mathcal{D}^N((e_i))$ whose coefficients are quantized by a 'finite alphabet'.

We investigate two natural approximation properties which (e_i) might have. The first of these, which we call the *Coefficient Quantization Property* (abbr. CQP), is defined roughly as follows: for every prescribed tolerance there exists a quantization such that every vector $x = \sum_{i\in E} a_i e_i$ in X that can be expressed as a finite linear combination of dictionary elements can be approximated by a quantized vector $y = \sum_{i\in E} d_i e_i$ with the same (or possibly smaller) support E. Thus, for each $\varepsilon > 0$, there exists N such for every x with finite support E there exists $y \in \mathcal{D}^N((e_i))$ supported in E such that $\|x - y\| \le \varepsilon$.

Precise definitions and some useful permanence properties are presented in Section 2. One of our main results (Theorem 2.4) is the perhaps surprising fact that quantization of the unit ball for some $\varepsilon < 1$ automatically implies quantization of the whole space.

Several examples of bases with the CQP, including the Schauder system for $C([0,1])$ and a class of bases for $C(K)$, where K is a countable compact metric space, are discussed in Section 3. On the other hand, it is shown that the Haar basis for $C(\Delta)$, where Δ denotes the Cantor set, is *not* a CQP basis. It turns out that all of the natural examples satisfy a stronger form of the CQP which we call the *Strong Coefficient Quantization Property.* Roughly, this means that the quantization of each coefficient can be an arbitrary δ-net, not necessarily a discrete subgroup of $\mathbb{R}$.

The main results of the paper are summarized in the following theorem.

Main Theorem. *Let X be a separable Banach space. Then X has a fundamental and total normalized minimal system with the CQP if and only if c_0 is isomorphic to a subspace of X. Moreover, if X has a basis then X has a normalized weakly null basis with the CQP if and only if X contains an isomorph of c_0.*

The sufficiency is proved in Section 4 (Theorem 4.1) and the necessity is proved in Section 6 (Theorem 6.1). The necessity result is stated more precisely as the following dichotomy : if (e_i) is a fundamental and total minimal system with the CQP then some subsequence of (e_i) is equivalent to the unit vector basis of c_0 or to the summing basis of c_0.

For the reader who wishes to make a beeline for the proof of the Main Theorem we suggest a shorter route through the paper. After absorbing the definitions of the CQP and SCQP in Section 2 and the NQP in Section 5, he or she should then read Section 4, Theorem 5.11 (which is very short), and Section 6.

The second natural approximation property, which we call the *Net Quantization Property* (abbr. NQP), is investigated in Section 5. We say that (e_i) has the NQP if for every $\varepsilon > 0$ there exists N such that $\mathcal{D}^N((e_i))$ is an ε-net for X. We prove that the NQP is a weaker property than the CQP. In particular, while the CQP is preserved under the operation of passing to a subsequence, this is not the case for the NQP. Indeed, we prove (Theorem 5.9) that *every* normalized monotone basic sequence may be embedded as a subsequence of a Schauder basis with the NQP. Another main result of Section 5 is related to the *greedy algorithm* in Banach spaces (see e.g. [7]). It is proved that the unit vector basis of c_0 is the only *quasi-greedy* NQP minimal system.

We do not know whether or not every space X with an NQP basis contains c_0. However, we are able to prove the weaker result that if X admits a minimal system with the NQP then the dual space of X contains an isomorphic copy of ℓ_1 (Theorem 5.18). In particular, X is necessarily non-reflexive.

The last section contains some examples and questions of a finite-dimensional character that are related to the CQP.

Standard Banach space notation and terminology are used throughout (see e.g. [14]). For the sake of clarity, however, we recall the notation that is used most heavily. Let $(X, \|\cdot\|)$ be a real Banach space with *dual space* X^*. The *unit ball* of X is the set $Ba(X) := \{x \in X \colon \|x\| \leq 1\}$. We write $Y \hookrightarrow X$ (where $(Y, \|\cdot\|)$ is another Banach space) if there exists a continuous linear isomorphism from Y into X.

Let (e_i) be a sequence in X. The closed linear span of (e_i) is denoted $[(e_i)]$. We say that (e_i) is *weakly Cauchy* if the scalar sequence $(x^*(e_i))$ converges for each $x^* \in X^*$. We say that (e_i) is *nontrivial weakly Cauchy* if (e_i) is weakly Cauchy but not weakly convergent, i.e. (e_i) congerges weak-star to an element of $X^{**} \setminus X$. We say that a sequence (e_i) of nonzero vectors is *basic* if there exists a positive constant K such that

$$\|\sum_{i=1}^{m} a_i e_i\| \leq K \|\sum_{i=1}^{n} a_i e_i\|$$

for all scalars (a_i) and all $1 \leq m \leq n \in \mathbb{N}$; the least such constant is called the *basis constant*; (e_i) is *monotone* if we can take $K = 1$; (e_i) is *C-unconditional*, where C is a positive constant C, if

$$\|\sum_{i=1}^{n} \varepsilon_i a_i e_i\| \leq C \|\sum_{i=1}^{n} a_i e_i\|$$

for all scalars (a_i), all choices of signs $\varepsilon_i = \pm 1$, and all $n \geq 1$. The least such constant is called the *constant of unconditionality*. We say that (e_i) is a (Schauder) basis for X if (e_i) is basic and $[(e_i)] = X$. Two basic sequences (e_i) and (f_i) are said to be *equivalent* if the mapping $e_i \mapsto f_i$ extends to a linear isomorphism from $[e_i]$ onto $[f_i]$.

For $1 \leq p < \infty$, ℓ_p is the space of real sequences (a_i) equipped with the norm $\|(a_i)\|_p = (\sum_{i=1}^{\infty} |a_i|^p)^{1/p}$. The space of sequences converging to zero (reps. bounded) equipped with the supremum norm $\|\cdot\|_\infty$ is denoted c_0 (reps. ℓ_∞). The linear space of eventually zero sequences is denoted c_{00}. For $(a_i) \in c_{00}$, the *support* of x, denoted $\operatorname{supp} x$, is the set $\{i \in \mathbb{N} \colon a_0 \neq 0\}$. The space of continuous functions on a compact Hausdorff space K equipped with the supremum norm $\|\cdot\|_\infty$ is denoted $C(K)$. For Banach spaces X and Y, the direct sum $X \oplus_\infty Y$ (resp. $X \oplus_1 Y$) is equipped with the maximum norm $\|(x, y)\|_\infty = \max(\|x\|, \|y\|)$ (resp. sum norm $\|(x, y)\|_1 = \|x\| + \|y\|$). Similarly, $(\sum_{n=1}^{\infty} \oplus X_n)_0$ and

$(\sum_{n=1}^\infty \oplus X_n)_1$ denote the c_0 and ℓ_1 sums of the Banach spaces $(X_n)_{n=1}^\infty$ equipped with their usual norms.

Finally, it is worth emphasizing that we consider only *real* Banach spaces in this paper.

2. The Coefficient Quantization Property

Throughout, X will denote a separable infinite-dimensional Banach space and (e_i) will denote a semi-normalized *dictionary* for X, i.e:

(i) there exist positive constants a and b such that $a \leq \|e_i\| \leq b$ $(i \in \mathbb{N})$;

(ii) (e_i) is a *fundamental* system for X, i.e. $[(e_i)] = X$.

We say that (e_i) is a *minimal system* (we shall always assume that the minimal system is semi-normalized and fundamental) if there exists a biorthogonal sequence (e_i^*) in X^* such that $e_i^*(e_j) = \delta_{ij}$. We say that (e_i) is *total* if $e_i^*(x) = 0$ for all $i \in \mathbb{N}$ implies that $x = 0$, and that (e_i) is *bounded* if $\sup \|e_i\|\|e_i^*\| = M < \infty$. Ovsepian and Pełczyński [18] showed that every separable Banach space possesses a total and bounded minimal system [18]. Pełczyński [19] proved later that one can take $M = 1 + \varepsilon$ for any $\varepsilon > 0$.

Recall that a subset S of a metric space (T, ρ) is a δ-*net* for $A \subseteq T$ (and is said to be δ-*dense* in A) if for every $x \in A$ there exists $y \in S$ such that $\rho(x, y) \leq \delta$. Also S is said to be δ-*separated* if the distance between distinct point of S is at least δ.

Definition 2.1. A dictionary (e_i) has the (ε, δ)-*Coefficient Quantization Property* (abbr. (ε, δ)-CQP) if for every $x = \sum_{i \in E} a_i e_i \in X$ (where E is a finite subset of $\mathbb{N}$) there exist $n_i \in \mathbb{Z}$ $(i \in E)$ such that

$$\|x - \sum_{i \in E} n_i \delta e_i\| \leq \varepsilon. \tag{2.1}$$

(b) (e_i) has the CQP if (e_i) has the (ε, δ)-CQP for some $\varepsilon > 0$ and $\delta > 0$.

Remark 2.2. Setting

$$\mathcal{F}_\delta((e_i)) := \{\sum_{i \in E} n_i \delta e_i \colon E \subset \mathbb{N} \text{ finite}, n_i \in \mathbb{Z}\},$$

note that (2.1) is equivalent to the following:

$$\mathcal{F}_\delta((e_i)_{i\in E}) \text{ is } \varepsilon\text{-dense in } [(e_i)_{i\in E}].$$

We begin with some elementary observations.

Proposition 2.3. *Let (e_i) be a a dictionary for X with the CQP and let $\varepsilon, \delta > 0$.*

(a) The following are equivalent:

(i) *(e_i) has the (ε, δ)-CQP.*

(ii) *(e_i) has the $(\lambda\varepsilon, \lambda\delta)$-CQP for all $\lambda > 0$.*

(iii) *$((1/\varepsilon)e_i)$ has the $(1, \delta/\varepsilon)$-CQP.*

Thus, if (e_i) has the CQP then there exists $c > 0$ such that (e_i) has the $(\varepsilon, c\varepsilon)$-CQP for all $\varepsilon > 0$.

(b) The mapping

$$\delta \mapsto \varepsilon(\delta) := \inf\{\varepsilon\colon (e_i) \textit{ has the } (\varepsilon, \delta)\textit{-CQP}\}.$$

is linear, i.e. $\varepsilon(\lambda\delta) = \lambda\varepsilon(\delta)$ for all $\delta > 0$ and $\lambda > 0$; moreover, if (e_i) is linearly independent then (e_i) has the $(\varepsilon(\delta), \delta)$-CQP.

Proof. (a) To prove the implication (i) $\Rightarrow$ (ii), let $\lambda > 0$ and $x = \sum_{i\in E} a_i e_i$, where E is finite. Since (e_i) has the (ε, δ)-CQP there exist $n_i \in \mathbb{Z}$ such that $\|x/\lambda - \sum_{i\in E} n_i\delta e_i\| \le \varepsilon$. Hence $\|x - \sum_{i\in E} n_i\lambda\delta e_i\| \le \lambda\varepsilon$, which proves (ii). The proofs of the other implications are similar.
(b) The first assertion is an immediate consequence of (a), and the second is an easy compactness argument. □

Now suppose that we relax Definition 2.1 by only requiring that one can approximate each element x of the *unit ball* of X instead of the whole space. Accordingly, for each $\delta > 0$, we define $\varepsilon^{(b)}(\delta)$ to be the infimum of those $\varepsilon > 0$ such that for all finite $E \subset \mathbb{N}$ we have that

$$\mathcal{F}_\delta((e_i)_{i\in E}) \text{ is } \varepsilon\text{-dense in } Ba([(e_i)_{i\in E}]).$$

The following theorem, which is the main result of this section, explains why the CQP has been defined in terms of quantization of the whole space instead of the unit ball.

Theorem 2.4. *Let (e_i) be a dictionary for X. The following are equivalent:*

(i) (e_i) *has the CQP;*

(ii) $\varepsilon^{(b)}(\delta_0) < 1$ *for some* $\delta_0 > 0$*;*

(iii) *there exists* $\delta_1 > 0$ *such that* $\varepsilon(\delta) = \varepsilon^{(b)}(\delta) < \infty$ *for all* $0 < \delta \leq \delta_1$.

