

COEFFICIENT QUANTIZATION FOR FRAMES IN BANACH SPACES

P. G. CASAZZA, S. J. DILWORTH, E. ODELL, TH. SCHLUMPRECHT, AND A. ZSÁK

ABSTRACT. Let (e_i) be a fundamental system of a Banach space. We consider the problem of approximating linear combinations of elements of this system by linear combinations using quantized coefficients. We will concentrate on systems which are possibly redundant. Our model for this situation will be frames in Banach spaces.

CONTENTS

1. Introduction	1
2. Frames in Hilbert spaces and Banach spaces	5
3. Three Examples	11
4. Quantization with Z -bounded coefficients	13
5. Quantization and Cotype	23
6. Concluding remarks and open problems	32
References	34

1. INTRODUCTION

Hilbert space frames provide a crucial theoretical underpinning for compression, storage and transmission of signals because they provide robust and stable representation of vectors. They also have applications in mathematics and engineering in a wide variety of areas including sampling theory [AG], operator theory [HL], harmonic analysis, nonlinear sparse approximation [DE], pseudo-differential operators [GH], and quantum computing [EF].

In many situations it is useful to think of a signal as being a vector x in a Hilbert space and being represented as a (finite or infinite) sequence $(\langle x_i, x \rangle)_{i=1}^{\infty}$, where

Key words and phrases. Coefficient Quantization; Banach spaces; Frames.

2000 Mathematics Subject Classification: Primary 46B20, Secondary 41A65.

The research of the first, second, third and fourth author was supported by the NSF. The first, second, third, and fifth authors were supported by the Linear Analysis Workshop at Texas A&M University in 2007. All authors were supported by the Banff International Research Station.

(x_i) is a *frame*, i.e. a sequence in H which satisfies for some $0 < a \leq b$,

$$(1) \quad a\|x\|^2 \leq \sum | \langle x_i, x \rangle |^2 \leq b\|x\|^2, \text{ whenever } x \in H.$$

Since the sequence (x_i) does not have to be (and usually is not) a basis for H , the representation of an $x \in H$ as the sequence $(\langle x_i, x \rangle)_{i=1}^{\infty}$ includes some redundancy, which, for example, can be used to correct errors in transmissions [GKK]. Using a Hilbert space as the underlying space has, *inter alia*, the advantage of an easy reconstruction formula. Nevertheless, there are circumstances which make it necessary to leave the confines of a Hilbert space, and generalize frames to the category of Banach spaces. One such instance occurs when we wish to replace the frame coefficients by *quantized coefficients*, i.e. by integer multiples of a given $\delta > 0$.

An example of such a situation is described by Daubechies and DeVore in [DD]: Let $f \in L_2(-\infty, \infty)$ be a *band-limited function*, to wit, the support of the Fourier Transform \hat{f} is contained in $[-\Omega, \Omega]$ for some $\Omega > 0$. For simplicity we assume that $\Omega = \pi$. Now we can think of \hat{f} as an element of $L_2[-\pi, \pi]$, write \hat{f} on $[-\pi, \pi]$ as a series in e^{-inx} , $n \in \mathbb{Z}$, and apply the inversion formula for the Fourier transform. This leads to the *sampling formula*

$$f(x) = \sum_{n \in \mathbb{Z}} f(n) \frac{\sin(x\pi - n\pi)}{x\pi - n\pi}, \quad x \in \mathbb{R}.$$

This series converges ‘badly’. In particular it is not absolutely convergent in general. Therefore we consider some $\lambda > 1$ and think of the space $L_2[-\pi, \pi]$ as being embedded (in the natural way) into $L_2[-\lambda\pi, \lambda\pi]$. The family of functions $(e^{-inx/\lambda})_{n \in \mathbb{Z}}$ forms an orthogonal basis for $L_2[-\lambda\pi, \lambda\pi]$, and it can be viewed as a frame for the ‘smaller’ space $L_2[-\pi, \pi]$ (see section 2). We write $\hat{f}(\xi) = \sqrt{2\pi}\hat{\rho}(\xi) \cdot \hat{f}(\xi)$, where $\hat{\rho} : \mathbb{R} \rightarrow [0, 1/\sqrt{2\pi}]$ is C_{∞} , $\hat{\rho}|_{[-\pi, \pi]} \equiv 1/\sqrt{2\pi}$, and $\hat{\rho}|_{(-\infty, \lambda\pi] \cup [\lambda\pi, \infty)} \equiv 0$. Now we can express \hat{f} on $[-\lambda\pi, \lambda\pi]$ as a series in $(e^{-inx/\lambda})$ and apply the inverse transform once again. This leads to the expansion

$$f(x) = \frac{1}{\lambda} \sum_{n \in \mathbb{Z}} f\left(\frac{n}{\lambda}\right) \rho\left(x - \frac{n}{\lambda}\right), \quad x \in \mathbb{R},$$

which not only converges faster, but is also absolutely unconditionally convergent, since $\hat{\rho}$ is C_{∞} , and, thus, ρ and all its derivatives are in $L_1(\mathbb{R})$.

Now assume that $\|f\|_{L_{\infty}} \leq 1$ (note that bandlimited functions are bounded in L_{∞}). It was shown in [DD] that the $\Sigma - \Delta$ -*quantization algorithm* can be used to find a

sequence $(q_n)_{n \in \mathbb{Z}} \subset \{-1, 1\}$ for which

$$\left| f(x) - \frac{1}{\lambda} \sum_{n \in \mathbb{Z}} q_n \rho\left(x - \frac{n}{\lambda}\right) \right| \leq \frac{1}{\lambda} \|\rho'\|_{L_1}, \text{ for } x \in \mathbb{R}.$$

This means that our approximation does not hold in L_2 (and it need not for the applications at hand) but it does hold in the Banach space L_∞ (in fact in $C(\mathbb{R})$).

We consider therefore a signal to be an arbitrary vector x in a Banach space X and ask if there is a dictionary (e_i) , e.g. some sequence (e_i) whose span is dense in X , so that x can be approximated in norm, up to some $\varepsilon > 0$, by a linear combination of the e_i 's using only coefficients from a discrete alphabet, i.e. the integer multiples of some given δ . The case that (e_i) is a non-redundant system, for example a basis, or, more generally, a total fundamental minimal system, was treated in [DOSZ]. It was shown there, for example, that if (e_i) is a semi-normalized fundamental and total minimal system which has the property that for some $\varepsilon, \delta > 0$ every vector of the form $x = \sum_{i \in E} a_i e_i$, with $E \subset \mathbb{N}$ finite, can be ε -approximated by a vector $\tilde{x} = \sum_{i \in E} \delta k_i e_i$, with $(k_i) \subset \mathbb{Z}$, then (e_i) must have a subsequence which is either equivalent to unit-vector basis of c_0 , or to the summing basis for c_0 . Conversely, every separable Banach space X containing c_0 admits such a total fundamental minimal system.

In this work we will concentrate on redundant dictionaries. Our model for redundant dictionaries will be frames in Banach spaces. In section 2 we shall recall their definition and make some elementary observations. Before we tackle the problem of coefficient quantization with respect to frames, we first have to ask ourselves what exactly we mean by a *meaningful coefficient quantization*. In section 3 we recall the notion *Net Quantization Property* (NQP) as introduced for fundamental systems in [DOSZ]. We shall then present several examples of systems which formally satisfy the NQP, but on the other hand clearly do not accomplish the goals of quantization, namely data compression and easy reconstruction. These examples will lead us to a notion of quantization which is more restrictive, and more meaningful, in the case of redundant systems.

In section 4 we ask under which circumstances one can approximate a vector in a Banach space X by a vector with quantized coefficients which are bounded in some *associated sequence space* Z with a basis (z_i) (see Definition 4.1). If Z has non trivial lower estimates this is only possible if one reconciles with the fact that the length of the frame increases exponentially with the dimension of the underlying space. We shall show this type of quantization cannot happen if (z_i) satisfies nontrivial lower and upper estimates. The proof of these facts utilizes volume arguments and must therefore be formulated first in the finite-dimensional case. An infinite-dimensional

argument proves directly that the associated space Z with a semi-normalized basis (z_i) cannot be reflexive. In particular, there is no semi-normalized frame (x_i) for an infinite-dimensional Hilbert space so that for some choice of $0 < \varepsilon, \delta < 1$ and $C \geq 1$, every $x \in H$, $\|x\| = 1$, can be ε -approximated by a vector $\tilde{x} = \sum \delta k_i x_i$, with $(k_i) \subset \mathbb{Z}$ and $\sum \delta^2 k_i^2 \leq C$.

In section 5 we consider conditions under which an n -dimensional space admits, for given $\varepsilon, \delta > 0$ and $C \geq 1$, a finite frame $(x_i)_{i=1}^N$, so that every element in the *zonotope* $\{\sum_{i=1}^N a_i x_i : |a_i| \leq 1\}$ can be ε -approximated by some element from $\{\sum_{i=1}^N \delta k_i x_i : k_i \in \mathbb{Z}, |k_i| \leq C/\delta\}$. Using results from convex geometry we shall show that this is only possible for spaces X with trivial cotype. Among others, we provide an answer to a question raised in [DOSZ] and prove that ℓ_1 does not have a semi-normalized basis with the NQP.

In the final section we will state some open problems.

All Banach spaces are considered to be spaces over the real field \mathbb{R} . S_X and B_X , denote the unit sphere and the unit ball of a Banach space X , respectively. For a set S we denote by $c_{00}(S)$, or simply c_{00} , if $S = \mathbb{N}$, the set of all families $x = (\xi_s)_{s \in S}$ with finite support, $\text{supp}(x) = \{s \in S : \xi_s \neq 0\}$. The unit vector basis of c_{00} , as well as the unit vector basis of ℓ_p , $1 \leq p < \infty$, and c_0 is denoted by (e_i) .