Proof. The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are clear. To prove the nontrivial implication (ii) $\Rightarrow$ (iii), let $q_0 := (\varepsilon^{(b)}(\delta_0) + 1)/2 < 1$. First we show that there exist $0 < q_1 < 1$ and $\delta_1 > 0$ such that for every $0 < \delta < \delta_1$, we have $\varepsilon^{(b)}(\delta) < q_1$.

Indeed, choose $n_1 \in \mathbb{N}$ and $0 < q_1 < 1$ such that

$$\frac{n_1 + 1}{n_1} q_0 < q_1 < 1,$$

and set $\delta_1 := \dfrac{\delta_0}{n_1}$. For $0 < \delta \leq \delta_1$ and $x = \sum_{i \in E} a_i e_i \in Ba(X)$, with $E \subset \mathbb{N}$ finite, choose $n \in \mathbb{N}$ such that $\dfrac{\delta_0}{n+1} < \delta \leq \dfrac{\delta_0}{n}$ (note that $n \geq n_1$) and choose $k_i \in \mathbb{Z}$ $(i \in E)$ such that

$$\Big\| \sum_{i \in E} a_i e_i \frac{\delta_0}{(n+1)\delta} - \sum_{i \in E} k_i \delta_0 e_i \Big\| < q_0.$$

Thus, since $n \geq n_1$,

$$\Big\| \sum_{i \in E} a_i e_i - \sum_{i \in E} k_i (n+1) \delta e_i \Big\| \leq q_0 \frac{(n+1)\delta}{\delta_0} \leq q_0 \frac{n+1}{n} < q_1,$$

which implies that $\varepsilon^{(b)}(\delta) < q_1$.

Suppose that $0 < \delta, \tilde{\delta} \leq \delta_1$ satisfy

$$q_1 \leq \frac{\delta}{\tilde{\delta}} \leq \frac{1}{q_1}. \tag{2.2}$$

We claim that

$$\frac{\varepsilon^{(b)}(\delta)}{\delta} \leq \frac{\varepsilon^{(b)}(\tilde{\delta})}{\tilde{\delta}}. \tag{2.3}$$

Once the claim is shown, it follows, by exchanging the roles of δ and $\tilde{\delta}$, that we also have

$$\frac{\varepsilon^{(b)}(\tilde{\delta})}{\tilde{\delta}} \leq \frac{\varepsilon^{(b)}(\delta)}{\delta},$$

which implies local linearity and, thus, linearity of $\varepsilon^{(b)}$ on $(0, \delta_1]$.

Let $x = \sum_{i\in E} a_i e_i \in Ba(X)$ with E finite. There exists $y = \sum_{i\in E} k_i \delta e_i \in \mathcal{F}_\delta((e_i))$ such that $\|x-y\| < q_1$. Note that $(\tilde{\delta}/\delta)(x-y) \in Ba(X)$ by (2.2). Hence, given $\eta > 0$, there exists $z = \sum_{i\in E} m_i \tilde{\delta} e_i \in \mathcal{F}_{\tilde{\delta}}((e_i))$ such that

$$\|\frac{\tilde{\delta}}{\delta}(x-y) - z\| < (1+\eta)\varepsilon^{(b)}(\tilde{\delta}),$$

i.e.

$$\|x - \sum_{i\in E}(k_i + m_i)\delta e_i\| < (1+\eta)\frac{\delta}{\tilde{\delta}}\varepsilon^{(b)}(\tilde{\delta}),$$

which yields (2.3) since $\eta > 0$ is arbitrary.

In order show that $\varepsilon(\cdot) = \varepsilon^{(b)}(\cdot)$ on $(0, \delta_1]$, let $0 < \delta \leq \delta_1$, let $x = \sum_{i\in E} a_i e_i$, with $E \subset \mathbb{N}$ finite, and let $\eta > 0$ be arbitrary. If $\|x\| \geq 1$ there exist $k_i \in \mathbb{Z}$ $(i \in E)$ such that

$$\Big\|\frac{x}{\|x\|} - \sum_{i\in E} k_i \frac{\delta}{\|x\|} e_i\Big\| < (1+\eta)\varepsilon^b\Big(\frac{\delta}{\|x\|}\Big) = (1+\eta)\frac{\varepsilon^{(b)}(\delta)}{\|x\|}$$

and thus

$$\Big\|x - \sum_{i\in E} k_i \delta e_i\Big\| \leq (1+\eta)\varepsilon^{(b)}(\delta). \tag{2.4}$$

If $\|x\| \leq 1$ we can of course also find $k_i \in \mathbb{Z}$ such that (2.4) holds. Since $\eta > 0$ is arbitrary, it follows that $\varepsilon(\cdot) \leq \varepsilon^{(b)}(\cdot)$ and, thus, $\varepsilon(\cdot) = \varepsilon^{(b)}(\cdot)$ on $(0, \delta_1]$. □

The following corollary is a quantitative version of the last result.

Corollary 2.5. *Let $0 < \varepsilon_0 < 1$ and $\delta > 0$. If $\mathcal{F}_\delta((e_i)_{i\in E})$ is ε_0-dense in $Ba([(e_i)_{i\in E}])$ for all finite $E \subset \mathbb{N}$ then $\mathcal{F}_\delta((e_i)_{i\in E})$ is ε_1-dense in $[(e_i)_{i\in E}]$ for all*

$$\varepsilon_1 > \Big(\Big\lfloor \frac{\varepsilon_0}{1-\varepsilon_0} \Big\rfloor + 1\Big)\varepsilon_0. \tag{2.5}$$

(Here $\lfloor x \rfloor$ denotes the integer part of x.) In particular, if $\varepsilon_0 < 1/2$, then $\mathcal{F}_\delta((e_i)_{i\in E})$ is ε_1-dense in $[(e_i)_{i\in E}]$ for all $\varepsilon_1 > \varepsilon_0$.

Proof. Using the notation of the last proof, we may take $n_1 = \lfloor \varepsilon_0/(1-\varepsilon_0) \rfloor + 1$. The last proof yields

$$\varepsilon(\delta/n_1) = \varepsilon^b(\delta/n_1) \leq \varepsilon_0.$$

Thus, $\varepsilon(\delta) \leq n_1 \varepsilon_0$, which gives the result. □

Remark 2.6. The assumption that (e_i) is semi-normalized is not required for the validity of Corollary 2.5. Moreover, if (e_i) is linearly independent then strict inequality in (2.5) may be replaced by non-strict inequality. Finally, the result is also valid for quasi-normed spaces.

In the finite-dimensional setting Corollary 2.5 can be formulated as a covering result of independent interest.

Theorem 2.7. *Let $K \subset \mathbb{R}^n$ be a compact zero-neighborhood that is star-shaped about zero (i.e. $\lambda K \subseteq K$ for all $0 \le \lambda \le 1$) and let $L \subset \mathbb{R}^n$ be a lattice (i.e. a discrete subgroup of $\mathbb{R}^n$). If $K \subset L + \varepsilon_0 K$, where $0 < \varepsilon_0 < 1$, then $\mathbb{R}^n = L + \varepsilon_1 K$, where $\varepsilon_1 = (\lfloor \varepsilon_0/(1-\varepsilon_0) \rfloor + 1)\varepsilon_0$.*

Proof. The gauge functional $\|x\|_K := \min\{t > 0\colon tx \in K\}$ is positively homogeneous, which is the only property of the norm that is used in the proof of Theorem 2.4. Hence, setting

$$\varepsilon_L^{(b)}(\delta) := \min\{\varepsilon\colon K \subset \delta L + \varepsilon K\},$$

the proof of Theorem 2.4 yields

$$\varepsilon_L^{(b)}(\delta) = n_1 \delta \varepsilon^{(b)}(1/n_1) \le n_1 \delta \varepsilon_0$$

for all $0 \le \delta \le 1/n_1$, where $n_1 := n_1(\varepsilon_0)$ is defined as in the proof of Corollary 2.5. The proof is concluded as before. □

The examples presented in the next section all have a formally stronger version of the CQP which we now define.

Definition 2.8. Let $\varepsilon > 0$ and let $\delta > 0$.
(a) A dictionary (e_i) has the (ε, δ)-*Strong Coefficient Quantization Property* (abbr. (ε, δ)-SCQP) if for every sequence $\overline{D} := (D_i)$ of δ-nets for $\mathbb{R}$, such that $0 \in D_i$, and for every $x = \sum_{i \in E} a_i e_i$ in X (where E is a finite subset of $\mathbb{N}$) there exist $d_i \in D_i$ ($i \in E$) such that

$$\|x - \sum_{i \in E} d_i e_i\| \le \varepsilon. \tag{2.6}$$

(b) (e_i) has the SCQP if (e_i) has the (ε, δ)-SCQP for some $\varepsilon > 0$ and $\delta > 0$.

Remarks 2.9. (i) If we set

$$\mathcal{F}_{\overline{D}}((e_i)) := \{\sum_{i \in E} d_i e_i\colon E \subset \mathbb{N} \text{ finite}, d_i \in D_i\},$$

then (2.6) is equivalent to the following:

$$\mathcal{F}_{\overline{D}}((e_i)_{i\in E}) \text{ is } \varepsilon\text{-dense in } [(e_i)_{i\in E}].$$

(ii) The obvious analogue for the SCQP of Proposition 2.3 is valid.
(iii) Note also the implication (ε,δ)-SCQP $\Rightarrow$ $(\varepsilon,2\delta)$-CQP since $2\delta\mathbb{Z}$ is a δ-net.
(iv) If (e_i) has the (ε,δ)-CQP, we say that (e_i) is an (ε,δ)-*CQP dictionary*, and similarly for the SCQP.
(v) To avoid repetition we shall assume henceforth *that every δ-net for $\mathbb{R}$ contains zero.*
(vi) Unless stated otherwise all sums of the form $\sum a_ie_i$ will be assumed to be *finite*.

The uniformity built into the definitions of the CQP and the SCQP is natural in view of the following uniform boundedness result.

Proposition 2.10. *Let (e_i) be a dictionary for X. The following are equivalent:*

(i) *(e_i) has the SCQP;*
(ii) *For all $\delta>0$ and for every sequence (D_i) of δ-nets there exists $M>0$ such that for every $x=\sum_{i\in E}a_ie_i\in X$ (where E is a finite subset of $\mathbb{N}$) there exist $d_i\in D_i$ $(i\in E)$ such that*
$$\|x-\sum_{i\in E}d_ie_i\|\leq M.$$
(iii) *Condition (ii) for $\delta=1$.*

Proof. Clearly, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). To prove (iii) $\Rightarrow$ (i), we argue by contradiction. Suppose that (i) does not hold. Then by (ii) of Remarks 2.9 (e_i) fails the $(M,1)$-SCQP for all $M>0$. First we construct by induction a sequence (E_n) of finite disjoint subsets of $\mathbb{N}$, a sequence $((D_i^n))$ of sequences of 1-nets, and vectors $x_n=\sum_{i\in E_n}a_i^ne_i\in X$ $(n\geq1)$ such that

$$\inf\{\|x_n-\sum_{i\in E_n}d_i^ne_i\|\colon d_i^n\in D_i^n\}>n \qquad (n\geq 1). \tag{2.7}$$

Suppose that $n_0\geq1$ and that the construction has been carried out for all $n<n_0$. Set $F:=\cup_{n<n_0}E_n$. Since (e_i) does not have the $(M,1)$-SCQP for $M=\operatorname{card}(F)\max_{i\in F}\|e_i\|+n_0$ there exist a sequence $(D_i^{n_0})$

of 1-nets, a finite set $G \subset \mathbb{N}$, and $x = \sum_{i\in G} a_i e_i \in X$ such that

$$\inf\{\|x - \sum_{i\in G} d_i^{n_0} e_i\| \colon d_i^{n_0} \in D_i^{n_0}\} > \operatorname{card}(F) \max_{i\in F} \|e_i\| + n_0. \tag{2.8}$$

Choose $d_i^{n_0} \in D_i^{n_0}$ such that $|a_i - d_i^{n_0}| \le 1$ for $i \in G \cap F$. Then

$$\|\sum_{i\in G\cap F} (a_i - d_i^{n_0}) e_i\| \le \operatorname{card}(G\cap F) \max_{i\in G\cap F} \|e_i\|,$$

and thus (2.8) yields

$$\inf\{\|\sum_{i\in G\setminus F} (a_i - d_i^{n_0}) e_i\| \colon d_i^{n_0} \in D_i^{n_0}\} > n_0.$$

Set $E_{n_0} := G \setminus F$ and $x_{n_0} = \sum_{i\in G\setminus F} a_i e_i$ to complete the induction. Now define a sequence (D_i) of 1-nets as follows:

$$D_i = \begin{cases} D_i^n & \text{if there exist } n \text{ such that } i \in E_n, \\ 2\mathbb{Z} & \text{otherwise.} \end{cases}$$

Then by (2.7) (D_i) does not satisfy (iii).

□

Our first permanence result ensures that the SCQP is preserved under linear isomorphisms.

Proposition 2.11. *Suppose that $T : X \to Y$ is a bounded operator. Suppose also that (e_i) is a dictionary for X with the property that $(T(e_i))$ is a dictionary for Y.*
(a) If (e_i) is an (ε, δ)-SCQP dictionary for X then $(T(e_i))$ is an $(\varepsilon\|T\|, \delta)$-SCQP dictionary for Y.
(b) If (e_i) has the SCQP then $(T(e_i))$ also has the SCQP.

Proof. (a) Let (D_i) be any family of δ-nets for $\mathbb{R}$. Consider $\sum_{i\in E} a_i T(e_i) \in Y$, where E is a finite subset of $\mathbb{N}$. Since (e_i) has the (ε, δ)-CQP there exist $d_i \in D_i$ such that

$$\|\sum_{i\in E} (a_i - d_i) e_i\| \le \varepsilon,$$

whence

$$\|\sum_{i\in E} (a_i - d_i) T(e_i)\| \le \|T\|\varepsilon$$

(b) This follows at once from (a). □

Remark 2.12. The analogue of Proposition 2.11 for the CQP is also valid.

The following useful result shows that the SCQP is preserved after normalization of the dictionary.

Proposition 2.13. *Suppose that* (e_i) *has the* (ε,δ)*-SCQP and that* $a \le \|e_i\| \le b$*. Then the normalized dictionary* $(e_i/\|e_i\|)$ *has the* (ε,δ')*-CQP for* $\delta' = a\delta$*.*

Proof. Let (D_i') be a family of δ'-nets for $\mathbb{R}$. Then each $D_i = \{d_i'/\|e_i\| \colon d_i' \in D_i'\}$ is a δ-net. Since (e_i) has the (ε,δ)-SCQP, it follows that for each $\sum_{i\in E} a_i(e_i/\|e_i\|)$ in X, where E is a finite subset of $\mathbb{N}$, there exist $d_i' \in D_i'$ $(i \in E)$ such that

$$\|\sum_{i\in E} \frac{a_i}{\|e_i\|} e_i - \sum_{i\in E} \frac{d_i'}{\|e_i\|} e_i\| \le \varepsilon.$$

□

We conclude this section with some open problems.

Problems 2.14. (1) For a given dictionary (e_i) is the SCQP equivalent to the CQP?
(2) Does the analogue of Theorem 2.4 for the SCQP hold?
(3) Does the analogue of Proposition 2.13 for the CQP hold?

Remark 2.15. We say that a dictionary (e_i) has property P if the following condition holds. There exists $\delta > 0$ such that for all δ-nets (D_i) and for all finite $E \subseteq N$ there exist $d_i \in D_i \setminus \{0\}$ $(i \in E)$ such that $\|\sum_{i\in E} d_i e_i\| \le 1$. To see that Property P implies the SCQP, let (D_i) be a sequence of δ-nets and consider $x = \sum_{i\in E} a_i e_i$. Clearly, each $D_i' := \{d_i - b_i \colon d_i \in D_i\} \cup \{0\}$ is a δ-net. Property P implies that there exist $d_i \in D_i$ $(i \in E)$ with $d_i \ne b_i$ such that $\|\sum_{i\in E}(d_i - b_i)e_i\| \le 1$, so (e_i) has the SCQP. When (e_i) is linearly independent, one can also show that the converse implication holds, i.e. that the SCP implies Property P. So for a linearly independent dictionary the first problem stated above is equivalent to the following: is the CQP equivalent to Property P?

3. Examples

3.1. **The unit vector basis of c_0.** The unit vector basis of c_0 has the $(\varepsilon, \varepsilon)$-SCQP. To see this, let (D_i) be a sequence of ε-nets. Given $x = \sum_{i\in E} a_i e_i$, simply choose $d_i \in D_i$ such that $|a_i - d_i| \le \varepsilon$. Then

$$\|\sum_{i\in E} a_i e_i - \sum_{i\in E} d_i e_i\| = \max_{i\in E} |a_i - d_i| \le \varepsilon.$$

It is instructive to note that if (e_i) is a bounded minimal system then the above procedure for choosing the approximation is only effective for the unit vector basis of c_0. To be precise, suppose that the δ-nets (D_i) are γ-separated for some $\gamma > 0$. Consider the following algorithm: choose d_i to be the *best approximation* to the coefficient a_i (or the best approximation of smallest absolute value when a_i is exactly half-way between two d_i values).

Proposition 3.1. *Let (e_i) be a bounded minimal system. The following are equivalent:*

(i) *(e_i) is equivalent to the unit vector basis of c_0.*
(ii) *(e_i) has the CQP and the algorithm described above is effective.*

Proof. $(i) \Rightarrow (ii)$ was proved above. For the proof of $(ii) \Rightarrow (i)$, suppose that the (ε, δ)-CQP for (e_i) is implemented by the aforementioned algorithm, where $0 < \varepsilon < 1$ and $\delta > 0$. Let $x = \sum_{i\in E} a_i e_i$ be a unit vector and suppose that $\max |a_i| < \delta/2$. According to the algorithm, we should approximate x by taking $d_i = 0$ for all $i \in E$, which yields the contradiction $1 = \|x\| \le \varepsilon < 1$. Hence

$$\frac{1}{M} \max |a_i| \le \|x\| \le \frac{2}{\delta} \max |a_i|,$$

where $M = \sup \|e_i^*\|$. Thus, (i) holds. □

3.2. **The summing basis of c_0.** The linear space of sequences (a_i) for which $\sum_{i=1}^{\infty} a_i$ converges is a Banach space when equipped with the following norm:

$$\|(a_i)\|_{sb} = \sup_n |\sum_{i=1}^{n} a_i|.$$

This space is isometrically isomorphic to the space c of convergent sequences with the supremum norm. The unit vector basis (e_i) is equivalent to a conditional basis of c_0 called the summing basis.

To see that (e_i) has the $(\varepsilon,\varepsilon)$-SCQP, let $(a_i) \in c_{00}$. Suppose that $(d_i)_{i=1}^k$ have been chosen so that $|\sum_{i=1}^j (a_i - d_i)| \le \varepsilon$ for $1 \le j \le k$. Then we continue by choosing $d_{k+1} \in D_{k+1}$ so that $d_{k+1} = 0$ if $a_{k+1} = 0$ and so that $|\sum_{i=1}^{k+1}(a_i - d_i)| \le \varepsilon$.

Let us generalize this example as follows. Let $N \in \mathbb{N}$. For each $1 \le n \le N$, let $(\varepsilon_i^n)_{i=1}^\infty$ be a sequence of signs $\varepsilon_i^n = \pm 1$. Consider the following norm on c_{00}:

$$\|(a_i)\| = \max_{1\le n\le N} \|(\varepsilon_i^n a_i)\|_{sb}.$$

For each $\eta = (\eta_n)_{n=1}^N \in \{-1,1\}^N$, let $A_\eta = \{m \in \mathbb{N} \colon \varepsilon_m^n = \eta_n, 1 \le n \le N\}$. Then (A_η) $(\eta \in \{1,-1\}^N)$ is a partition of $\mathbb{N}$. Note that for $(a_i), (d_i) \in c_{00}$, the triangle inequality gives

$$\|(a_i - d_i)\| \le \sum_{\eta\in\{1,-1\}^N} \|(a_i - d_i)_{i\in A_\eta)}\|_{sb} \tag{3.9}$$

Now suppose that (D_i) is a sequence of $\varepsilon/2^N$-nets for $\mathbb{R}$. For each $\eta \in \{1,-1\}^N$, choose $d_i \in D_i$ for $i \in A_\eta$ so that $\|(a_i - d_i)_{i\in A_\eta}\|_{sb} \le \varepsilon/2^N$. This is possible since the summing basis has the $(\varepsilon/2^N, \varepsilon/2^N)$-SCQP. It follows from (3.9) that $\|(a_i - d_i)\| \le \varepsilon$. Hence $\|\cdot\|$ has the $(\varepsilon, \varepsilon/2^N)$-SCQP.

3.3. **The Schauder basis.** Let us recall the definition of the classical Schauder basis $(f_i)_{i\ge 0}$ for $C([0,1])$: $f_0(t) = 1$, $f_1(t) = t$, and for $i = 2^k + l, 0 \le l < 2^k$, f_i is the piecewise-linear function supported on $[l2^{-k}, (l+1)2^{-k}]$ satisfying $f_i(l2^{-k}) = f_i((l+1)2^{-k}) = 0$ and $f_i((2l+1)2^{-k-1}) = 1$.

Theorem 3.2. *The Schauder basis for $C([0,1])$ has the $(\varepsilon,\varepsilon)$-SCQP for all $\varepsilon > 0$.*

Proof. Let (D_i) be a sequence of ε-nets. Suppose that $N \ge 0$ and that $x = \sum_{i=0}^N a_i f_i$. We shall prove that there exist $d_i \in D_i$ such that

$$\|\sum_{i=0}^k (a_i - d_i) f_i\|_\infty \le \varepsilon \tag{3.10}$$

for $0 \le k \le N$ and such that $d_i = 0$ if $a_i = 0$. Choose $d_0 \in D_0$ such that $|a_0 - d_0| \le \varepsilon$ and choose $d_1 \in D_1$ such that $|a_0 + a_1 - d_0 - d_1| \le \varepsilon$ (with $d_i = 0$ if $a_i = 0$). This establishes (3.10) for $k = 0$ and $k = 1$. Suppose that $2 \le n \le N$ and that $d_0, \dots, d_{n-1}$ have been chosen so that (3.10) holds for $0 \le k \le n-1$. Let the support of f_n be the dyadic interval $[a,b]$ and consider

$$g(x) = |\sum_{i=0}^{n} a_i f_i(x) - \sum_{i=0}^{n-1} d_i f_i(x)|.$$

Then g is piecewise-linear on $[a,b]$ with nodes at a, b, and $(a+b)/2$. So g must attain its maximum at one of these three points. If the maximum occurs at either $x=a$ or $x=b$, then, since $f_n(a) = f_n(b) = 0$, it follows from the case $k = n-1$ of (3.10) that

$$\max_{x\in[a,b]} g(x) \le \max_{x\in[0,1]} \|\sum_{i=0}^{n-1}(a_i - d_i)f_i\|_\infty \le \varepsilon.$$

Then, setting $d_n = 0$, (3.10) will be satisfied for $k = n$. So suppose that the maximum is attained at $(a+b)/2$. Choose $d_n \in D_n$ such that

$$|\sum_{i=0}^{n-1}(a_i - d_i)f_i(\frac{a+b}{2}) + a_n f_n(\frac{a+b}{2}) - d_n| \le \varepsilon.$$

With this choice of d_n, we see that (3.10) is again satisfied for $k = n$. □

Remark 3.3. Let K be an uncountable compact metric space. Then $C(K)$ is uniformly isomorphic to $C([0,1])$ by Milutin's Theorem [16]. Since the Schauder basis of $C([0,1])$ has the $(\varepsilon,\varepsilon)$-SCQP, it follows from Propositions 2.11 and 2.13 that $C(K)$ has a normalized $(\varepsilon, c\varepsilon)$-SCQP basis for some absolute constant $c > 0$.