A *Schauder basis*, or simply a *basis*, of a Banach space X is a sequence (x_n) , which has the property that every x can be uniquely written as a norm converging series $x = \sum a_i x_i$. It follows then from the Uniform Boundedness Principle that the coordinate functionals (x_n^*) ,

$$x_n^* : X \rightarrow \mathbb{R}, \quad \sum a_i x_i \mapsto a_n$$

are bounded (cf.[FHHMPZ]) and the projections P_n , with

$$P_n : X \rightarrow X, \quad x = \sum a_i x_i \mapsto \sum_{i=1}^n a_i x_i, \quad \text{for } n \in \mathbb{N}$$

are continuous and uniformly bounded in the operator norm. We call $C = \sup_{n \in \mathbb{N}} \|P_n\|$ the *basis constant of (x_i)* and $K = \sup_{0 \leq m \leq n} \|P_n - P_m\|$ ($P_0 \equiv 0$) the *projection constant of (x_i)* . Note that $C \leq K \leq 2C$. We call (x_n) *monotone* if $C = 1$ and *bimonotone* if also $K = 1$. A basis (x_n) is called *unconditional* if for any $x \in X$ the unique representation $x = \sum a_n x_n$ converges unconditionally. This is equivalent (cf. [FHHMPZ]) to the property that for all $(a_i) \in c_{00}$

$$K_u = \sup \left\{ \left\| \sum \pm a_i x_i \right\| : \left\| \sum a_i x_i \right\| = 1 \right\} < \infty.$$

If X is a finite dimensional space we can represent it isometrically as $(\mathbb{R}^n, \|\cdot\|)$ where $\|\cdot\|$ is a norm function on \mathbb{R}^n . With this representation we consider the Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$ and denote it by $\text{Vol}(A)$. Of course $\text{Vol}(A)$ depends on the representation of X . Nevertheless, if we only consider certain ratios of volumes this is not the case. Therefore, the quotient $\text{Vol}(A)/\text{Vol}(B)$ is well defined even in abstract finite dimensional spaces without any specific representation.

2. FRAMES IN HILBERT SPACES AND BANACH SPACES

In this section we give a short review of the concept of frames in Banach spaces, and make some preparatory observations. Let us start with the well known notion of Hilbert space frames.

Definition 2.1. Let H be a (finite or infinite dimensional) Hilbert space. A sequence $(x_j)_{j \in \mathbb{J}}$ in H , $\mathbb{J} = \mathbb{N}$ or $\mathbb{J} = \{1, 2, \dots, N\}$, for some $N \in \mathbb{N}$, is called a *frame of H* or *Hilbert frame for H* if there are $0 < a \leq b < \infty$ so that

$$(2) \quad a\|x\|^2 \leq \sum_{j \in \mathbb{N}} |\langle x, x_j \rangle|^2 \leq b\|x\|^2 \text{ for all } x \in H.$$

For a frame $(x_j)_{j \in \mathbb{J}}$ of H we consider the operator

$$\Theta : H \rightarrow \ell_2(\mathbb{J}), \quad x \mapsto (\langle x, x_j \rangle)_{j \in \mathbb{J}},$$

its adjoint

$$\Theta^* : \ell_2(\mathbb{J}) \rightarrow H, \quad (\xi_j)_{j \in \mathbb{J}} \mapsto \sum_{j \in \mathbb{J}} \xi_j x_j$$

and their product

$$I = \Theta^* \circ \Theta : H \rightarrow H, \quad x \mapsto \sum_{j \in \mathbb{J}} \langle x, x_j \rangle x_j.$$

Since

$$a\|x\|^2 \leq \sum_{j \in \mathbb{N}} |\langle x, x_j \rangle|^2 = \left\langle x, \sum_{j \in \mathbb{N}} \langle x, x_j \rangle x_j \right\rangle = \langle x, I(x) \rangle \leq b\|x\|^2,$$

I is a positive and invertible operator with $a\text{Id}_H \leq I \leq b\text{Id}_H$ and thus,

$$x = I^{-1} \circ I(x) = \sum_{j \in \mathbb{N}} \langle x, x_j \rangle I^{-1}(x_j), \text{ or}$$

$$x = I \circ I^{-1}(x) = \sum_{j \in \mathbb{N}} \langle I^{-1}(x), x_j \rangle x_j = \sum_{j \in \mathbb{N}} \langle x, I^{-1}(x_j) \rangle x_j.$$

For an introduction to the theory of Hilbert space frames we refer the reader to [Ca1] and [Ch]. We follow [HL] and [CHL] for the generalization of frames to Banach spaces.

Definition 2.2. (Schauder Frame) Let X be a (finite or infinite dimensional) separable Banach space. A sequence $(x_j, f_j)_{j \in \mathbb{J}}$, with $(x_j)_{j \in \mathbb{J}} \subset X$, $(f_j)_{j \in \mathbb{J}} \subset X^*$, and $\mathbb{J} = \mathbb{N}$ or $\mathbb{J} = \{1, 2, \dots, N\}$, for some $N \in \mathbb{N}$, is called a (Schauder) frame of X if for every $x \in X$

$$(3) \quad x = \sum_{j \in \mathbb{J}} f_j(x)x_j.$$

In case that $\mathbb{J} = \mathbb{N}$, we mean that the series in (3) converges in norm, i.e. that $x = \lim_{n \rightarrow \infty} \sum_{j=1}^n f_j(x)x_j$.

An *unconditional frame of X* is a frame $(x_i, f_i)_{i \in \mathbb{N}}$ for X for which the convergence in (3) is unconditional.

We call a frame (x_i, f_i) *bounded* if

$$\sup_i \|x_i\| < \infty \text{ and } \sup_i \|f_i\| < \infty,$$

and semi-normalized if (x_i) and (f_i) are semi-normalized, i.e. if $0 < \inf_i \|x_i\| \leq \sup_i \|x_i\| < \infty$ and $0 < \inf_i \|f_i\| \leq \sup_i \|f_i\| < \infty$.

In the following Remark we make some easy observations.

Remark 2.3. Let $(x_i, f_i)_{i \in \mathbb{N}}$ be a frame of X .

- a) If $\inf_{i \in \mathbb{N}} \|x_i\| > 0$, then $f_i \xrightarrow{w^*} 0$ as $i \rightarrow \infty$.
- b) Using the Uniform Boundedness Principle we deduce that

$$K = \sup_{x \in B_X} \sup_{m \leq n} \left\| \sum_{i=m}^n f_i(x)x_i \right\| < \infty.$$

This implies that if $\inf_{i \in \mathbb{N}} \|x_i\| > 0$ then (f_i) is bounded and if $\inf_{i \in \mathbb{N}} \|f_i\| > 0$ then (x_i) is bounded.

We call K the *projection constant of (x_i, f_i)* . The projection constant for finite frames is defined accordingly.

- c) For all $f \in X^*$ and $x \in X$ it follows that

$$f(x) = f\left(\sum_{i=1}^{\infty} f_i(x)x_i\right) = \sum_{i=1}^{\infty} f_i(x)f(x_i) = \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n f(x_i)f_i\right)(x),$$

and, thus,

$$f = w^* - \sum_{i=1}^{\infty} f(x_i)f_i.$$

Moreover, for $m \leq n$ in \mathbb{N} it follows that

$$(4) \quad \left\| \sum_{i=m}^n f(x_i) f_i \right\| = \sup_{x \in B_X} \left| \sum_{i=m}^n f(x_i) f_i(x) \right| \leq \|f\| \sup_{x \in B_X} \left\| \sum_{i=m}^n f_i(x) x_i \right\| \leq K \|f\|,$$

and

$$(5) \quad \begin{aligned} \left\| \sum_{i=m}^{\infty} f(x_i) f_i \right\| &= \sup_{x \in B_X} \left| \sum_{i=m}^{\infty} f(x_i) f_i(x) \right| \\ &= \sup_{x \in B_X} f \left(\sum_{i=m}^{\infty} f_i(x) x_i \right) \\ &\begin{cases} \leq \sup_{z \in \text{span}(x_i: i \geq m), \|z\| \leq K} f(z) = K \|f|_{\text{span}(x_i: i \geq m)}\| \\ \geq \|f|_{\{f_1, f_2, \dots, f_m\}^\perp}\|. \end{cases} \end{aligned}$$

d) If (x_i, f_i) is an unconditional frame it follows from the Uniform Boundedness Principle that

$$K_u = \sup_{x \in B_X} \sup_{(\sigma_i) \subset \{\pm 1\}} \left\| \sum \sigma_i f_i(x) x_i \right\| < \infty.$$

We call K_u the *unconditional constant* of (x_i, f_i) .

The following Proposition is a slight variation of [CHL, Theorem 2.6].

Proposition 2.4. *Let X be a separable Banach space and let $(x_i)_{i \in \mathbb{J}} \subset X$ and $(f_i)_{i \in \mathbb{J}} \subset X^*$, with $\mathbb{J} = \mathbb{N}$ or $\mathbb{J} = \{1, 2, \dots, N\}$ for some $N \in \mathbb{N}$.*

a) $(x_i, f_i)_{i \in \mathbb{J}}$ is a Schauder frame of X if and only if there is a Banach space Z with a Schauder basis $(z_i)_{i \in \mathbb{J}}$ and corresponding coordinate functionals (z_i^*) , an isomorphic embedding $T : X \rightarrow Z$ and a bounded linear surjective map $S : Z \rightarrow X$, so that $S \circ T = Id_X$ (i.e. X is isomorphic to a complemented subspace of Z), and $S(z_i) = x_i$, for $i \in \mathbb{J}$, and $T^*(z_i^*) = f_i$, for $i \in \mathbb{J}$, with $x_i \neq 0$.

Moreover S and T can be chosen so that $\|S\| = 1$ and $\|T\| \leq K$, where K is the projection constant of (x_i, f_i) , and (z_i) can be chosen to be a bimonotone basis with $\|z_i\| = \|x_i\|$ if $i \in \mathbb{J}$, with $x_i \neq 0$.

b) $(x_i, f_i)_{i \in \mathbb{J}}$ is an unconditional frame of X if and only if there is a Banach space Z with an unconditional basis (z_i) and corresponding coordinate functionals (z_i^*) , an isomorphic embedding $T : X \rightarrow Z$ and a surjection $S : Z \rightarrow X$, so that $S \circ T = Id_X$, $S(z_i) = x_i$, for $i \in \mathbb{J}$, and $T^*(z_i^*) = f_i$ for $i \in \mathbb{J}$, with $x_i \neq 0$.

Proof. (a) part “ \Rightarrow ” Assume that $(x_i, f_i)_{i \in \mathbb{J}}$ is a frame of X and let K be the projection constant of $(x_i, f_i)_{i \in \mathbb{J}}$. We put $\tilde{\mathbb{J}} = \{i \in \mathbb{J} : x_i \neq 0\}$, denote the unit vector basis of $c_{00}(\mathbb{J})$ by (z_i) and define on $c_{00}(\mathbb{J})$ the following norm $\|\cdot\|_Z$.

$$(6) \quad \left\| \sum_{i \in \mathbb{J}} a_i z_i \right\|_Z = \max_{m \leq n} \left\| \sum_{i \in \tilde{\mathbb{J}} \cap \{m, m+1, \dots, n\}} a_i x_i \right\|_X + \left(\sum_{i \in \mathbb{J} \setminus \tilde{\mathbb{J}}} a_i^2 \right)^{1/2} \text{ for } (a_i) \subset \mathbb{R}.$$

It follows easily that (z_i) is a bimonotone basic sequence and, thus, a basis of the completion of $c_{00}(\mathbb{J})$ with respect to $\|\cdot\|_Z$, which we denote by Z .

The map

$$S : Z \rightarrow X, \quad \sum a_j z_j \mapsto \sum a_j x_j,$$

is linear and bounded with $\|S\| = 1$. Secondly, define

$$T : X \rightarrow Z, \quad x = \sum_{i \in \mathbb{J}} f_i(x) x_i = \sum_{i \in \tilde{\mathbb{J}}} f_i(x) x_i \mapsto \sum_{i \in \tilde{\mathbb{J}}} f_i(x) z_i.$$

Remark 2.3 (b) yields for $x \in X$

$$\begin{aligned} \left\| \sum_{i \in \tilde{\mathbb{J}}} f_i(x) z_i \right\|_Z &= \sup_{m \leq n} \left\| \sum_{i \in \tilde{\mathbb{J}} \cap \{m, m+1, \dots, n\}} f_i(x) x_i \right\|_X \\ &= \sup_{m \leq n} \left\| \sum_{i \in \tilde{\mathbb{J}} \cap \{m, m+1, \dots, n\}} f_i(x) x_i \right\|_X \leq K \|x\|, \end{aligned}$$

and, thus, that T is linear and bounded with $\|T\| \leq K$. Clearly it follows that $S \circ T = Id_X$, which implies that T is an isomorphic embedding and that S is a surjection. Finally, if (z_i^*) are the coordinate functionals of (z_i) we deduce for $x \in X$ and $i \in \mathbb{J}$ that

$$T^*(z_i^*)(x) = z_i^*(T(x)) = \begin{cases} f_i(x) & \text{if } x_i \neq 0 \\ 0 & \text{if } x_i = 0, \end{cases}$$

which finishes the proof of “ \Rightarrow ”.

In order to show the converse in (a), assume that Z is a space with a basis $(z_i)_{i \in \mathbb{J}}$ and that $S : Z \rightarrow X$ is a bounded linear surjection, and $T : X \rightarrow Z$ an isomorphic embedding, with $S \circ T = Id_X$. Put $x_i = S(z_i)$ and $f_i = T^*(z_i^*)$, for $i \in \mathbb{J}$. Then for $x \in X$,

$$x = S \circ T(x) = S \left(\sum z_i^*(T(x)) z_i \right) = \sum T^*(z_i^*)(x) S(z_i) = \sum f_i(x) x_i,$$

which implies that $(x_i, f_i)_{i \in \mathbb{J}}$ is a frame of X .

For the proof of (b) we replace (6) by

$$(7) \quad \left\| \sum_{i \in \mathbb{J}} a_i z_i \right\|_Z = \max_{(\sigma_i) \subset \{\pm 1\}} \left\| \sum_{i \in \mathbb{J}} \sigma_i a_i x_i \right\|_X + \left(\sum_{i \in \mathbb{J} \setminus \bar{\mathbb{J}}} a_i^2 \right)^{1/2}.$$

and note that arguments similar to those in the proof of (a) yield (b). \square

Definition 2.5. Let (x_i, f_i) be a frame of a Banach space X and let Z be a space with a basis (z_i) and corresponding coordinate functionals (z_i^*) . We call $(Z, (z_i))$ an *associated space* to (x_i, f_i) or a *sequence space associated to (x_i, f_i)* and (z_i) an *associated basis*, if

$$S : Z \rightarrow X, \quad \sum a_i z_i \mapsto \sum a_i x_i \quad \text{and} \quad T : X \rightarrow Z, \quad x = \sum f_i(x) x_i \mapsto \sum_{x_i \neq 0} f_i(x) z_i$$

are bounded operators. We call S the *associated reconstruction operator* and T the *associated decomposition operator* or *analysis operator*.

In this case, following [Gr] we call the triple $((x_i), (f_i), Z)$ an *atomic decomposition* of X .

Remark 2.6. By Proposition 2.4 the property of Banach space X to admit a frame is equivalent to the property of X being isomorphic to a complemented subspace of a space Z with basis. It was shown independently by Pełczyński [Pe] and Johnson, Rosenthal and Zippin [JRZ] (see also [Ca2][Theorem 3.13]) that the later property is equivalent to X having the *Bounded Approximation Property*. We say X has the *Bounded Approximation Property* if there is a $\lambda \geq 1$, so that for every $\varepsilon > 0$ and every compact set $K \subset X$ there is a finite rank operator $T : X \rightarrow X$ with $\|T\| \leq \lambda$ so that $\|T(x) - x\| \leq \varepsilon$ whenever $x \in K$.

Remark 2.7. Let $(x_j)_{j \in \mathbb{J}}$ be a Hilbert frame of a Hilbert space H and let Θ and I be defined as in the paragraph following Definition 2.1. We choose Z to be $\ell_2(\mathbb{J})$, $S = \Theta^*$ and

$$T = \Theta \circ I^{-1} : H \rightarrow \ell_2, \quad x \mapsto \sum_{j \in \mathbb{J}} \langle I^{-1}(x), x_j \rangle e_j = \sum_{j \in \mathbb{J}} \langle x, I^{-1}(x_j) \rangle e_j$$

and observe that $S \circ T = \text{Id}_H$, and for $j \in \mathbb{J}$ it follows that $S(e_j) = \Theta^*(e_j) = x_j$, and

$$T^*(e_j)(x) = \langle x, I^{-1}(x_j) \rangle, \quad (x \in H) \quad \text{and, thus} \quad T^*(e_j) = I^{-1}(x_j).$$

Thus, if (x_i) is a Hilbert frame, then $((x_i), (I^{-1}(x_i)))$ is a Schauder frame for which $Z = \ell_2(\mathbb{J})$ together with its unit vector basis is an associated space.

Conversely, let (x_i, f_i) be a Schauder frame of a Hilbert space H and assume that $Z = \ell_2(\mathbb{J})$ with its unit vector basis is an associated space. Denote by $T : H \rightarrow$

$\ell_2(\mathbb{J})$ and $S : \ell_2(\mathbb{J}) \rightarrow H$ the associated decomposition, respectively reconstruction operator. Then it follows that for all $x \in H$

$$\sum \langle x_i, x \rangle^2 = \sum \langle S(e_i), x \rangle^2 = \sum \langle e_i, S^*(x) \rangle^2 = \|S^*(x)\|^2.$$

Thus, since S^* is an isomorphic embedding of H into ℓ_2 , it follows that (x_i) is a Hilbert frame.

In the following observation we show, that we can always *expand a frame by a bounded linear operator*.

Proposition 2.8. *Let (x_i, f_i) be a frame of a Banach space X and let Z be a space with a basis (z_i) which is associated to (x_i, f_i) . Furthermore assume that Y is another space with a basis (y_i) and let $V : Y \rightarrow X$ be linear and bounded.*

Let $\tilde{Z} = Z \oplus_\infty Y$ and define $(\tilde{z}_i) \subset \tilde{Z}$, $(\tilde{x}_i) \subset X$ and $(\tilde{f}_i) \subset X^$ by*

$$\tilde{z}_i = \begin{cases} (z_{i/2}, \lambda_{i/2} y_{i/2}) & \text{if } i \text{ even} \\ (z_{(i+1)/2}, -\lambda_{(i+1)/2} y_{(i+1)/2}) & \text{if } i \text{ odd,} \end{cases} \quad \tilde{x}_i = \begin{cases} x_{i/2} + \lambda_{i/2} V(y_{i/2}) & \text{if } i \text{ even} \\ x_{(i+1)/2} - \lambda_{(i+1)/2} V(y_{(i+1)/2}) & \text{if } i \text{ odd,} \end{cases}$$

$$\tilde{f}_i = \begin{cases} \frac{1}{2} f_{i/2} & \text{if } i \text{ even} \\ \frac{1}{2} f_{(i+1)/2} & \text{if } i \text{ odd,} \end{cases}$$

where $\lambda_i = \|z_i\|/\|y_i\|$, for $i \in \mathbb{N}$. Then $(\tilde{x}_i, \tilde{f}_i)$ is a frame of X , (\tilde{z}_i) is a basis for \tilde{Z} and $(\tilde{Z}, (\tilde{z}_i))$ is an associated space for $(\tilde{x}_i, \tilde{f}_i)$.

Proof. Let $T : X \rightarrow Z$ and $S : Z \rightarrow X$ be the associated decomposition and reconstruction operator, respectively. Note that the operators

$$\tilde{S} : Z \oplus_\infty Y \rightarrow X, \quad (z, y) \mapsto S(z) + V(y) \quad \text{and} \quad \tilde{T} : X \rightarrow Z \oplus_\infty Y, \quad x \mapsto (T(x), 0)$$

are bounded and linear and that $\tilde{S} \circ \tilde{T} = Id_X$ and $\tilde{S}(\tilde{z}_i) = \tilde{x}_i$, for $i \in \mathbb{N}$. It is easy to verify that (\tilde{z}_i) is a basis of \tilde{Z} for which its coordinate functionals (\tilde{z}_i^*) are given by (denote the coordinate functionals of (y_i) by (y_i^*))

$$\tilde{z}_i^* = \begin{cases} \frac{1}{2}(z_{i/2}^*, \frac{1}{\lambda_i} y_{i/2}^*) & \text{if } i \text{ even} \\ \frac{1}{2}(z_{(i+1)/2}^*, -\frac{1}{\lambda_i} y_{(i+1)/2}^*) & \text{if } i \text{ odd.} \end{cases}$$

it follows for $x \in X$ that

$$\tilde{T}^*(\tilde{z}_i^*)(x) = \tilde{z}_i^*(\tilde{T}(x)) = \begin{cases} f_{i/2}(x)/2 & \text{if } i \text{ even} \\ f_{(i+1)/2}(x)/2 & \text{if } i \text{ odd} \end{cases} = \tilde{f}_i(x),$$

which yields $\tilde{T}^*(\tilde{z}_i^*) = \tilde{f}_i$, for $i \in \mathbb{N}$. Thus, the claim follows from Proposition 2.4. \square

3. THREE EXAMPLES

In [DOSZ] the following notion of quantization was introduced and studied for non redundant systems.