3.4. **Tree spaces.** By a *tree* we shall mean a partially ordered set $(\mathcal{T}, \le)$ with the property that each *node* $\alpha \in \mathcal{T}$ has finitely many linearly ordered predecessors (with respect to $\le$). We say that $\mathcal{T}$ is *rooted* if there is exactly one node without an immediate predecessor. The tree $\mathcal{T}_\infty$ is the rooted tree with the property that every node has countably infinitely many immediate successors. We equip $c_{00}(\mathcal{T})$ with the following norm:

$$\|x\| = \max_{\beta\in\mathcal{T}} |\mathcal{S}_\beta(x)|,$$

where $\mathcal{S}_\beta(x) = \sum_{\alpha\le\beta} x(\alpha)$. Let $\mathcal{S}(\mathcal{T})$ denote the completion of the normed space $(c_{00}(\mathcal{T}), \|\cdot\|)$.

Henceforth we shall assume that $\mathcal{T}$ is countably infinite. Suppose that $(\alpha(i))$ is any enumeration of $\mathcal{T}$ which respects the ordering of $\mathcal{T}$, i.e. such that

$$\alpha(i) \le \alpha(j) \Rightarrow i \le j.$$

Clearly, $(e_{\alpha(i)})$ is a normalized monotone basis for $\mathcal{S}(\mathcal{T})$.

Proposition 3.4. *(a) Suppose that $\mathcal{T}$ is rooted. Then $\mathcal{S}(\mathcal{T})$ is isometrically isomorphic to $C(K)$, where K is the weak-star closure of $\{\mathcal{S}_\beta \colon \beta \in \mathcal{T}\}$ in $Ba(\mathcal{S}(\mathcal{T}^*))$.*
(b) If K is a countable compact metric space then $C(K)$ is isometrically isomorphic to $\mathcal{S}(\mathcal{T})$ for some rooted tree $\mathcal{T}$.
(c) $\mathcal{S}(\mathcal{T}_\infty)$ is isometrically isomorphic to $C(\Delta)$, where Δ denotes the Cantor set.

Proof. (a) It is easily seen that $c_{00}(\mathcal{T})$ is a separating subalgebra of $C(K)$. Since $\mathcal{S}_\alpha(e_\emptyset) = 1$ for all $\alpha \in \mathcal{T}$, where $\emptyset$ is the root node, it follows that $\chi_K \in c_{00}(\mathcal{T})$, and hence by the Stone-Weierstraß theorem that $c_{00}(\mathcal{T})$ is dense in $C(K)$.
(b) It is well-known that every countable compact metric space is homeomorphic to an ordinal interval $[0, \alpha]$, for some countable ordinal α, with the order topology. We prove the result by transfinite induction. Suppose the result holds for $K = [0, \beta]$ for all $0 \le \beta < \alpha$. There exist $1 \le n \le \infty$ and countable ordinals $\alpha_j < \alpha$ $(0 \le j < n)$ such that $K := [0, \alpha]$ is homeomorphic to the one-point compactificiation of the disjoint union of the ordinal intervals $K_j := [0, \alpha_j]$ $(0 \le j < n)$. By hypothesis there exist trees $\mathcal{T}_j$ $(0 \le j < n)$ such that $\mathcal{S}(\mathcal{T}_j)$ is isometrically isomorphic to $C(K_j)$. Let $\mathcal{T}$ be the rooted tree which has each $\mathcal{T}_j$ $(0 \le j < n)$ as a subtree immediately succeeding the root node. Then $\mathcal{S}(\mathcal{T})$ is easily seen to be isometrically isomorphic to $C(K)$.
(c) In this case K is easily seen to be a perfect and totally disconnected compact metric space, and thus homeomorphic to Δ. ☐

Theorem 3.5. *$(e_\alpha)_{\alpha\in\mathcal{T}}$ has the $(\varepsilon, \varepsilon) - SCQP$ in $\mathcal{S}(\mathcal{T})$ for all $\varepsilon > 0$.*

Proof. Let $(\alpha(i))$ be any ordering of the basis which respects the ordering of $\mathcal{T}$. Let $\varepsilon > 0$ and let $(D_\alpha)_{\alpha\in\mathcal{T}}$ be a family of ε-nets and suppose that $\sum_{i\in E} x_{\alpha(i)} \in c_{00}(\mathcal{T})$. We define $d_\alpha \in D_\alpha$ inductively. Suppose that $n \ge 0$ and that $d_{\alpha(1)}, \dots, d_{\alpha(n)}$ have been chosen such that

$$|\mathcal{S}_\gamma(\sum_{i=1}^{n}(x_{\alpha(i)} - d_{\alpha(i)})e_{\alpha(i)})| \le \varepsilon \tag{3.11}$$

for all $\gamma \in \mathcal{T}$ (This condition is vacuous for $n = 0$.) If $x_{\alpha(n+1)} = 0$, set $d_{\alpha(n+1)} = 0$. Otherwise choose $d_{\alpha(n+1)} \in D_{\alpha(n+1)}$ such that

$$|\sum_{\beta<\alpha(n+1)} (x_\beta - d_\beta) + x_{\alpha(n+1)} - d_{\alpha(n+1)}| \leq \varepsilon,$$

noting that if $\beta < \alpha(n+1)$ then $\beta = \alpha(j)$ for some $j \leq n$. Now we verify the inductive hypothesis for $n+1$. If $\gamma \geq \alpha(n+1)$ then

$$|\mathcal{S}_\gamma(\sum_{i=1}^{n+1}(x_{\alpha(i)} - d_{\alpha(i)}e_{\alpha(i)}))| = |\sum_{\beta\leq\alpha(n+1)} (x_\beta - d_\beta)| \leq \varepsilon.$$

On the other hand, if $\gamma < \alpha(n+1)$ or if γ and $\alpha(n+1)$ are incomparable, then

$$|\mathcal{S}_\gamma(\sum_{i=1}^{n+1}(x_{\alpha(i)} - d_{\alpha(i)})e_{\alpha(i)})| = |\mathcal{S}_\gamma(\sum_{i=1}^{n}(x_{\alpha(i)} - d_{\alpha(i)})e_{\alpha(i)})| \leq \varepsilon$$

by the inductive assumption (3.11). This completes the verification of the inductive step. It follows that

$$\|\sum_{n\in E}(x_{\alpha(n)} - d_{\alpha(n)})e_{\alpha(n)}\| \leq \varepsilon.$$

Thus, $(e_\alpha)_{\alpha\in\mathcal{T}}$ has the $(\varepsilon,\varepsilon)$-SCQP. □

Corollary 3.6. *If K is a countable compact metric space or if $K = \Delta$ then $C(K)$ has a monotone basis with the $(\varepsilon,\varepsilon)$-CQP for all $\varepsilon > 0$.*

Remark 3.7. In all of the above examples the dictionary (e_i) has the *neighborly CQP*, i.e. for every $x = \sum_{i\in E} a_i e_i$ with finite support, the approximation $y = \sum_{i\in E} n_i \delta e_i$ satisfies $|a_i - n_i\delta| \leq \delta$. We do not know whether this holds in general, i.e. whether the CQP implies the neighborly CQP.

3.5. **The Haar Basis for $C(\Delta)$.** We have already seen that $C(\Delta)$ has a monotone basis with the $(\varepsilon,\varepsilon)$-CQP. Surprisingly, however, the natural basis of $C(\Delta)$), namely the Haar basis, does *not* have the ε-CQP for any $\varepsilon > 0$. Let us recall the definition of the Haar basis. Let $\Delta_0 := \Delta$, and, for $k \geq 0$, let Δ_{2k+1} and Δ_{2k+2} be the left-hand and right-hand halves of Δ_k obtained by removing the 'middle third' in the classical construction of the Cantor set. Then

$$h_i = \begin{cases} \chi_\Delta & \text{for i=0} \\ \chi_{\Delta_{2i-1}} - \chi_{\Delta_{2i}} & \text{for } i > 0. \end{cases}$$

clearly, $(h_i)_{i=0}^{\infty}$ is a monotone basis for $C(\Delta)$. For $k=1,2,\dots$, we say that the 2^{k-1} Haar functions $\{h_i\colon 2^{k-1}\leq i<2^k\}$ are on the k-th level.

Proposition 3.8. *Let $0<\varepsilon<1$ and let $\delta>0$. Then $\mathcal{F}_\delta((h_i))$ is not an ε-net for the unit ball of $C(\Delta)$. In particular (h_i) does not have the CQP.*

Proof. For $N\in\mathbb{N}$, let $x_N=(1/N)\sum_{i=1}^{2^N-1}h_i$ and let $y\in\mathcal{F}_\delta((h_i))$. Note that $\|x_N\|=1$. We shall prove that $\|x-y\|\geq 1$ provided $N\geq 2/\delta$. Since (h_i) is a monotone basis, we may assume that $y\in\operatorname{span}\{h_i\colon 0\leq i\leq 2^N-1\}$. Since x_N and $-x_N$ have the same distribution, we may also assume that the coefficient of h_0 in the expansion of y is $-\alpha$, where $\alpha\geq 0$. Let $k_1\geq 1$ be the first level (if there are any) of the Haar system for which the *leftmost* Haar function has a nonzero coefficient in the expansion of y. Let this Haar function be h_{i_1} and let a_1 be the corresponding coefficient. Note that $|a_1|\geq\delta$. By considering the left-hand and the right-hand halves of the support of h_{i_1}, and using the monotonicity of the Haar basis, we see that

$$\max_{t\in I_1}(x-y)(t)\geq\frac{k_1-2}{N}+\alpha+\delta=\frac{k_1}{N}+\alpha+(\delta-\frac{2}{N}),$$

where I_1 is the (left-hand or right-hand) half of the support of h_{i_1} on which $a_1h_{i_1}$ takes a *negative* value. Now we repeat the argument for I_1. Suppose that the next level for which there is a nonzero coefficient in the leftmost Haar function whose support is entirely contained in I_1 is the (k_1+k_2)-th level, where $k_2\geq 1$. Let h_{i_2} denote this Haar function and let a_2 be the corresponding coefficient. Then, by the same reasoning as above, we get

$$\max_{t\in I_2}(x-y)(t)>\frac{k_1+k_2}{N}+\alpha+2(\delta-\frac{2}{N}),$$

where I_2 is the half of the support of h_{i_2} on which $a_1h_{i_2}$ takes a *negative* value. This process terminates after $J\geq 0$ steps at level $k_1+\dots+k_J$ with a set I_J (half of the support of h_{i_J}) such that

$$\max_{t\in I_J}(x-y)(t)>\frac{k_1+\dots+k_J}{N}+\alpha+J(\delta-\frac{2}{N}).$$

Finally, let I be the left-hand half of the leftmost Haar function on the N-th level whose support is entirely contained in I_J. Since the inductive process has terminated after J steps, we obtain

$$(x-y)(t) \geq 1+\alpha+J(\delta-\frac{2}{N}) \geq 1 \qquad (t \in I)$$

provided $N \geq 2/\delta$.

□

Remarks 3.9. (i) The proof of Proposition 3.8 actually shows that if $\delta \geq 2/N$ then $\mathcal{F}_\delta((h_i))$ is not an ε-net for the unit ball of $\ell_\infty^{2^N}$ for any $0<\varepsilon<1$.
(ii) In the terminology of Section 5 below, Proposition 3.8 shows that (h_i) does not have the Net Quantization Property.

4. An Existence Result

Theorem 4.1. *Suppose that $c_0 \hookrightarrow X$. Then X has a normalized, bounded, total, weakly null minimal system which has the $(\varepsilon, c\varepsilon)$-SCQP for all $\varepsilon>0$, where c is an absolute constant (independent of X and ε). Moreover, if X has a basis, then X has a normalized weakly null $(\varepsilon, c\varepsilon)$-SCQP basis.*

First let us explain the construction that is used in the proof of Theorem 4.1. To that end, let $(e_j)_{j=1}^{n+1}$ denote the unit vector basis of ℓ_∞^{n+1}. Define a new basis $(f_j)_{j=1}^{n+1}$ as follows:

$$f_j = e_j + \frac{e_{n+1}}{n} \qquad (1 \leq j \leq n)$$

and

$$f_{n+1} = e_1 + e_2 + \cdots + e_n.$$

The following lemma is easily verified.

Lemma 4.2. *$(f_j)_{j=1}^{n+1}$ is a normalized basis for ℓ_∞^{n+1} with basis constant at most 3.*

Proof of Theorem 4.1. By Sobczyk's theorem [22] that c_0 is 2-complemented in any separable superspace and James's theorem [12] that every Banach space isomorphic to c_0 contains an almost isometric copy of c_0, it follows that X is uniformly isomorphic to $X \oplus_\infty c_0$. So by Proposition 2.11 and Proposition 2.13, it suffices to prove the result for $X \oplus_\infty c_0$. Let (ϕ_i) be a normalized total minimal system (resp. normalized basis) for X.

For convenience, we regard c_{00} as the space of all finitely supported sequences (a_j^n) doubly indexed by $n \in \mathbb{N}$ and $1 \leq j \leq n^2+1$. Let (e_j^n) denote the standard basis for this realization of c_{00} and order the basis elements lexicographically (i.e., $e_1^1, e_2^1, e_1^2, e_2^2 \dots$). Define a norm $\|\cdot\|_Y$ on c_{00} as follows:

$$\|(a_j^n)\|_Y = \max\left\{ \sup_{n\geq 1} \|(a_j^n + a_{n^2+1}^n)_{j=1}^{n^2}\|_\infty, \left\| \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\sum_{j=1}^{n^2} a_j^n \right) \phi_n \right\|_X \right\},$$

and let Y denote the completion of $(c_{00}, \|\cdot\|)$. It is easily seen that Y is *isometrically isomorphic* to $X \oplus_\infty (\sum_{n=1}^\infty \oplus \ell_\infty^{n^2})_0$, which in turn is isometrically isomorphic to $X \oplus_\infty c_0$, and that (e_j^n) is a normalized bounded and total minimal system for Y. Moreover, for each $n \in \mathbb{N}$, $(e_j^n)_{j=1}^{n^2+1}$ is isometrically equivalent to the basis $(f_j)_{j=1}^{n^2+1}$ described above. Thus, for the case in which (ϕ_i) is a basis for X, it follows easily from Lemma 4.2 that (e_j^n) is a basis for Y.

Let us next check that (e_j^n) is weakly null. Under the isometric isomorphism of Y with $X \oplus_\infty (\sum_{n=1}^\infty \oplus \ell_\infty^{n^2})_0$, the basis vector e_j^n corresponds to

$$\begin{cases} (\phi_n/n^2, g_j^n), & \text{if } 1 \leq j \leq n^2 \\ (0, \sum_{i=1}^{n^2} g_i^n), & \text{if } j = n^2+1, \end{cases} \tag{4.12}$$

where $(g_i^n)_{i=1}^{n^2}$ denotes the unit vector basis of $\ell_\infty^{n^2}$. Thus it sufffices to check that the sequence defined by (4.12) is weakly null. But this is readily verified directly using the fact that $(X \oplus_\infty (\sum_{n=1}^\infty \oplus \ell_\infty^{n^2})_0)^*$ is isometrically isomorphic to $X^* \oplus_1 (\sum_{n=1}^\infty \oplus \ell_1^{n^2})_1$.

To see that (e_j^n) has the $(\varepsilon, c\varepsilon)$-CQP, let $\delta > 0$ and let (D_j^n) be a doubly-indexed family of δ-nets and let $(a_j^n) \in c_{00}$. For each $n \in \mathbb{N}$, choose $d_j^n \in D_j^n$, with $d_j^n = 0$ if $a_j^n = 0$, such that

$$\left| \sum_{j=1}^{k} (a_j^n - d_j^n) \right| \leq \delta \qquad (1 \leq k \leq n^2) \tag{4.13}$$

and

$$|a_{n^2+1}^n - d_{n^2+1}^n| \leq \delta. \tag{4.14}$$

From (4.13) and the triangle inequality, we see that

$$|a_j^n - d_j^n| \leq 2\delta \qquad (1 \leq j \leq n^2). \tag{4.15}$$

Combining (4.13), (4.14), and (4.15), we obtain

$$\begin{aligned}\|\sum(a_j^n - d_j^n)e_j^n\|_Y &\le \sup_n \max_{1\le j\le n^2} |a_j^n - d_j^n + a_{n^2+1}^n - d_{n^2+1}^n| \\ &\quad + \sum_{n=1}^{\infty}\frac{1}{n^2}|\sum_{j=1}^{n^2}(a_j^n - d_j^n)| \\ &\le 3\delta + \delta\cdot\frac{\pi^2}{6}.\end{aligned}$$

This shows that (e_j^n) is a minimal system (resp. basis) for Y with the $(\varepsilon, c\varepsilon)$-CQP for $c = (3+\pi^2/6)^{-1}$. Finally, since Y is uniformly isomorphic to X, Propositions 2.11 and 2.13 conclude the proof. □

Remark 4.3. The construction used in the proof of Theorem 4.1 was also used by Wojtaszczyk [24] to construct 'RUC' systems. The dual construction was used recently in [6] to construct a *quasi-greedy* basis for $L_1([0,1])$.

5. The Net Quantization Property

In this section we discuss a natural quantization property which is more general than the CQP.

Definition 5.1. Let $\varepsilon > 0$ and let $\delta > 0$.
(a) A dictionary (e_i) has the (ε,δ)-*Net Quantization Property* (abbr. (ε,δ)-NQP) if for every $x \in X$ there exist a finite subset $E \subset \mathbb{N}$ and $n_i \in \mathbb{Z}$ $(i \in E)$ such that

$$\|x - \sum_{i\in E} n_i\delta e_i\| \le \varepsilon. \tag{5.16}$$

(b) (e_i) has the NQP if (e_i) has the (ε,δ)-NQP for some $\varepsilon > 0$ and $\delta > 0$.

Remarks 5.2. (i) Note that (5.16) simply says that $\mathcal{F}_\delta((e_i))$ is an ε-net for X. In particular, choosing $x = \sum_{i\in F} a_i e_i$ in (5.16), it is important to emphasize that the set E is *not* required to be contained in F. This suggests that the the NQP property should be weaker than the CQP property, and we prove below that this is indeed the case.
(ii) The analogue of Proposition 2.3 remains valid for the NQP.

The analogue of Theorem 2.4 for the NQP which is stated below remains valid with essentially the same proof.

Theorem 5.3. *Let (e_i) be a dictionary for X. The following are equivalent:*

(i) *(e_i) has the NQP;*
(ii) *there exist $0 < \varepsilon < 1$ and $\delta > 0$ such that $\mathcal{F}_\delta((e_i))$ is an ε-net for $Ba(X)$.*

Corollary 5.4. *Let X be a separable Banach space. There exists a dictionary (e_i) with the NQP such that $\mathcal{F}_1((e_i))$ is M-dense in X and $(1/M)$-separated for some $M > 0$.*

Proof. Let $(x_n)_{n=1}^\infty$ be a semi-normalized fundamental bounded minimal system for X with $\|x_n\| \leq 1/3$ for all n. Let (y_n) be dense in the unit ball of X with $y_n \in \langle x_i \rangle_{i=1}^{n-1}$, and let $e_n = x_n + y_n$. Then (e_n) is semi-normalized and 1/2-dense in $Ba(X)$. So by Theorem 5.3 $\mathcal{F}_1((e_i))$ is an M-net for X for some $M > 0$. Using the fact that (x_i) is a bounded minimal system it is easily verified that $\mathcal{F}_1((e_i))$ is $(1/M)$-separated for sufficiently large M. □

The counterpart to Corollary 2.5 takes the following form. This result seems to be of interest even when X is finite-dimensional.

Theorem 5.5. *Let $0 < \varepsilon_0 < 1$, $\delta > 0$, and let (e_i) be a (not necessarily semi-normalized) fundamental sequence in X. If $\mathcal{F}_\delta((e_i))$ is ε_0-dense in $Ba(X)$ then $\mathcal{F}_\delta((e_i))$ is ε_1-dense in X for all*

$$\varepsilon_1 > \left(\left\lfloor \frac{\varepsilon_0}{1-\varepsilon_0} \right\rfloor + 1\right)\varepsilon_0.$$

In particular, if $\varepsilon_0 < 1/2$, then $\mathcal{F}_\delta((e_i))$ is ε_1-dense in X for all $\varepsilon_1 > \varepsilon_0$.

Next we introduce the analogue of the SCQP.

Definition 5.6. Let $\varepsilon > 0$ and let $\delta > 0$.
(a) A dictionary (e_i) has the (ε, δ)-*Strong Net Quantization Property* (abbr. (ε, δ)-SNQP) if $\mathcal{F}_{\overline{D}}((e_i))$ is an ε-net for X for every sequence $\overline{D} = (D_i)$ of δ-nets.
(b) (e_i) has the SNQP if (e_i) has the (ε, δ)-NQP for some $\varepsilon > 0$ and $\delta > 0$.

The proof of Proposition 2.10 for a general dictionary does not seem to transfer to the SNQP. However, when (e_i) is a Schauder basis it is easy to modify the proof to get the following uniform boundedness result.

Proposition 5.7. *Let (e_i) be a Schauder basis for X. The following are equivalent:*

(i) *(e_i) has the SNQP;*
(ii) *for all $\delta > 0$ and for every sequence $\overline{D} = (D_i)$ of δ-nets there exists $M := M(\overline{D}) > 0$ such that $\mathcal{F}_{\overline{D}}((e_i))$ is an M-net for X;*
(iii) *condition (ii) for $\delta = 1$.*

Remark 5.8. The analogues of Propositions 2.11 and 2.13 remain valid for the SNQP.

Trivially, every separable Banach space has a dictionary with the $(\varepsilon, c\varepsilon)$-SNQP for all $0 < c < 1$. Indeed, simply take (e_i) to be dense in the unit sphere of X. By a more careful choice of dense set in the unit sphere of ℓ_2, it is not difficult to construct an NQP dictionary for ℓ_2 which is *not* a CQP dictionary. Our next result, the construction of an SNQP *Schauder basis* which does not have the CQP, is more involved. It is a consequence of the following general embedding theorem..

Theorem 5.9. *Let (e_i) be a normalized monotone basis for a Banach space E. Given $\eta > 0$ there exists a Banach space U with a normalized monotone basis (u_i) with the following properties:*

(a) *(u_i) has the $(\varepsilon, \varepsilon/3)$-SNQP;*

(b) *there exists a subsequence (u_{n_i}) of (u_i) that is $(1+\eta)$-equivalent to (e_i).*

Before proceeding with the proof, let us see how it implies the existence of an SNQP basis which is not a CQP basis. The CQP is inherited by subsequences, so if we apply Theorem 5.9 to any basis (e_i) which does not have the CQP (e.g. the unit vector basis of ℓ_2) then the constructed basis (u_i) will have the SNQP but not the CQP.