Definition 3.1. Let $(x_i)_{i \in \mathbb{J}}$ be a fundamental system for X , with $\mathbb{J} = \mathbb{N}$ or $\mathbb{J} = \{1, 2, \dots, N\}$, for some $N \in \mathbb{N}$ (i.e. $\overline{\text{span}(e_i : i \in \mathbb{J})} = X$) and let $\varepsilon > 0$ and $\delta > 0$ be given. We say that $(x_i)_{i \in \mathbb{J}}$ has the (ε, δ) -Net Quantization Property (abbr. (ε, δ) -NQP) if for any $x = \sum_{i \in E} a_i x_i \in X$, $E \subset \mathbb{J}$ finite, there exists a sequence $(k_i) \subset \mathbb{Z}$, $\text{supp}(k_i) = \{i \in \mathbb{N} : k_i \neq 0\}$ is finite, such that

$$(8) \quad \left\| x - \sum k_i \delta x_i \right\| \leq \varepsilon.$$

We say that (x_i) has the NQP if (x_i) has the (ε, δ) -NQP for some $\varepsilon > 0$ and $\delta > 0$.

When we ask whether or not in a certain representation of vectors the coefficients can be replaced by quantized coefficients, we are often interested in memorizing data as economically as possible, and reconstructing them with as little error as possible. With this in mind, we will exhibit in this section several examples, which show that it is not always meaningful to apply the notion of NQP word for word to redundant systems like frames. These examples will then also guide us to more appropriate quantization concepts for frames.

The first example is a *tight* Hilbert frame (f_i) in S_{ℓ_2} (i.e. $a = b$) consisting of normalized vectors so that for every $x \in \ell_2$ there is a sequence $(k_i) \subset \mathbb{Z}$ so that $\|x - \sum_{i \in \mathbb{N}} k_i f_i\| < 1$. The second example is a semi-normalized Hilbert frame (f_i) in ℓ_2 which has the property that for every $x \in \ell_2$ there is a sequence $(k_i) \subset \mathbb{Z}$ so that $\|x - \sum_{i \in \mathbb{N}} k_i f_i\| < 1$ and (k_i) has the additional property that $\max_{i \in \mathbb{N}} |k_i| \leq 1$, if $x \in B_{\ell_2}$. The third example is a Schauder frame (f_i) of ℓ_2 which has the property that for every $x \in \ell_2$ there is a sequence $(k_i) \subset \mathbb{Z}$, so that not only $\max_{i \in \mathbb{N}} |k_i| \leq \|x\|$ and $\|x - \sum_{i \in \mathbb{N}} k_i f_i\| < 1$, but so that also the support of (k_i) , i.e. the set $\{i \in \mathbb{N} : k_i \neq 0\}$ is uniformly bounded.

Example 3.2. Let $0 < \varepsilon_i < 1/2$. For $i \in \mathbb{N}$ define the following vectors f_{2i-1} and f_{2i} in S_{ℓ_2} .

$$f_{2i-1} = \sqrt{1 - \varepsilon_i^2} e_{2i-1} + \varepsilon_i e_{2i} \text{ and } f_{2i} = -\varepsilon_i e_{2i-1} + \sqrt{1 - \varepsilon_i^2} e_{2i}.$$

Clearly (f_i) is an orthonormal basis for ℓ_2 and we let $\mathcal{F} = \{e_i : i \in \mathbb{N}\} \cup \{f_i : i \in \mathbb{N}\}$. Then \mathcal{F} is a tight frame (as is any finite union of orthonormal bases) and the sequence

(z_i) with

$$\begin{aligned} z_{2i-1} &= e_{2i-1} - f_{2i-1} = (1 - \sqrt{1 - \varepsilon_i^2})e_{2i-1} - \varepsilon_i e_{2i} \\ z_{2i} &= e_{2i} - f_{2i} = \varepsilon_i e_{2i-1} + (1 - \sqrt{1 - \varepsilon_i^2})e_{2i}, \text{ for } i \in \mathbb{N} \end{aligned}$$

is an orthogonal basis and $\|z_{2i-1}\| = \|z_{2i}\| = O(\varepsilon_i)$. Thus, if (ε_i) converges fast enough to 0 it follows that for any $x \in \ell_2$ there is a family $(k_i) \subset \mathbb{Z}$, with $|\text{supp}(k_i)| < \infty$, so that

$$\left\| x - \sum k_i z_i \right\| = \left\| x - \sum k_i (e_i - f_i) \right\| < 1.$$

Example 3.3. Our second example is a semi-normalized Hilbert frame (x_i) in ℓ_2 so that

$$D = \left\{ \sum_{i=1}^{\infty} k_i x_i : k_i \in \{-1, 0, 1\} \text{ and } \{i : k_i \neq 0\} \text{ is finite} \right\}$$

is dense in B_{ℓ_2} .

Put $(c_i)_{i=1}^{\infty} = (1/2^i)$ and partition the unit vector basis (e_i) of ℓ_2 into infinitely many subsequences of infinite length, say $(e(i, j) : i, j \in \mathbb{N})$. Then our frame (f_k) is defined to be the sequence:

$$\begin{aligned} f_1 &= c_1 e_1 + e(1, 1), f_2 = e(1, 1), \\ f_3 &= c_1 e_2 + e(1, 2), f_4 = e(1, 2), f_5 = c_2 e_1 + e(2, 1), f_6 = e(2, 1), \\ f_7 &= c_1 e_3 + e(1, 3), f_8 = e(1, 3), f_9 = c_2 e_2 + e(2, 2), \\ f_{10} &= e(2, 2), f_{11} = c_3 e_1 + e(3, 1), f_{12} = e(3, 1), \\ f_{13} &= c_1 e_4 + e(1, 4), f_{15} = e(1, 4), \dots, f_{19} = c_4 e_1 + e(4, 1), f_{20} = e(4, 1), \\ &\vdots \end{aligned}$$

Note that the set of vectors $x \in B_{\ell_2}$ of the form

$$x = \sum_{i,j \in \mathbb{N}} \varepsilon(i, j) c_i e_j = \sum_{i,j \in \mathbb{N}} \varepsilon(i, j) (c_i e_j + e(i, j)) - \varepsilon(i, j) e(i, j)$$

where $(\varepsilon(i, j)) \subset \{-1, 0, 1\}$ so that the set $\{i, j \in \mathbb{N} : \varepsilon(i, j) \neq 0\}$ is finite, are dense in B_{ℓ_2} . This implies that every $x \in B_{\ell_2}$ is the limit of vectors (x_n) with

$$(x_n) \subset \left\{ \sum \varepsilon_i f_i : (\varepsilon_i) \subset \{-1, 0, 1\}, \{i \in \mathbb{N} : \varepsilon_i \neq 0\} \text{ is finite} \right\}.$$

The sequence (f_n) is a frame. Indeed for any $x = \sum x_i e_i \in \ell_2$ we have

$$\begin{aligned} \sum \langle f_i, x \rangle^2 &= \sum_{i \text{ even}} \langle f_i, x \rangle^2 + \sum_{i \text{ odd}} \langle f_i, x \rangle^2 = \sum x_i^2 + \sum_{i,j \in \mathbb{N}} (c_i x_j + \langle e(i,j), x \rangle)^2 \\ &\begin{cases} \geq \|x\|^2 \\ \leq \sum x_i^2 + \sum_{i,j \in \mathbb{N}} 2c_i^2 x_j^2 + 2\langle e(i,j), x \rangle^2 \leq (3 + 2 \sum c_i^2) \|x\|^2. \end{cases} \end{aligned}$$

Example 3.4. We construct a Schauder frame (x_i, f_i) of ℓ_2 so that (x_n) is dense in B_{ℓ_2} .

Let (z_n) be dense in B_{ℓ_2} and choose for each $n \in \mathbb{N}$

$$x_{2n-1} = z_n + e_n, x_{2n} = z_n, f_{2n-1} = e_n \text{ and } f_{2n} = -e_n.$$

Clearly, for every $x \in \ell_2$

$$x = \sum \langle e_i, x \rangle e_i = \sum \langle f_{2n-1}, x \rangle x_{2n-1} + \langle f_{2n}, x \rangle x_{2n}$$

(the above sum is conditionally converging). It follows that $((x_n), (f_n))$ is a Schauder frame of ℓ_2 . It is clear that (x_n) is not a Hilbert frame.

Remark 3.5. All three examples satisfy the conditions (NQP) if we extend this notion word for word to frames. Nevertheless these examples do not satisfy our understanding of what quantization of coefficients should mean.

In Example 3.2 every $x \in \ell_2$ can be approximated by an expansion with respect to a frame using only integer coefficients, but these coefficients might get arbitrarily large for elements in B_{ℓ_2} . This means that we would need an infinite alphabet to approximate vectors which are in B_{ℓ_2} . Therefore it is not enough (as in the non redundant case) to assume that our frame is semi-normalized.

In the Examples 3.3 and 3.4 we achieve the approximation of any vector in ℓ_2 by a quantized expansion whose coefficients are bounded by a fixed multiple of the norm of the vector, but in order to approximate even the vectors of a given finite dimensional subspace (for example the space generated by two elements of the unit vector basis elements of ℓ_2) we need an infinite dictionary.

4. QUANTIZATION WITH Z -BOUNDED COEFFICIENTS

One way to avoid examples like the ones mentioned in Section 3 is to impose boundedness conditions on the quantized coefficients within an associated space Z .

Definition 4.1. Assume $(x_i, f_i)_{i \in \mathbb{J}}$, $\mathbb{J} = \mathbb{N}$ or $\mathbb{J} = \{1, 2, \dots, N\}$, for some $N \in \mathbb{N}$, is a frame of a Banach space X . Let Z be a space with basis (z_i) which is associated to $(x_i, f_i)_{i \in \mathbb{J}}$. Let $\varepsilon, \delta > 0$, $C \geq 1$.

We say that (x_i, f_i) satisfies the (ε, δ, C) -Net Quantization Property with respect to $(Z, (z_i))$ or (ε, δ, C) -Z-NQP, if for all $x \in X$ there exists a sequence $(k_i)_{i \in \mathbb{J}} \subset \mathbb{Z}$ with finite support so that

$$(9) \quad \left\| \sum k_i \delta z_i \right\|_Z \leq C \|x\| \quad \text{and} \quad \left\| x - \sum \delta k_i x_i \right\|_X \leq \varepsilon.$$

We say that (x_i, f_i) satisfies the NQP with respect to $(Z, (z_i))$ if it satisfies the (ε, δ, C) -Z-NQP for some choice of $\varepsilon, \delta > 0$ and $C \geq 1$.