Proof of Theorem 5.9. Choose integer reciprocals $\eta_i \downarrow 0$ such that for each j the set

$$S_j := \{\sum_{i=1}^{j} k_i \eta_i e_i^* \colon k_i \in \mathbb{Z}\} \cap Ba(E^*)$$

is $(1-\eta)$-norming for $\langle e_i \rangle_{i=1}^{j}$. Note that if $j \le k$, then each element g of S_k is an "extension" of an element g' of S_j (i.e. $g(e_i) = g'(e_i)$ for

$1 \le i \le j$). We shall construct a subset $\mathcal{G} \subset Ba(c_{00})$ containing the unit vector basis of c_{00} such that $P_n(\mathcal{G}) \subset \mathcal{G}$ for all $n \in \mathbb{N}$, where (P_n) is the sequence of basis projections in c_{00}. Then we define U to be the Banach space with Schauder basis (u_i) whose norm is given by

$$\|\sum a_i u_i\| = \sup_{f \in \mathcal{G}} |\sum f(i) a_i|.$$

The conditions on $\mathcal{G}$ ensure that (u_i) is a normalized monotone basis for U. The construction of $\mathcal{G}$ and the sequence (n_i) is inductive. Set $n_1 = 1$ and

$$\mathcal{G}_1 := \{(k_1\eta_1, 0, 0, \dots) \colon k_1\eta_1 e_1^* \in S_1\}.$$

Suppose $j_0 \ge 1$ and that n_j and $\mathcal{G}_j$ have been defined for each $j \le j_0$ such that every $f \in \mathcal{G}_j$ is supported on $[1, n_j]$, $P_n(\mathcal{G}_j) \subset \mathcal{G}_j$ for all $n \in \mathbb{N}$, and $P_{n_j}(\mathcal{G}_{j+1}) \subset \mathcal{G}_j$, i.e. every element of $\mathcal{G}_{j+1} \setminus \mathcal{G}_j$ is an extension on $[n_j + 1, n_{j+1}]$ of some element of $\mathcal{G}_j$, and such that if $f \in \mathcal{G}_j$ then there exists a $\tilde{g} := \tilde{g}(f) \in S_j$ such that $f(n_i) = \tilde{g}(e_i)$ for all $1 \le i \le j$. We now proceed to the definition of n_{j_0+1} and $\mathcal{G}_{j_0+1}$. Let

$$T_{j_0} := \{(f, g) \in \mathcal{G}_{j_0} \times S_{j_0+1} \colon \ g \text{ extends } \tilde{g}(f)\} \subset \mathcal{G}_{j_0} \times S_{j_0+1}.$$

Let $n_{j_0+1} := n_{j_0} + \operatorname{card} T_{j_0} + 1$ and define a bijection $(f, g) \to i((f, g))$ from T_{j_0} onto $[n_{j_0} + 1, n_{j_0+1} - 1]$. For each $(f, g) \in T_{j_0}$, define $f' := f'((f, g))$ by

$$f'(i) = \begin{cases} f(i) & \text{if } 1 \le i \le n_{j_0} \\ 1 & \text{if } i = i((f, g)) \\ g(e_{j_0+1}) & \text{if } i = n_{j_0+1} \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$\mathcal{G}_{j_0+1} := \{P_n(f'((f, g))) \colon (f, g) \in T_{j_0}, n \le n_{j_0+1}\}.$$

Finally, set $\mathcal{G} = \cup_{j \ge 1} \mathcal{G}_j$. Then $\mathcal{G}$ contains the unit vector basis of c_{00} and satisfies $P_n(\mathcal{G}) \subset \mathcal{G}$ ($n \in \mathbb{N}$) as claimed. Thus, (u_i) is a normalized monotone basis for U. Henceforth, we identify $\mathcal{G}$ with a norming subset of $Ba(U^*)$ and use the notation $f(\sum a_i u_i) := \sum f(i) a_i$ for $f \in \mathcal{G}$. It is clear from the construction that

$$\|\sum_{i=1}^m a_i u_{n_i}\| = \sup_{g \in S_m} g(\sum_{i=1}^m a_i e_i),$$

and so (u_{n_i}) is $(1 + \eta)$-equivalent to (e_i), which verifies (b).

Let us now turn to the verification of (a). Let $\varepsilon > 0$ and let (D_i) be a sequence of $\varepsilon/3$ nets. To show that $\mathcal{F}_{\overline{D}}((u_i))$ is an ε-net for U, it suffices to show that for every $x = \sum_{i \in A} a_i u_i \in U$, where $A \subset \mathbb{N}$ is finite, there exists $y = \sum_{i \in E} d_i e_i$ $(d_i \in D_i)$, where $E \subset \mathbb{N}$ is finite, such that $\|x - y\| \leq 2\varepsilon/3$ (since the collection of all such x is dense in U). We may assume that $A \subset [1, n_j]$ for some j. The proof is by induction on j. The case $j = 1$ is clear: $n_1 = 1$, so $x = a_1 u_1$ in this case, and we simply choose $d_1 \in D_1$ with $|a_1 - d_1| \leq \varepsilon/3$, so that $\|x - d_1 u_1\| \leq \varepsilon/3$.

Suppose the inductive hypothesis holds for $j = j_0$. For the inductive step, suppose that $x = \sum_{i=1}^{n_{j_0+1}} a_i u_i$ and let $x' = \sum_{i=1}^{n_{j_0}} a_i u_i$. By the inductive hypothesis there exists $y' = \sum_{i=1}^{n_{j_0}} d_i u_i$ such that $\|x' - y'\| \leq 2\varepsilon/3$. Let $y = \sum_{i=1}^{n_{j_0+1}} d_i u_i$ be an extension of y' to $[1, n_{j_0+1}]$. Then

$$|f(x - y)| = |f(x' - y')| \leq \varepsilon \quad \text{for all } f \in \mathcal{G}_j \text{ when } j \leq j_0.$$

Since $P_{n_{j_0+1}}(\mathcal{G}_j) = \mathcal{G}_{j_0+1}$ when $j \geq j_0$, it suffices to choose the extension y such that $|f(x - y)| \leq 2\varepsilon/3$ for all $f \in \mathcal{G}_{j_0+1} \setminus \mathcal{G}_{j_0}$. To that end, for each $(f, g) \in T_{j_0}$, setting $i' := i((f, g))$ choose $d_{i'} \in D_{i'}$ such that

$$|f(x' - y') + a_{i'} - d_{i'}| \leq \varepsilon/3.$$

This defines d_i for $n_{j_0}+1 \leq i \leq n_{j_0+1}-1$. Finally, choose $d_{n_{j_0+1}} \in D_{n_{j_0+1}}$ such that $|a_{n_{j_0+1}} - d_{n_{j_0+1}}| \leq \varepsilon/3$. This completes the definition of y. Suppose that $f' = f'((f, g))$ for some $(f, g) \in T_{j_0}$. Then

$$\begin{aligned} |f'(x - y)| &= |f(x' - y') + a_{i'} - d_{i'} + g(e_{j_0+1})(a_{n_{j_0+1}} - d_{n_{j_0+1}})| \\ &\leq |f(x' - y') + a_{i'} - d_{i'}| + |(a_{n_{j_0+1}} - d_{n_{j_0+1}})| \\ &\leq \varepsilon/3 + \varepsilon/3 = 2\varepsilon/3. \end{aligned}$$

Moreover,

$$|(P_n f')(x - y)| = |f(x' - y') + a_{i'} - d_{i'}| \leq \varepsilon/3 \qquad (i' \leq n \leq n_{j_0+1} - 1)$$

and

$$|(P_n f')(x - y)| = |(P_n f)(x' - y')| \leq 2\varepsilon/3 \qquad (1 \leq n < i')$$

by inductive hypothesis since $P_n f \in \mathcal{G}_{j_0}$ for $1 \leq n < i'$. This completes the proof of the inductive step. □

In some of our results in Section 6 it is possible to replace the CQP by the formally weaker assumption that every subsequence has the NQP. When (e_i) is a Schauder basis, however, our next result shows that this assumption is in fact equivalent to the CQP.

Theorem 5.10. *Let (e_i) be a semi-normalized basic sequence which fails the CQP. Then some subsequence fails the NQP for its closed linear span.*

Proof. Let K be the basis constant of (e_i). We may assume without loss of generality that $\|e_i\| \le 1$ for all i.

Claim 1: For every $\delta > 0$ there exists $M \subset \mathbb{N}$ such that $(e_i)_{i\in M}$ fails the $(1,\delta)$-NQP.

Proof of Claim 1: Suppose not. Then there exists $\delta > 0$ such that $(e_i)_{i\in M}$ has the $(1,\delta)$-NQP for every $M \subset \mathbb{N}$. Let $x = \sum_{i\in E} a_i e_i$ and let $n = \max E$. Since $M := E \cup (n,\infty)$ has the $(1,\delta)$-CQP there exists $y \in \mathcal{F}_\delta((e_i)_{i\in M})$ such that $\|x - y\| \le 1$. Then $\|x - P_E y\| \le K$. Thus (e_i) has the (K,δ)-CQP, which is a contradiction.

Claim 2: For all $n \in \mathbb{N}$ there exist a finite set $F_n \subset [n+1,\infty)$ and $x_n = \sum_{i\in F_n} a_i e_i$ such that $\|y - x_n\| > 2K$ for all $y \in \mathcal{F}_{1/n}((e_i)_{i\in F_n})$.

Proof of Claim 2: Let $\delta_n = 1/n$. By Claim 1 there exists $M_n \subset \mathbb{N}$ such that $(e_i)_{i\in M_n}$ fails the $(2K+1,\delta_n)$-NQP. So there exists $z_n = \sum_{i\in M_n} a_i e_i$ with $\|z_n - y\| > 2K+1$ for all $y \in \mathcal{F}_{1/n}((e_i)_{i\in M_n})$. Let $x_n = z_n|_{[n+1,\infty)}$. Note that every vector supported on $[1,n]\cap M_n$ (in particular, the vector $z_n - x_n$) can be 1-approximated by an element of $\mathcal{F}_{1/n}((e_i)_{i\in M_n})$ simply by approximating each of the (at most n) nonzero coordinates to within $\delta_n = 1/n$. Setting $F_n := \operatorname{supp} x_n$, it follows that $\|x_n - y\| > 2K$ for all $y \in \mathcal{F}_{1/n}((e_i)_{i\in F_n})$. Thus, x_n and F_n verify Claim 2.

Now pass to a subsequence so that the sets F_{n_k} satisfy $\max F_{n_k} < \min F_{n_{k+1}}$ for all $k \in \mathbb{N}$. Let $M = \cup_{k\ge1} F_{n_k}$.

Claim 3: $(e_i)_{i\in M}$ fails the NQP.

Proof of Claim 3: Suppose that $(e_i)_{i\in M}$ has the $(1,\delta)$-NQP (and hence the $(1,1/n)$-NQP provided $1/n < \delta$). Choose k with $1/n_k < \delta$. Then there exists $y \in \mathcal{F}_{1/n_k}((e_i)_{i\in M})$ such that $\|y - x_{n_k}\| \le 1$. But this implies that $\|P_{F_{n_k}}(y) - x_{n_k}\| \le 2K$, which contradicts the choice of x_{n_k} and F_{n_k}.

□

We turn now to discuss the relationship between the NQP and unconditionality.

Theorem 5.11. *Suppose that X has a semi-normalized unconditional basis (e_i) with the NQP. Then (e_i) is equivalent to the unit vector basis of c_0.*

Proof. Let K be the constant of unconditionality of (e_i) and choose $\varepsilon > 0$ such that $K < \dfrac{1-\varepsilon}{\varepsilon}$. There exists $\delta > 0$ such that $\mathcal{F}_\delta((e_i))$ is ε-dense in X. Suppose $x = \sum e_i^*(x)e_i \in X$ with $\|x\| = 1$ and $\|x\|_\infty := \sup |e_i^*(x)| = \alpha < \delta$. Choose $y \in \mathcal{F}_\delta((e_i))$ with $\|x - y\| \le \varepsilon$. Then

$$\|y\| \ge \|x\| - \|x - y\| \ge 1 - \varepsilon.$$

Since $\sup |e_i^*(x)| \le \alpha$ and since $y \in \mathcal{F}_\delta((e_i))$, it follows that $y = \sum \lambda_i e_i^*(x - y)e_i$ for a multiplier sequence (λ_i) satisfying

$$\sup |\lambda_i| \le \frac{\delta}{\delta - \alpha}.$$

Hence by K-unconditionality of (e_i), we have

$$1 - \varepsilon \le \|y\| \le K \sup |\lambda_i| \|y - x\| \le K \frac{\delta}{\delta - \alpha} \varepsilon. \tag{5.17}$$

If (e_i) is not equivalent to the unit vector basis of c_0 then α may be chosen to be arbitrarily small. But then (5.17) yields $K \ge \dfrac{1-\varepsilon}{\varepsilon}$, which contradicts the choice of ε.

□

Weaker notions of unconditionality (see [5]), especially that of a *quasi-greedy* basis, have recently attracted attention in connection with greedy algorithms for data compression. Our next goal is to show that every quasi-greedy basis with the NQP is equivalent to the unit vector basis of c_0. The relevant definitions are given next. For further information on the topic of greedy algorithms in Banach spaces, we refer the reader to [13, 6, 7, 5, 23].

Definition 5.12. Let (e_i) be a dictionary for X and let $\delta > 0$.
(a) Denote by $L((e_i), \delta)$ the least constant $L \in [1, \infty]$ with the property that whenever $\|\sum a_i e_i\| \le 1$ and $F \subset \{i \colon |a_i| \ge \delta\}$ then

$$\|\sum_{i \in F} a_i e_i\| \le L.$$

(b) We say that (e_i) is *Elton-unconditional* if

$$L((e_i), \delta) < \infty \quad \text{for all } \delta > 0.$$

(c) Denote by $K((e_i),\delta)$ the least constant $K\in[0,\infty]$ with the property that whenever $\|\sum a_ie_i\|\leq 1$ and $F=\{i\colon |a_i|\geq\delta\}$ then

$$\|\sum_{i\in F}a_ie_i\|\leq K.$$

(d) We say that (e_i) is *quasi-greedy* if

$$K((e_i)):=\sup_{\delta>0}K((e_i),\delta)<\infty.$$

Remark 5.13. Clearly, $K((e_i),\delta)\leq L((e_i),\delta))$. Note that (e_i) is unconditional if and only if $\sup_{\delta>0}L((e_i),\delta)<\infty$. It is known that every quasi-greedy basic sequence is Elton-unconditional (in fact a seminormalized Schauder basis (e_i) is Elton-unconditional if and only if $K((e_i),\delta)<\infty$ for all $\delta>0$) and that there exist Elton-unconditional bases which are not quasi-greedy [6].

Lemma 5.14. *Let (e_i) be a minimal system for X. Suppose that there exist $0<\varepsilon<1$, $\delta>0$, and $\lambda>0$ such that $\mathcal{F}_\delta((e_i))\cap\lambda Ba(X)$ is an ε-net for $Ba(X)$ and such that $L((e_i),\delta/\lambda)<\infty$. Then (e_i) is equivalent to the unit vector basis of c_0.*

Proof. Clearly, (e_i) has the NQP. So by Theorem 5.11 it suffices to show that (e_i) is unconditional. Let $S:=\mathcal{F}_\delta((e_i))\cap\lambda Ba(X)$. Since S is an ε-net for $Ba(X)$ it follows from the Hahn-Banach theorem that S is $(1-\varepsilon)$-norming for X^*, i.e.

$$\|x^*\|\leq\frac{1}{1-\varepsilon}\sup\{|x^*(x)|\colon x\in S\}\qquad(x^*\in X^*).$$

Moreover, if $x=\sum_E k_i\delta e_i\in S$ and $F\subseteq E$, then (since $x/\lambda\in Ba(X)$) $\|\sum_{i\in F}k_i\delta e_i\|\leq\lambda L((e_i),\delta/\lambda)$. Hence

$$\tilde{S}:=\{\sum_{i\in F}k_i\delta e_i\colon \sum_{i\in E}k_i\delta e_i\in S, F\subseteq E\}\subseteq\lambda L((e_i),\delta/\lambda)Ba(X).$$

Now suppose that $\sum_{i\in E} a_i e_i^* \in X^*$ and that $F \subseteq E$. Then

$$\begin{aligned}\|\sum_{i\in F} a_i e_i^*\| &\le \frac{1}{1-\varepsilon}\sup\{\sum_{i\in F} a_i e_i^*(x) \colon x \in S\}\\ &\le \frac{1}{1-\varepsilon}\sup\{\sum_{i\in E} a_i e_i^*(x) \colon x \in \tilde{S}\}\\ &\le \frac{\lambda}{1-\varepsilon} L((e_i),\delta/\lambda)\|\sum_{i\in E} a_i e_i^*\|.\end{aligned}$$

Thus, (e_i^*) is K-unconditional for $K = \lambda L((e_i),\delta/\lambda)/(1-\varepsilon)$, and hence by duality (e_i) is also K-unconditional. □

The following substantial strengthening of Theorem 5.11 is an immediate consequence of the last result.

Theorem 5.15. *Suppose that (e_i) is a minimal system with the NQP. If (e_i) is Elton-unconditional (in particular, if (e_i) is quasi-greedy) then (e_i) is equivalent to the unit vector basis of c_0.*

The main open question of this section is the following.

Problem 5.16. Suppose that X has an NQP basis. Does $c_0 \hookrightarrow X$?

In fact, we do not know whether or not ℓ_1 provides a negative answer to Problem 5.16.

Problem 5.17. Does ℓ_1 have an NQP basis (resp. minimal system)?

We conclude this section with some partial results concerning Problem 5.16.

Theorem 5.18. *Suppose that (e_i) is a bounded NQP minimal system for X. Then $\ell_1 \hookrightarrow X^*$. In fact, a subsequence of (e_i^*) is equivalent to the unit vector basis of ℓ_1 or $\ell_\infty \hookrightarrow X^*$.*

Proof. Since (e_i) has the NQP, there exists $\delta > 0$ such that $\mathcal{F}_\delta((e_i)) \cap (3/2)Ba(X)$ is a 1/2-net for $Ba(X)$. Thus,

$$\frac{1}{2}\|x^*\| \le \sup\{|x^*(x)| \colon x \in \mathcal{F}_\delta((e_i)) \cap \frac{3}{2}Ba(X)\} \le \frac{3}{2}\|x^*\| \qquad (x^* \in X^*).$$

So $X^* \hookrightarrow C(K)$, where K is the weak-star closure of $\mathcal{F}_\delta((e_i)) \cap (3/2)Ba(X)$ in X^{**}. By Rosenthal's ℓ_1 theorem [20], (e_i^*) has a subsequence equivalent to the unit vector basis of ℓ_1 or a weakly

Cauchy subsequence (f_i^*). The former obviously implies that $\ell_1 \hookrightarrow X^*$. In the latter case, let $g_i^* = f_{2i}^* - f_{2i-1}^*$. Then (g_i^*) is weakly null in X^*, and, since the range of $f_i^*|_K \subset \mathbb{Z}\delta$, we have

$$g_i^*(k) \neq 0 \Rightarrow |g_i^*(k)| \geq \delta \qquad (i \in \mathbb{N}, k \in K).$$

Thus, for each $k \in K$, the sequence $(g_i^*(k))$ is eventually zero, so the series $\sum g_i^*$ is *extremely weakly unconditionally Cauchy*, i.e. $\sum_{i=1}^{\infty} |g_i^*(k)|$ converges (trivially!) for every $k \in K$. By a theorem of Elton [9] (see also [11]), $c_0 \hookrightarrow [(g_i^*)]$. But this implies that ℓ_∞ is isomorphic to a complemented subspace of X^* [3], and a fortiori that $\ell_1 \hookrightarrow X^*$. $\square$

Corollary 5.19. *Suppose that X is reflexive. Then X does not contain a bounded minimal system with the NQP.*

6. Containment of c_0

The main result of this section is the following converse to Theorem 4.1.

Theorem 6.1. *Let (e_i) be a semi-normalized basic sequence with the CQP. Then (e_i) has a subsequence that is equivalent to the unit vector basis of c_0 or to the summing basis of c_0.*

As the proof is quite long we shall break it down into several parts. We shall frequently refer to the excellent survey article [1] for the proofs of certain assertions.

First we prove a result which is of independent interest.

Theorem 6.2. Let (e_i) be a semi-normalized nontrivial weakly Cauchy basis for X. Then there exists a subsequence (e_{n_i}) such that either

(a) (e_{n_i}) is equivalent to the summing basis of c_0, or
(b) $(e_{n_i}^*)$ is weakly null in $[(e_{n_i})]^*$.

Proof. Let $x^{**} \in X^{**} \setminus X$ be the weak-star limit of (e_i). By passing to a subsequence we may assume that (e_i) dominates the summing basis, i.e. $\|\sum a_i e_i\| \geq c\|(a_i)\|_{sb}$ for some $c > 0$ [1, Prop. II.1.5] If $x^{**} \in X^{**} \setminus D(X)$, where $D(X)$ denotes the collection of all elements of $X^{**} \setminus X$ whose restrictions to $Ba(X^*)$ (equipped with the weak-star topology) are differences of semi-continuous functions (see [1]), then by [21, Theorem 1.8] (e_i) has a *strongly summing* subsequence (e_{n_i}). In particular, $(\sum_{i=1}^m e^*_{n_i})$ is a nontrivial weakly Cauchy sequence in $[(e_{n_i})]^*$ [1, Lemma II.2.6], and so $(e^*_{n_i})$ is weakly null in $[(e_{n_i})]^*$, which yields (b).

Now suppose that $x^{**} \in D(X)$. Then there exists a sequence $(x_i) \subset X$ that is equivalent to the summing basis of c_0 such that $x_i \to x^{**}$ weak-star [1, Theorem II.1.2]. Note that $(e_i - x_i)$ is weakly null. If some subsequence of $(e_i - x_i)$ is norm-null then (a) follows by a standard perturbation argument. So we may assume that $(e_i - x_i)$ is a semi-normalized weakly null sequence. By a theorem of Elton [8, 17], $(e_i - x_i)$ has either a subsequence equivalent to the unit vector basis of c_0 or a basic subsequence whose sequence of biorthogonal functionals is weakly null (in the dual of the closed linear span of that basic subsequence). If the first alternative holds, let $(e_{n_i} - x_{n_i})$ be the c_0 subsequence. Then

$$\begin{aligned} c\|(a_i)\|_{sb} &\leq \|\sum a_i e_{n_i}\| \\ &\leq \|\sum a_i(e_{n_i} - x_{n_i})\| + \|\sum a_i x_{n_i}\| \\ &\leq C_1 \sup_i |a_i| + C_2\|(a_i)\|_{sb} \leq C_3\|(a_i)\|_{sb}, \end{aligned}$$

for certain constants C_1, C_2, C_3. Hence (e_{n_i}) is equivalent to the summing basis of c_0. If the second alternative holds, let $(e_{n_i} - x_{n_i})$ be a basic subsequence with weakly null biorthogonal functionals. To prove that $(e^*_{n_i})$ is weakly null in $[(e_{n_i})]^*$, it suffices to show that $a_i \to 0$ whenever (a_i) satisfies $\sup_m \|\sum_{i=1}^m a_i(e_{n_i})\| = K < \infty$. Now

$$\|\sum_{i=1}^m a_i x_{n_i}\| \leq C_2\|(a_i)\|_{sb} \leq c^{-1}C_2\|\sum_{i=1}^m a_i(e_{n_i})\| \leq c^{-1}C_2K,$$

and hence by the triangle inequality

$$\sup_m \|\sum_{i=1}^m a_i(e_{n_i} - x_{n_i})\| \leq K + c^{-1}C_2K.$$

Since the sequence of biorthogonal functionals to $(e_{n_i} - x_{n_i})$ is weakly null, we deduce fnally that $a_i \to 0$. □

Proposition 6.3. *Suppose X has a minimal system (e_i) with the NQP. Then no subsequence of (e^*_i) is weakly null.*

Proof. Let $0 < \varepsilon < 1$. There exists $\delta > 0$ such that $\mathcal{F}_\delta((e_i)) \cap 2Ba(X)$ is an ε-net for $Ba(X)$. Thus,

$$\frac{1-\varepsilon}{2}\|x^*\| \leq \sup\{|x^*(x)| \colon x \in \mathcal{F}_\delta((e_i)) \cap 2Ba(X)\} \leq 2\|x^*\| \qquad (x^* \in X^*).$$

So $X^* \hookrightarrow C(K)$, where K is the weak-star closure of $\mathcal{F}_\delta((e_i)) \cap 2Ba(X)$ in X^{**}. Suppose that $(e^*_{n_i})$ is a weakly null subsequence of (e^*_i), whence $\sup_i \|e^*_{n_i}\| = C < \infty$. Thus,

$$\{|e^*_{n_i}(k)| \colon k \in K\} \subset \{0\} \cup [\delta, 2C] \qquad (i \geq 1),$$

so $(e^*_{n_i})$ has an unconditional basic subsequence [5, Theorem 23] (see also [10] and [15]). Relabel this unconditional subsequence as $(e^*_{n_i})$ and let $Y := [(e^*_{n_i})] \subset X^*$. Observe that $(e_{n_i}|_Y)$ is a semi-normalized unconditional basic sequence in Y^* whose biorthogonal sequence is $(e^*_{n_i}) \subset Y$. We claim that (e_{n_i}) has the NQP for its closed linear span in Y^*. To prove the claim, let $x = \sum_{i \in A} a_i e_i$, where $A \subset \{n_i \colon i \geq 1\}$ is finite. Since (e_i) has the NQP for X there exists $y = \sum_{i \in B} m_i \delta e_i$ with $\|x - y\| \leq \varepsilon$, where $B \subseteq \mathbb{N}$ is finite and $m_i \in \mathbb{Z}$ for each i. Let $z = \sum_{i \in B'} m_i e_i$, where $B' = B \cap \{n_i \colon i \geq 1\}$. Then $y|_Y = z|_Y$ and

$$\|x - z\|_{Y^*} \leq \|x - y\| \leq \varepsilon,$$

which proves the claim. Since $(e_{n_i}|_Y) \subset Y^*$ is an unconditional basic sequence with the NQP, it follows from Theorem 5.11 that $(e_{n_i}|_Y)$ is equivalent to the unit vector basis of c_0. But this implies that $(e^*_{n_i})$ is equivalent to the unit vector basis of ℓ_1, which contradicts the assumption that $(e^*_{n_i})$ is weakly null! □