It is easy to see that the property (ε, δ, C) -NQP with respect to some associated space is homogenous in (ε, δ) , meaning that a frame (x_i, f_i) satisfies the (ε, δ, C) -NQP if and only if for some $\lambda > 0$ (or for all λ) (x_i, f_i) satisfies the $(\lambda\varepsilon, \lambda\delta, C)$ -NQP. The following result, analogous to [DOSZ, Theorem 2.4], shows that it is enough to verify that one can quantize the coefficients of elements x which are in B_X to deduce the NQP.

Proposition 4.2. *Assume that (x_i) and (z_i) are some sequences in Banach spaces X and Z , respectively, and assume that there are $C_0 < \infty$, $\delta_0 > 0$ and $0 < q_0 < 1$, so that for all $x \in B_X$ there is a sequence $(k_i) \subset \mathbb{Z}$, $(k_i) \in c_{00}$ with*

$$(10) \quad \left\| \sum \delta_0 k_i z_i \right\| \leq C_0 \quad \text{and} \quad \left\| x - \sum \delta_0 k_i x_i \right\| \leq q_0.$$

Then there are $\delta_1 > 0$ and $C_1 < \infty$, depending only on δ_0, q_0 and C_0 , so that for all $x \in X$ there is a sequence $(k_i) \subset \mathbb{Z}$, $(k_i) \in c_{00}$, with

$$(11) \quad \left\| \sum \delta_1 k_i z_i \right\| \leq C_1 \|x\| \quad \text{and} \quad \left\| x - \sum \delta_1 k_i x_i \right\| \leq 1.$$

Proof. Choose $n_1 \in \mathbb{N}$ and q_1 so that

$$(12) \quad \frac{n_1 + 1}{n_1} q_0 = q_1 < 1.$$

and put $\delta_1 = \delta_0/n_1$.

We first claim that for any $0 < \delta \leq \delta_1$ and any $x \in B_X$ there is a sequence $(k_i) \in \mathbb{Z}^{\mathbb{N}} \cap c_{00}$ so that

$$(13) \quad \left\| \sum k_i \delta z_i \right\| \leq 2C_0 \quad \text{and} \quad \left\| x - \sum \delta k_i x_i \right\| \leq q_1.$$

Indeed, let $\delta \leq \delta_1$ and $x \in B_X$ and choose $n \geq n_1$ in \mathbb{N} so that $\frac{\delta_0}{n+1} < \delta \leq \frac{\delta_0}{n}$ and $(k_i) \subset \mathbb{Z}$ so that

$$\left\| \sum k_i \delta_0 z_i \right\| \leq C_0 \quad \text{and} \quad \left\| \frac{\delta_0}{\delta(n+1)} x - \sum k_i \delta_0 x_i \right\| \leq q_0.$$

Thus, since $n \geq n_1$,

$$\left\| \sum k_i(n+1)\delta z_i \right\| = \left\| \sum k_i\delta_0 z_i \right\| \frac{\delta(n+1)}{\delta_0} \leq \frac{n+1}{n} C_0 \leq 2C_0$$

and

$$\left\| x - \sum k_i\delta(n+1)x_i \right\| \leq \left\| \frac{\delta_0}{(n+1)\delta} x - \sum k_i\delta_0 x_i \right\| \frac{\delta(n+1)}{\delta_0} \leq q_0 \frac{\delta}{\delta_0} (n+1) \leq q_1.$$

By induction on $n \in \mathbb{N}$ we show that for any $\delta \leq q_1^{n-1}\delta_1$ and any $x \in B_X$ there is a $(k_i) \subset \mathbb{Z}$, $(k_i) \in c_{00}$, so that

$$(14) \quad \left\| \sum k_i\delta z_i \right\| \leq 2C_0 \sum_{i=0}^{n-1} q_1^i \text{ and } \left\| x - \sum k_i\delta x_i \right\| \leq q_1^n.$$

For $n = 1$ this is just (13). Assume our claim to be true for n and let $\delta \leq \delta_1 q_1^n$ and $x \in B_X$. By our induction hypothesis, we can find $(k_i) \subset \mathbb{Z}$, $(k_i) \in c_{00}$, so that

$$\left\| \sum k_i\delta z_i \right\| \leq 2C_0 \sum_{i=0}^{n-1} q_1^i \text{ and } \left\| x - \sum k_i\delta x_i \right\| \leq q_1^n,$$

Since $q_1^{-n}(x - \sum k_i\delta x_i) \in B_X$ and since $\delta q_1^{-n} \leq \delta_1$, we can use our first claim and choose $(\tilde{k}_i) \in \mathbb{Z}^{\mathbb{N}} \cap c_{00}$ so that

$$\left\| \sum \tilde{k}_i\delta q_1^{-n} z_i \right\| \leq 2C_0 \text{ and } \left\| q_1^{-n} \left(x - \sum k_i\delta x_i \right) - \sum \delta q_1^{-n} \tilde{k}_i x_i \right\| \leq q_1.$$

Thus

$$\left\| \sum (k_i + \tilde{k}_i)\delta z_i \right\| \leq 2C_0 \sum_{i=0}^{n-1} q_1^i + 2C_0 q_1^n,$$

and

$$\left\| x - \sum \delta k_i x_i - \sum \delta \tilde{k}_i x_i \right\| \leq q_1^{n+1},$$

which finishes the induction step.

Now define $C_1 = 2C_0 \sum_{n=0}^{\infty} q_1^n$ and let $x \in X$ be arbitrary.

If $\|x\| \geq 1$ (this is the only case left to consider) we choose $n \in \mathbb{N}$ with $q_1^n < \frac{1}{\|x\|} \leq q_1^{n-1}$ and, by (14) we can choose $(k_i) \in \mathbb{Z}^{\mathbb{N}} \cap c_{00}$ so that

$$\left\| \sum k_i \frac{\delta_1}{\|x\|} z_i \right\| \leq 2C_0 \sum_{i=0}^{n-1} q_1^i \leq C_1 \text{ and } \left\| \frac{x}{\|x\|} - \sum \frac{k_i\delta_1}{\|x\|} x_i \right\| \leq q_1^n,$$

which yields

$$\left\| \sum k_i\delta_1 z_i \right\| \leq C_1 \|x\| \text{ and } \left\| x - \sum k_i\delta_1 x_i \right\| \leq q_1^n \|x\| \leq 1.$$

□

In the following result we consider a finite frame $(x_i, f_i)_{i=1}^N$ of an n -dimensional Banach space X , and exploit the fact that, if $(x_i, f_i)_{i=1}^N$ has the (ε, δ, C) -NQP with respect to some space Z having a basis $(z_i)_{i=1}^N$, then the volume of B_X can be at most as large as the sum of volumes of the ε -balls around the elements of

$$\mathcal{F}_{(\delta, C)}(x_i) = \left\{ \sum n_j \delta x_j : \left\| \sum n_j \delta z_j \right\| \leq C \right\}.$$

Thus the value

$$\varepsilon^{-n} = \text{Vol}(B_X) / \text{Vol}(\varepsilon B_X)$$

must be smaller than the cardinality of $\mathcal{F}_{(\delta, C)}(x_i)$.

Proposition 4.3. *Assume $f, g : (0, \infty) \rightarrow (0, \infty)$ are strictly increasing functions so that*

$$(15) \quad \lim_{n \rightarrow \infty} f(n) = \infty \text{ and } \lim_{n \rightarrow \infty} \frac{g(n) \ln n}{n} = 0,$$

and $C, B, R \geq 1$, and $0 \leq \delta, \varepsilon < 1$.

Assume that $(x_i, f_i)_{i=1}^N$ is a frame of a Banach space X with $\dim(X) = n < \infty$ and that Z is an N -dimensional space, $N \in \mathbb{N}$, with basis $(z_i)_{i=1}^N$, which is associated to $(x_i, f_i)_{i=1}^N$. Let $T : X \rightarrow Z$ and $S : Z \rightarrow X$ be the associated decomposition and reconstruction operator, denote by K_Z the projection constant of (z_i) .

Assume

$$(16) \quad \mathcal{F}_{(\delta, C)}(x_i) \text{ is } \varepsilon\text{-dense in } B_X, \text{ and}$$

$$(17) \quad f(\#A) \leq \left\| \sum_{j \in A} n_j z_j \right\|, \text{ if } A \subset \{1, 2, \dots, N\} \text{ and } (n_j)_{j \in A} \subset \mathbb{Z} \setminus \{0\}.$$

Then

$$(18) \quad \ln N \geq \frac{n \ln(1/\varepsilon)}{f^{-1}(C/\delta)} - \ln \left(\frac{2K_Z C}{\delta} + 1 \right).$$

In addition there is an $n_0 \in \mathbb{N}$ (depending on $C, B, R, \varepsilon, \delta, f$ and g) so that if, moreover, $K_Z \leq B$, $\|S\| \leq R$ and

$$(19) \quad g(\#A) \geq \sup \left\| \sum_{j \in A} \pm z_j \right\|, \text{ whenever } A \subset \{1, 2, \dots, N\},$$

then $n \leq n_0$.

Proof. First note that, if $A \subset \{1, 2, \dots, N\}$ and $(n_j)_{j \in A} \subset \mathbb{Z} \setminus \{0\}$ with $C \geq \left\| \sum_{j \in A} n_j \delta z_j \right\|_Z \geq \delta f(\#A)$, then $\#A \leq f^{-1}(C/\delta)$ and $|n_j| \leq K_Z C / \delta$ for $j \in A$. Thus, (16) and the

volume argument, mentioned before the statement of our proposition, yields

$$\varepsilon^{-n} \leq \#\mathcal{F}_{(\delta,C)}(x_i) \leq \binom{N}{\lfloor f^{-1}(C/\delta) \rfloor} \left(\frac{2K_Z C}{\delta} + 1 \right)^{\lfloor f^{-1}(C/\delta) \rfloor} \leq \left[N \left(\frac{2K_Z C}{\delta} + 1 \right) \right]^{f^{-1}(C/\delta)},$$

which, after taking $\ln(\cdot)$ on both sides, implies (18).