Proposition 6.4. *Suppose that (e_i) is a weakly null dictionary for X. If every subsequence of (e_i) has the NQP for its closed linear span (in particular, if (e_i) has the CQP) then (e_i) has a subsequence equivalent to the unit vector basis of c_0.*

Proof. By the aforementioned theorem of Elton (e_i) has a subsequence equivalent to the unit vector basis of c_0 or a basic subsequence (e_{n_i}) such that $(e^*_{n_i})$ is weakly null in $[(e_{n_i})]^*$. But the latter cannot happen by Proposition 6.3. □

Proposition 6.5. *Suppose that (e_i) is a nontrivial weakly Cauchy dictionary for X. If every subsequence of (e_i) has the NQP for its closed* *linear span (in particular, if* (e_i) *has the CQP) then* (e_i) *has a subsequence equivalent to the summing basis of* c_0*.*

Proof. By Theorem 6.2 either (e_i) has a subsequence equivalent to the summing basis or a basic subsequence (e_{n_i}) such that $(e^*_{n_i})$ is weakly null in $[e_{n_i}]^*$. But (e_{n_i}) has the NQP for its closed linear span, so the latter alternative cannot happen by Proposition 6.3. □

Proof of Theorem 6.1. By Rosenthal's ℓ_1 theorem [20], either (e_i) has a subsequence that is equivalent to the unit vector basis of ℓ_1 or a weakly Cauchy basic subsequence. The first possibility cannot occur since the unit vector basis of ℓ_1 does not have the NQP. For the second possibility, either the subsequence is weakly null or it is nontrivial weakly Cauchy. In the former case there is a subsequence equivalent to the unit vector basis of c_0 by Proposition 6.4, and in the latter there is a subsequence equivalent to the summing basis by Proposition 6.5. □

Combining Theorem 4.1 and Theorem 6.1 we obtain a new characterization of separable Banach spaces containing c_0

Theorem 6.6. *Let* X *be a separable Banach space. The following are equivalent:*

(a) $c_0 \hookrightarrow X$*;*
(b) X *has a weakly null bounded and total minimal system with the SCQP;*
(c) X *has a total minimal system* (e_i) *with the CQP;*
(d) X *has a dictionary* (e_i) *with no nonzero weak limit point such that every subsequence of* (e_i) *has the NQP for its closed linear span.*

Proof. (a) ⇒ (b) follows from Theorem 4.1; (b) ⇒ (c) is trivial; (c) ⇒ (d) follows from the fact that a total minimal system has no nonzero subsequential weak limit point. To prove (d) ⇒ (a), note that (e_i) has a weakly Cauchy basic subsequence, so the result follows from Propositions 6.4 and 6.5.

□

We conclude this section with some results about NQP minimal systems that are motivated by Problem 5.16 above.

Proposition 6.7. *Let (e_i) be a minimal system for X with the NQP. Then no subsequence of (e_i^*) is nontrivial weakly Cauchy.*

Proof. Suppose that (e_i) has the (ε,δ)-NQP and that $(e_{n_i}^*)$ is nontrivial weakly Cauchy. After passing to a subsequence of $(e_{n_i}^*)$ we may assume that (f_i) is a weakly null basis for $Y=[(f_i)]=[(e_{n_i}^*)]$, where $f_1=e_{n_1}^*$ and $f_i=e_{n_i}^*-e_{n_{i-1}}^*$ for $i\geq 2$ [1, Prop. II.1.7]. Setting $e=\sum e_{n_i}|_Y$ (the sum converging weak-star in Y^*) and setting $e_0=0$, the sequence of biorthogonal functionals $(f_i^*)\subset Y^*$ is given by $f_i^*=e-\sum_{j=0}^{i-1}e_{n_j}|_Y$. We claim that (f_i^*) has the NQP for its closed linear span in Y^*. To check this claim, let $x=\sum_{i\in A}a_i f_i^*$, where $A\subseteq N$ is finite. Then we may rewrite the expression for x in the form

$$x=bf_1^*+\sum_{i\in B}b_ie_{n_i}|_Y \tag{6.18}$$

for some finite $B\subset\mathbb{N}$ and scalars b,b_i. Since (e_i) has the (ε,δ)-NQP there exists $z=\sum_{i\in C}m_i\delta e_i$ $(m_i\in\mathbb{Z})$, where C is a finite subset of $\mathbb{N}$, such that $\|\sum_{i\in B}b_ie_{n_i}-z\|\leq\varepsilon$. Since $e_{n_i}|_Y=f_i^*-f_{i+1}^*$, it follows that

$$z|_Y=\sum_{i\in C'}m_i'\delta f_i^* \tag{6.19}$$

for some finite $C'\subset\mathbb{N}$ and $m_i'\in\mathbb{Z}$. Choose $m\in\mathbb{Z}$ such that $|b-m\delta|\leq\delta$. From (6.18) and (6.19), we obtain

$$\begin{aligned}\|x-(m\delta f_1^*+\sum_{i\in C'}m_i'\delta f_i^*)\|_{Y^*}&\leq|m\delta-b|\|f_1^*\|+\|\sum_{i\in B}b_ie_{n_i}|_Y-z|_Y\|_{Y^*}\\&\leq\delta\|f_1^*\|+\|\sum_{i\in B}b_ie_{n_i}-z\|\\&\leq\delta\|f_1^*\|+\varepsilon,\end{aligned}$$

which verifies the claim. Thus (f_i^*) has the NQP for its closed linear span and its biorthogonal sequence (f_i) is weakly null. But this contradicts Proposition 6.3. □

Theorem 6.8. *Let (e_i) be a seminormalized basis with the NQP. Then every subsequence of (e_i^*) has a further subsequence equivalent to the unit vector basis of ℓ_1.*

Proof. By Proposition 6.3 no subsequence of (e_i^*) is weakly null, and by Proposition 6.7 no subsequence is nontrivial weakly Cauchy. Thus, by Rosenthal's ℓ_1 theorem, every subsequence of (e_i^*) has a further subsequence equivalent to the unit vector basis of ℓ_1. □

7. Some Notions Related to the CQP

There seems to be very little known about the relationships between the different quantization properties introduced in the previous sections. Let us recast some of the questions we formulated in previous sections.

Throughout this section (e_i) and (e_i^*) is a bounded minimal system of a Banach space X and we assume that (e_i) (and, thus, also (e_i^*)) are semi normalized.

Question 7.1. Let $\varepsilon, \delta > 0$.

(1) If (e_i) satisfies the (ε, δ)-CQP, does it satisfy the $(\varepsilon, \delta/2)$-SCQP, does it satisfy (ε, δ)-neighborly CQP (see Remark 3.7)?

In the case that the answer to our aforementioned questions are negative do they at least have qualitative positive answers, i.e. does the CQP imply the SCQP, does the CQP imply the neighborly CQP?

(2) In our next example we will exhibit that for some $\varepsilon, \delta > 0$ the (ε, δ)-NQP does not imply the $(\varepsilon, \delta/2)$-SNQP. But we do not know whether or not the NQP implies the SNQP.

One can reformulate these questions into finite dimensional ones. Assume that $n \in \mathbf{N}$ and that $K \subset \mathbf{R}^{\mathrm{n}}$ is a symmetric and convex body (i.e. $0 \in K^\circ$).

Let us consider the following properties K may have

$$\text{(P1)} \qquad \bigcup_{z \in \mathbb{Z}^n} z + K = \mathbf{R}^{\mathrm{n}}$$

$$\text{(P2)} \qquad \bigcup_{z \in \prod_{i=1}^n D_i} z + K = \mathbf{R}^{\mathrm{n}}$$

$$\text{whenever } D_i \subset R, 0 \in D_i, D_i \text{ is } \frac{1}{2}\text{-net for } i = 1, 2, \cdots n$$

$$\text{(P3)} \qquad [0,1]^n \subset \bigcup_{\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots \varepsilon_n) \in \{0,1\}^n} \varepsilon + K$$

Note that (P3) means that, not only is every point of $\mathbf{R}^{\mathrm{n}}$ an element of some translate of K by some point p having integer coordinates, but that p can be chosen so that $\max_{i=1,2\cdots n}|x_i - p_i| \le 1$.

It is easy to see that (e_i) satisfy (ε,δ)-CQP, $(\varepsilon,\delta/2)$-SCQP or (ε,δ)-neighborly CQP, if and only if for any finite $I \subset \mathbf{N}$ the set

$$K_I = \frac{\varepsilon}{\delta} B_X \cap [e_i : i \in I],$$

satisfies (P1), (P2) or (P3) respectively.

If we do not assume that (e_i) is a monotone basis a similar statement for NQP and SNQP is slightly more complicated.

First if $E = (R^n, \|\cdot\|)$ is finite dimensional then the unit vector basis (e_i) has the (ε,δ)-NQP or the $(\varepsilon,\delta/2)$-SNQP if and only if $\frac{\varepsilon}{\delta} B_E$ satisfies (P1) and or (P2). If for all $n \in \mathbf{N}$ $K_{\{1,2,\cdots n\}}$ (defined as above) satisfies (P1) or (P2) then for any $\eta > 0$ (e_i) has the $(\varepsilon,\delta-\eta)$-NQP or the $(\varepsilon,\delta-\eta)$-SNQP respectively. Conversely, if (e_i) is a monotone basis which satisfies the (ε,δ)-NQP or the (ε,δ)-SNQP, then for all $n \in \mathbf{N}$ the set $K_{\{1,2,\cdots n\}}$ satisfies (P1) or (P2), respectively.

The following example shows that $(P1) \not\Rightarrow (P2)$.

Example. In $\mathbf{R}^2$ let K be the convex hull of the points

$$P_1 = (\frac{1}{4}, 1),\ P_2 = (\frac{3}{4}, 1),\ P_3 = (-\frac{1}{4}, -1),\ \text{ and } P_4 = (-\frac{3}{4}, -1).$$

Instead of a formal proof, we leave it to the reader to verify the following by drawing a picture:

a) K is a parallelogram which tiles R^n, i.e.

$$\bigcup_{z\in\mathbb{Z}^2} z + K = \mathbf{R}^2 \text{ and } (\mathrm{z} + \mathrm{K}^\circ) \cap (\mathrm{z}' + \mathrm{K}^\circ) \text{ whenever } \mathrm{z} \ne \mathrm{z}' \text{ are in } \mathbb{Z}^2.$$

b) For $\mathbb{Q} = \frac{3}{4}P_2 + \frac{1}{4}P_3$ we have

$$Q \in [(0,0) + K] \cap [(1,1) + K]\,.$$

c) For small enough $\eta > 0$

$$P - (0, \eta/4) \notin \bigcup_{z\in\mathbb{Z}\times(1-\eta)\mathbb{Z}} z + K.$$

(thus K does not satisfy $(P2)$).

The aformentioned questions can be now reformulated as follows.

Question 7.2. Let $K \subset \mathbf{R}^{\mathrm{n}}$ be convex and symmetric and put for $I \subset \{1, 2, \cdots, n\}$

$$K_I = \big\{(x_1, x_2, \cdots, x_n) \in K : x_i = 0 \text{ for } i \in \{1, 2 \cdots, n\} \setminus I\big\}.$$

(1) If K_I satisfies $(P1)$ for all $I \subset \{1, 2 \cdots n\}$, does it satisfy $(P2)$ or $(P3)$?
(2) Is there at least a universal constant $c \geq 1$ so that if K_I satisfies $(P1)$ for all $I \subset \{1, 2 \cdots n\}$, then it satisfies $(P2)$ or $(P3)$?
(3) Is there a universal constant $c \geq 1$ so that if K satisfies $(P1)$ then it satisfies $(P2)$?

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