Now assume that also (19) is satisfied. Let $(e_i, e_i^*)_{i=1}^n$ be an Auerbach basis of X , i.e. $\|e_i\| = \|e_i^*\| = 1$ and $e_i^*(e_j) = \delta_{(i,j)}$. Such a basis always exists (c.f [FHHMPZ, Theorem 5.6]). Choose $0 < \eta < \infty$ so that $\varepsilon(1 + 1/\eta) < 1$ and define for $i = 1, 2, \dots, n$

$$A_i = \left\{ j \in \{1, 2, \dots, N\} : |e_i^*(x_j)| \geq \frac{\varepsilon}{\eta K_Z C f^{-1}(C/\delta)n} \right\}$$

Then it follows for the right choice of $\sigma_j = \pm 1$, $j \in A_i$ that

$$\begin{aligned} g(\#A_i) &\geq \left\| \sum_{j \in A_i} \pm z_j \right\| \\ &\geq \frac{1}{\|S\|} \sup \left\| \sum_{j \in A_i} \pm x_j \right\| \\ &\geq \frac{1}{\|S\|} e_i^* \left(\sum_{j \in A_i} \sigma_j x_j \right) \geq \frac{\#A_i \varepsilon}{\eta \|S\| K_Z C f^{-1}(C/\delta)n} \end{aligned}$$

and thus

$$(20) \quad \frac{\#A_i}{g(\#A_i)} \leq \frac{\eta \|S\| K_Z C f^{-1}(C/\delta)n}{\varepsilon}.$$

Put $A = \bigcup_{i \leq n} A_i$. If $(n_j)_{j \leq N} \subset \mathbb{Z}$ is such that $\sum_{j=1}^N \delta n_j x_j \in \mathcal{F}_{(\delta,C)}(x_j)$, then

$$\left\| \sum_{j \in A^c} \delta n_j x_j \right\| \leq \sum_{j \in A^c, n_j \neq 0} \left(\sum_{i=1}^n |e_i^*(x_j)| \right) \max_{j \leq N} \delta |n_j| \leq \#\{j : n_j \neq 0\} \frac{\varepsilon}{\eta f^{-1}(C/\delta)} \leq \frac{\varepsilon}{\eta},$$

where the second inequality follows from the definition of the A_i 's and the observation at the beginning of the proof, and the last inequality follows from the fact that

$$(21) \quad f(\#\{j : n_j \neq 0\}) \leq \left\| \sum n_j z_j \right\| \leq C/\delta.$$

This implies together with (16) that the set

$$\tilde{\mathcal{F}}_{(\delta,C)} = \left\{ \sum_{j \in A} n_j \delta x_j : (n_j) \subset \mathbb{Z}, \left\| \sum_{j=1}^N n_j \delta z_j \right\| \leq C \right\}$$

is $\varepsilon(1 + \frac{1}{\eta})$ -dense in B_X . Hence, our usual argument comparing volumes and (21) yields

$$\#A^{f^{-1}(C/\delta)} \left(\frac{2K_Z C}{\delta} + 1 \right)^{f^{-1}(C/\delta)} \geq \binom{\#A}{\lfloor f^{-1}(C/\delta) \rfloor} \left(\frac{2K_Z C}{\delta} + 1 \right)^{\lfloor f^{-1}(C/\delta) \rfloor} \geq \frac{1}{\varepsilon^n (1 + 1/\eta)^n},$$

Taking $\ln(\cdot)$ on both sides and letting $r(\ell) = \ln(\ell)g(\ell)/\ell$ for $\ell \in \mathbb{N}$, we conclude by (20) and since $\varepsilon(1 + 1/\eta) < 1$ that

$$\begin{aligned} & n \ln \left(\frac{1}{\varepsilon(1 + 1/\eta)} \right) \\ & \leq f^{-1}(C/\delta) \left(\ln(\#A) + \ln \left(\frac{2K_Z C}{\delta} + 1 \right) \right) \\ & \leq f^{-1}(C/\delta) \left(\ln n + \max_{i \leq n} \ln(\#A_i) + \ln \left(\frac{4K_Z C}{\delta} \right) \right) \\ & = f^{-1}(C/\delta) \left(\ln n + \max_{i \leq n} r(\#A_i) \left(\frac{\#A_i}{g(\#A_i)} \right) + \ln \left(\frac{4K_Z C}{\delta} \right) \right) \\ & \leq f^{-1}(C/\delta) \left(\ln n + nr(\#A_{i_0}) \frac{\eta \|S\| K_Z C f^{-1}(C/\delta)}{\varepsilon} + \ln \left(\frac{4K_Z C}{\delta} \right) \right) \end{aligned}$$

where $i_0 \leq n$ is chosen so that $\#A_{i_0}$ is maximal. By our assumption (19) on g we can find an $\ell_0 \in \mathbb{N}$ so that

$$f^{-1}(C/\delta) \frac{r(\ell)\eta \|S\| K_Z C f^{-1}(C/\delta)}{\varepsilon} \leq \frac{1}{2} \ln \left(\frac{1}{\varepsilon(1 + 1/\eta)} \right), \text{ whenever } \ell \geq \ell_0.$$

If $\#A_{i_0} \leq \ell_0$ then by the third line in above chain of inequalities

$$n \ln \left(\frac{1}{\varepsilon(1 + 1/\eta)} \right) \leq f^{-1}(C/\delta) \left[\ln n + \ln \ell_0 + \ln \left(\frac{2K_Z C}{\delta} \right) \right],$$

which implies that n is bounded by a number which only depends on $\varepsilon, \delta, C, f, g$ and K_Z . If $\#A_{i_0} > \ell_0$, then it follows that

$$\frac{n}{2} \ln \left(\frac{1}{\varepsilon(1 + 1/\eta)} \right) \leq f^{-1}(C/\delta) \left(\ln n + \ln \left(\frac{4K_Z C}{\delta} \right) \right).$$

which implies our claim also in that case. \square

We shall formulate a corollary of Proposition 4.3 for the infinite dimensional situation. We need first to introduce some notation and make some observations.

Let $(x_i, f_i)_{i \in \mathbb{N}}$ be a frame of X . Furthermore assume that X has the π_λ -property, which means that there is a sequence $\bar{P} = (P_n)$ of finite rank projections, whose norms are uniformly bounded, and which *approximate the identity*, i.e.

$$(22) \quad x = \lim_{n \rightarrow \infty} P_n(x), \text{ in norm for all } x \in X.$$

For example, if X has a basis (e_i) we could choose for $n \in \mathbb{N}$ the projection onto the first n coordinates, i.e.

$$P_n : X \rightarrow X, \quad \sum a_i e_i \mapsto \sum_{i=1}^n a_i e_i.$$

It is easy to see that $(P_n(x_i), f_i|_{P_n(X)})$ is a frame of the space $X_n = P_n(X)$. Moreover, condition (22) and a straightforward compactness argument shows that for any $n \in \mathbb{N}$ and any $\frac{1}{2} < r < 1$ there is an $M_n = M(r, n)$ so that it follows that

$$(23) \quad \left\| x - \sum_{i=1}^N \langle f_i, x \rangle P_n(x_i) \right\| \leq (1-r)\|x\|, \text{ whenever } x \in X_n \text{ and } N \geq M_n$$

It follows that the operators (Q_n) , with

$$Q_n : X_n \rightarrow X_n, \quad x \mapsto \sum_{i=1}^{M_n} \langle P_n^*(f_i), x \rangle P_n(x_i),$$

are uniformly bounded ($\|Q_n\| \leq 2$, for $n \in \mathbb{N}$), invertible and their inverses are uniformly bounded ($\|Q_n^{-1}\| \leq \frac{1}{r}$, for $n \in \mathbb{N}$). For $x \in X_n$ we write

$$x = Q_n^{-1}Q_n(x) = \sum_{i=1}^{M_n} \langle P_n^*(f_i), x \rangle (Q_n^{-1} \circ P_n)(x_i)$$

and deduce therefore that

$$(y_i^{(n)}, g_i^{(n)})_{i=1}^{M_n} := ((Q_n^{-1} \circ P_n)(x_i), P_n^*(f_i))_{i=1}^{M_n}$$

is a finite frame of X_n .

Let Z be a space with basis (z_i) which is associated to the frame (x_i, f_i) . It follows easily that the operators (S_n) and (T_n)

$$S_n : Z_n = [z_i : i \leq M_n] \rightarrow X_n, \quad z = \sum_{i=1}^{M_n} a_i z_i \mapsto \sum_{i=1}^{M_n} a_i (Q_n^{-1} \circ P_n)(x_i)$$

$$T_n : X_n \rightarrow Z_n, \quad x = \sum_{i=1}^{M_n} f_i(x) (Q_n^{-1} \circ P_n)(x_i) \mapsto \sum_{i=1}^{M_n} f_i(x) z_i$$

are uniformly bounded, and thus Z_n is an associated space for the frame $(y_i^{(n)}, g_i^{(n)})_{i \leq M_n}$ while T_n and S_n are the associated decomposition and reconstruction operators, respectively.

Finally assume that the frame (x_i, f_i) satisfies the NQP with respect to Z . Again by compactness and using Proposition 4.2 we can choose $M_n = M(r, n)$ large enough

so that it also satisfies for some (ε, δ, C) , independent of n ,

(24) For all $n \in \mathbb{N}$ and $x \in X_n$ there is a sequence $(k_i)_{i=1}^{M_n} \subset \mathbb{Z}$ so that

$$\left\| \sum_{i=1}^{M_n} k_i \delta z_i \right\| \leq C \|x\| \text{ and } \left\| x - \sum_{i=1}^{M_n} \delta k_i x_i \right\| \leq \varepsilon.$$

After changing $\varepsilon > 0$ and δ proportionally, if necessary, and since $r > \frac{1}{2}$, we can assume that $q = \frac{1-r}{r} + \sup_n \|P_n\| \frac{\varepsilon}{r} < 1$. For n in \mathbb{N} and $x \in B_{X_n}$ we can therefore choose $(k_i)_{i=1}^{M_n} \subset \mathbb{Z}$ so that $\left\| \sum_{i=1}^{M_n} \delta k_i z_i \right\| \leq C$ and

$$\begin{aligned} \left\| x - \sum_{i=1}^{M_n} \delta k_i (Q_n^{-1} \circ P_n)(x_i) \right\| &\leq \|Q_n^{-1}\| \cdot \left\| Q_n(x) - \sum_{i=1}^{M_n} \delta k_i P_n(x_i) \right\| \\ &\leq \|Q_n^{-1}\| \cdot \|Q_n(x) - x\| + \|Q_n^{-1}\| \cdot \left\| x - \sum_{i=1}^{M_n} \delta k_i P_n(x_i) \right\| \\ &\leq \|Q_n^{-1}\| \cdot \|Q_n(x) - x\| + \|Q_n^{-1}\| \cdot \|P_n\| \cdot \left\| x - \sum_{i=1}^{M_n} \delta k_i x_i \right\| \\ &\leq \frac{1-r}{r} + \sup_n \|P_n\| \frac{\varepsilon}{r} = q < 1. \end{aligned}$$

Thus, for every $n \in \mathbb{N}$ the frame $(P_n^*(f_i), (Q_n^{-1} \circ P_n)(x_i))_{i=1}^{M_n}$ satisfies condition (16) of Proposition 4.3 (for $(\varepsilon, \delta, C) = (q, \delta, C)$). Therefore we deduce the following Corollary.

Corollary 4.4. *Let $(x_i, f_i)_{i \in \mathbb{N}}$ be a frame of an infinite dimensional Banach space X for which there is a uniformly bounded sequence (P_n) of finite rank projections which approximate the identity. Assume that $(x_i, f_i)_{i \in \mathbb{N}}$ satisfies the (ε, δ, C) -NQP with respect to a space Z with basis (z_i) for some choice of $\varepsilon > 0$, $\delta > 0$ and C so that $q = \frac{1-r}{r} + \sup_n \|P_n\| \frac{2\varepsilon}{r} < 1$ with $\frac{1}{2} < r < 1$. Let (M_n) be any sequence in \mathbb{N} which satisfies (23) and (24).*

Finally assume that (z_i) satisfies the following lower estimate

$$\liminf_{n \rightarrow \infty} \left\{ \left\| \sum_{j \in A} n_j z_j \right\| : (n_j)_{j \in A} \subset \mathbb{Z} \setminus \{0\}, A \subset \mathbb{N}, \#A = n \right\} = \infty.$$

Then

- a) (M_n) increases exponentially with the dimension of X_n , i.e. there is a $c > 1$, so that $M_n \geq c^{\dim(X_n)}$ eventually,
- b) $\limsup_{n \rightarrow \infty} \frac{\ln(n)}{n} \sup \left\{ \left\| \sum_{i \in A} \pm z_i \right\| : A \subset \mathbb{N}, \#A = n, \right\} = \infty.$

Let us simplify the conditions in Corollary 4.4 and observe that it implies the following.

Corollary 4.5. *Assume that X is an infinite dimensional Banach space with the π_λ -property and that Z is a Banach space with a basis (z_i) satisfying for some choice of $1 < q < p < \infty$ lower ℓ_p and upper ℓ_q estimates, which means that for some C and all $(a_i) \in c_{00}$*

$$\frac{1}{C} \left(\sum |a_i|^p \right)^{1/p} \leq \left\| \sum a_i z_i \right\| \leq C \left(\sum |a_i|^q \right)^{1/q}.$$

Then no frame of X has the NQP with respect to $(Z, (z_i))$.

The following proposition shows how to construct a frame with respect to a space Z which contains ℓ_1 .

Proposition 4.6. *Let $(x_i, f_i)_{i \in \mathbb{N}}$ be any frame of a Banach space X and let Z be a space with semi-normalized basis (z_i) , which is associated to (x_i, f_i) . Then there is a frame $(\tilde{x}_i, \tilde{f}_i)_{i \in \mathbb{N}}$ of X and a basis (\tilde{z}_i) of $\tilde{Z} = Z \oplus_\infty \ell_1$ so that $(\tilde{x}_i, \tilde{f}_i)_{i \in \mathbb{N}}$ has \tilde{Z} as an associated space and has the NQP with respect to \tilde{Z} . Moreover, $(\tilde{x}_i, \tilde{f}_i)_{i \in \mathbb{N}}$ is semi-normalized if $(x_i, f_i)_{i \in \mathbb{N}}$ has this property (for example if (x_i) is a normalized basis of X).*

Proof. Assume, without loss of generality that $\|z_i\| = 1$ for $i \in \mathbb{N}$. Choose a quotient map $Q : \ell_1 \rightarrow X$ so that $(Q(e_i) : i \in \mathbb{N})$ is a $\frac{1}{2}$ -net in B_X and so that $\|Q(e_i) \pm x_i\| > \frac{1}{4}$ for $i \in \mathbb{N}$ (which is easy to accomplish). Finally we apply Proposition 2.8 to $Y = \ell_1$ with its unit vector basis (e_i) and $V = Q$, and observe that the frame $(\tilde{x}_i, \tilde{f}_i)$ and basis (\tilde{z}_i) of \tilde{Z} , as constructed there, has the property that for any $x \in B_X$ there is an $i \in \mathbb{N}$ so that

$$\left\| x - \frac{\tilde{x}_{2i} - \tilde{x}_{2i-1}}{2} \right\| = \|x - Q(e_i)\| \leq \frac{1}{2} \text{ and } \left\| \frac{1}{2}(\tilde{z}_{2i} - \tilde{z}_{2i-1}) \right\| = 1$$

which implies by Proposition 4.2 that $(x_i, f_i)_{i \in \mathbb{N}}$ has the NQP with respect to (z_i) .

By construction of $(\tilde{x}_i, \tilde{f}_i)$ in Proposition 2.8 it follows that (\tilde{f}_i) is semi-normalized if (f_i) has this property and since $\|Q(e_i) \pm x_i\| > \frac{1}{4}$, for $i \in \mathbb{N}$, it follows that

$$\frac{1}{4} \leq \|\tilde{x}_i\| \leq \sup_{j \in \mathbb{N}} \|x_j\| + 1,$$

which implies that $(\tilde{x}_i, \tilde{f}_i)$ is semi-normalized if (x_i, f_i) has this property. \square

Finally we present an infinite dimensional argument implying that if Z is a reflexive space with basis it cannot be the associated space of a frame $(x_i, f_i)_{i \in \mathbb{N}}$, with $\|x_i\| = 1$, for $i \in \mathbb{N}$, which satisfies the NQP.

Proposition 4.7. *Assume that Z is a reflexive space with a basis (z_i) , with $\inf_{i \in \mathbb{N}} \|z_i\| > 0$, and assume that $(x_i, (f_i))$ is a frame of an infinite dimensional Banach space X with associated space Z .*

Then $((x_i), (f_i))$ cannot have the NQP with respect to Z .

The following result follows from Proposition 4.7 as well as from Corollary 4.5.

Corollary 4.8. *A semi-normalized frame of an infinite dimensional Hilbert space H cannot have the NQP with respect to the associated Hilbert space $\ell_2(\mathbb{N})$.*

Proof of Proposition 4.7. We assume w.l.o.g. that (z_i) is bimonotone and let $T : X \rightarrow Z$ and $S : Z \rightarrow X$ be the associated decomposition and reconstruction operator, respectively.

For $C < \infty$ and $\delta > 0$ define

$$\mathcal{B}_{(C,\delta)} = \left\{ \sum_{i=1}^{\infty} \delta k_i z_i \in Z : (k_i) \subset \mathbb{Z}, \left\| \sum_{i=1}^{\infty} \delta k_i z_i \right\| \leq C \right\}.$$

Assume that (x_i, f_i) has the NQP with respect to Z . Then we can choose $\delta > 0$ small enough and $C \geq 1$ large enough so that $S(B_{(C,\delta)})$ is ε -dense in B_X for some $0 < \varepsilon < 1$.

Since (z_i) is semi-normalized and Z is reflexive, $\mathcal{B}_{(C,\delta)}$ is weakly compact. Indeed, assume that for $n \in \mathbb{N}$,

$$y_n = \sum_{i=1}^{\infty} \delta k_i^{(n)} z_i \in \mathcal{B}_{(C,\delta)}.$$

After passing to a subsequence we can assume that for all $i \in \mathbb{N}$ there is a $k_i \in \mathbb{N}$ so that $k_i^{(n)} = k_i$ whenever $n \geq i$. Thus, by bimonotonicity, it follows that $\left\| \sum_{i=1}^n \delta k_i z_i \right\| \leq C$, for all $n \in \mathbb{N}$, and, thus, since (z_i) is boundedly complete $\sum_{i=1}^{\infty} \delta k_i z_i \in Z$ and $\left\| \sum_{i=1}^{\infty} \delta k_i z_i \right\| \leq C$. Thus, $\sum_{i=1}^{\infty} \delta k_i z_i$ is in $B_{(C,\delta)}$, and since $k_i^{(n)}$ converges point-wise to (k_i) and (z_i) is shrinking it is the weak limit of $y^{(n)}$. The support of each element in $B_{(C,\delta)}$ is finite since (z_i) is a semi-normalized basis, and thus $B_{(C,\delta)}$ is countable. Since $S(B_{(C,\delta)})$ is ε -dense in X , it follows that the map

$$E : X^* \rightarrow C(\mathcal{B}_{(C,\delta)}), \text{ with } E(x^*) \left(\sum \delta k_i z_i \right) = \sum \delta k_i S^*(x^*)(z_i),$$

is an isomorphic embedding. Indeed, for $x^* \in B_X^*$ there is an $x \in B_X$ with $|x^*(x)| = 1$ and $\sum \delta k_i z_i \in \mathcal{B}_{(C,\delta)}$ so that $\|x - \sum \delta k_i z_i\| \leq \varepsilon$, and thus

$$\|E(x^*)\| \geq \left| E(x^*) \left(\sum \delta k_i z_i \right) \right| = x^* \left(\sum \delta k_i z_i \right) = 1 + x^* \left(\sum \delta k_i z_i - x \right) \geq 1 - \varepsilon.$$

But this yields that X^* is isomorphic to a subspace of the space of continuous functions on a countable compact space, and, so X must contain c_0 . This is impossible since X is a quotient of a reflexive space and thus also reflexive. \square

5. QUANTIZATION AND COTYPE

In this section we consider a quantization concept for Schauder frames, which is independent of an associated space.

Definition 5.1. Let $(x_i, f_i)_{i \in \mathbb{J}}$ be a frame of a (finite or infinite dimensional) Banach space X , $\mathbb{J} = \mathbb{N}$ or $\mathbb{J} = \{1, 2, \dots, N\}$, for some $N \in \mathbb{N}$, and let $0 < \varepsilon$, $0 < \delta \leq 1$ and $1 \leq C < \infty$. We say that $(x_i, f_i)_{i \in \mathbb{J}}$ satisfies the (ε, δ, C) -Bounded Coefficient Net Quantization Property or (ε, δ, C) -BCNQP if for all $(a_i)_{i \in \mathbb{J}} \in [-1, 1]^{\mathbb{J}} \cap c_{00}(\mathbb{J})$ there is a $(k_i)_{i \in \mathbb{J}} \in \mathbb{Z}^{\mathbb{J}} \cap c_{00}(\mathbb{J})$ so that

$$\left\| \sum_{i \in \mathbb{J}} a_i x_i - \sum_{i \in \mathbb{J}} \delta k_i x_i \right\| \leq \varepsilon \text{ and } \max_{i \in \mathbb{J}} |k_i| \leq \frac{C}{\delta}.$$

Remark 5.2. Let $(x_i, f_i)_{i \in \mathbb{J}}$ be a frame of X and let $0 < \varepsilon$, $0 < \delta \leq 1$ and $1 \leq C < \infty$.

- a) Since for any $(a_i)_{i \in \mathbb{J}} \in c_{00}(\mathbb{J})$ and any $i \in \mathbb{J}$ we can write $a_i = m_i \delta + \tilde{a}_i$ with $m_i \in \mathbb{Z}$, $|m_i| \delta \leq |a_i|$ and $|\tilde{a}_i| \leq \delta$, for $i \in \mathbb{J}$, (x_i, f_i) satisfies the (ε, δ, C) -BCNQP implies that

(25) for all $(a_i)_{i \in \mathbb{J}} \in c_{00}(\mathbb{J})$ there is a $(k_i)_{i \in \mathbb{J}} \in \mathbb{Z}^{\mathbb{J}} \cap c_{00}(\mathbb{J})$ so that

$$\left\| \sum_{i \in \mathbb{J}} a_i x_i - \sum_{i \in \mathbb{J}} \delta k_i x_i \right\| \leq \varepsilon \text{ and } \max_{i \in \mathbb{J}} |k_i| \leq \frac{1}{\delta} \max_{i \in \mathbb{J}} (|a_i| + C).$$

(a) immediately implies

- b) If (x_i, f_i) satisfies (ε, δ, C) -BCNQP and $0 < \lambda \leq 1$ then (x_i, f_i) satisfies $(\lambda \varepsilon, \lambda \delta, 1 + \lambda C)$ -BCNQP.
- c) If (x_i) is a semi-normalized basis of X and (f_i) are the coordinate functionals with respect to (x_i) and (x_i) satisfies the (ε, δ) -NQP (Definition 3.1), then (x_i, f_i) satisfies the (ε, δ, C) -BCNQP with $C = 1 + \varepsilon \sup_{i \in \mathbb{J}} \|f_i\|$. Indeed, for $x \in X$, $x = \sum_{i=1}^n a_i x_i$, with $|a_i| \leq 1$, there is a sequence $(k_i) \in \mathbb{Z}$ with finite support so that $\|x - \sum \delta k_i x_i\| \leq \varepsilon$ and

$$\delta \max |k_i| \leq \max_i \left(|f_i(x)| + \left| f_i \left(x - \sum \delta k_i x_i \right) \right| \right) \leq 1 + \varepsilon \sup_{i \in \mathbb{J}} \|f_i\|.$$

We will connect the property BCNQP with properties of the *cotype* of the Banach space.

Definition 5.3. Let $p \leq 2$. A Banach space X has type p if there is a $c < \infty$ so that for all $n \in \mathbb{N}$ and all vectors $x_1, x_2, \dots, x_n \in X$

$$\left(\text{ave} \left\| \sum_{i=1}^n \pm x_i \right\|^2 \right)^{1/2} = \left(2^{-n} \sum_{(\sigma_i)_{i=1}^n \in \{\pm 1\}^n} \left\| \sum_{i=1}^n \sigma_i x_i \right\|^2 \right)^{1/2} \leq c \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p}.$$

In that case the smallest such c will be denoted by $T_p(X)$.

Let $q \geq 2$. A Banach space X has *cotype* q if there is a $c < \infty$ so that for all $n \in \mathbb{N}$ and all $x_1, x_2, \dots, x_n \in X$:

$$\left(\sum_{i=1}^n \|x_i\|^q \right)^{1/q} \leq c \left(\text{ave} \left\| \sum_{i=1}^n \pm x_i \right\|^2 \right)^{1/2} = c \left(2^{-n} \sum_{(\sigma_i)_{i=1}^n \in \{\pm 1\}^n} \left\| \sum_{i=1}^n \sigma_i x_i \right\|^2 \right)^{1/2}.$$

The smallest of all these constants will be denoted by $C_q(X)$.

We say that X has only trivial type, or only trivial cotype if $T_p(X) = \infty$ for all $p > 1$, or $C_q(X) = \infty$, for all $q < \infty$.

Basic properties of spaces with type and cotype can be found for example in [DJT] or [Pi2]. We are mainly interested in estimates of the volume ratio of the unit ball B_X of a finite dimensional space X using $C_q(X)$ and the connection between finite cotype and the lack of containing ℓ_∞^n 's uniformly.

Assume that X is an n -dimensional space which we identify with $(\mathbb{R}^n, \|\cdot\|)$. Let E be the John ellipsoid of the unit ball B_X of X , i.e. the ellipsoid contained in B_X having maximal volume. It was show in [Jo] (see also [Pi2, Chapter 3]) that E is unique. We call the ratio $\text{Vol}^{1/n}(B_X)/\text{Vol}^{1/n}(E)$ the *volume ratio* of B_X . Combining [Ro, Theorem 6], which establishes an upper estimate for the volume ratio using $T_p(X^*)$, with a result of Maurey and Pisier [MP1, MP2] (see also [DJT, Proposition 13.17]) estimating $T_p(X^*)$ and a result of Pisier ([Pi1] (see also [Pi2, Theorem 2.5]) estimating the K -convexity constant $K(X)$ of X , we obtain the connection between the volume ratio of B_X and the cotype constant of X .

Theorem 5.4. *There is a universal constant d so that for all finite dimensional Banach spaces X , with $n = \dim(X) \geq 2$, and all $2 \leq q < \infty$.*

$$(26) \quad \left(\frac{\text{Vol}(B_X)}{\text{Vol}(E)} \right)^{1/n} \leq d C_q(X) n^{\alpha(q)} \ln n,$$

where $E \subset X$ is the John ellipsoid of B_X and $\alpha(q) := \frac{1}{2} - \frac{1}{q}$.

We will also need a second upper estimate for the volume ratio due to Milman and Pisier [MiP].

Theorem 5.5. [MiP] (see also [Pi2, Theorem 10.4]) *There is a universal constant A so that for any finite dimensional Banach space X ,*

$$(27) \quad \left(\frac{\text{Vol}(B_X)}{\text{Vol}(E)} \right)^{1/n} \leq g(C_2(X)) := A C_2(X) \ln(1 + C_2(X))$$

where $E \subset X$ is the John ellipsoid of B_X .

The next result describes the connection between the property of having a cotype $q < \infty$ and the lack of containing ℓ_∞^n 's uniformly.

Theorem 5.6. [MP1, MP2] *For every $N \in \mathbb{N}$ there exists a $q(N) \in (2, \infty)$ and a $C(N) < \infty$ so that:*

(28) *For any (finite or infinite dimensional) Banach space X which does not contain a 2-isomorphic copy of ℓ_∞^N we have that $C_{q(N)}(X) \leq C(N)$.*

Finally we will need the following result from [Os]. It is implicitly already contained in [GMP, pp.95–97].

In order to state it we will need the following notation. Let $m \leq n \in \mathbb{N}$ and let $L \subset \mathbb{R}^n$ be an m -dimensional subspace. Let Q_n be the unit cube in \mathbb{R}^n , i.e. the unit ball of ℓ_∞^n , $[-1, 1]^n$. By a simple compactness argument there is a projection $P : \mathbb{R}^n \rightarrow L$ for which $\text{Vol}(P(Q_n))$ is minimal. In that case we call the image $P(Q_n)$ a *minimal-volume projection of Q_n onto L* .

Theorem 5.7. [Os, Theorem 1] *Let L be a linear subspace of \mathbb{R}^n , and let \mathcal{M} be the set of all minimal-volume projections of Q_n onto L .*

Then \mathcal{M} contains a parallelepiped.

We are now in the position to state and to prove the connection between cotype and BCNQP in the finite dimensional case.

Theorem 5.8. *There is a map $n_0 : [1, \infty)^2 \rightarrow [0, \infty)$ so that for all finite dimensional Banach spaces X the following holds.*

If $(x_i, f_i)_{i=1}^N$ is a frame of X , with $\|x_i\| = 1$, for $i \leq N$, and which satisfies $(1, \delta, C)$ -BCNQP for some $0 < \delta < 1$ and $C \geq 1$, then for all $2 \leq q < \infty$

$$N \geq \dim(X) \ln(\dim(X)) \frac{1}{2q \ln\left(1 + 2\frac{C}{\delta}\right)}, \text{ whenever } \dim(X) \geq n_0\left(\frac{C}{\delta}, KC_q(X)\right),$$

where K is the projection constant of $(x_i, f_i)_{i=1}^N$.

Proof. Let Z be the space with a basis (z_i) and let $T : X \rightarrow Z$ and $S : Z \rightarrow X$ be the associated decomposition and the reconstruction operator as constructed in the proof of Proposition 2.4 (a) “ \Rightarrow ”. Since the x_i 's are normalized the z_i 's are also of norm 1. After a linear transformation we can assume that $Z = \mathbb{R}^N$ and $(z_i)_{i=1}^N$ is the unit vector basis of \mathbb{R}^N . Since (z_i) is a bimonotone basis it follows that $\|z_i^*\| = 1$, for $i \leq N$. Hence $B_Z \subset Q_N$, where Q_N denotes the unit cube in \mathbb{R}^N . Define $L = T(X)$ and put $n = \dim(X) = \dim(L)$. Since $S \circ T = Id_X$ it follows that $P = T \circ S$ is a

projection from Z onto L . If we denote the John ellipsoid of $T(B_X)$ by E , and we deduce that (recall that by Proposition 2.4 (a) $\|T\| \leq K$)

$$(29) \quad E \subset T(B_X) = P \circ T(B_X) \subset P(K \cdot B_Z) \subset P(K \cdot Q_N).$$

By Theorem 5.7 there is a minimal -volume projection M of Q_N onto L which is a parallelepiped. Let B_n denote the n dimensional Euclidean ball in \mathbb{R}^n . Since there is a universal constant c so that

$$\text{Vol}(B_n) \leq \left(\frac{c}{\sqrt{n}} \right)^n,$$

and since

$$\frac{1}{K}E \subset \frac{1}{\|T\|}E \subset L \cap B_Z \subset L \cap Q_N \subset M,$$

we deduce from the fact that B_n is the John ellipsoid of the unit cube in \mathbb{R}^n [Jo] (see also [Pi2, Chapter 3]), that

$$K \left(\frac{\text{Vol}(P(Q_N))}{\text{Vol}(E)} \right)^{1/n} = \left(\frac{\text{Vol}(P(Q_N))}{\text{Vol}(\frac{1}{K}E)} \right)^{1/n} \geq \left(\frac{\text{Vol}(M)}{\text{Vol}(\frac{1}{K}E)} \right)^{1/n} \geq \frac{\sqrt{n}}{c}$$

The last inequality follows from applying a linear transformation A to L so that $A(M)$ is a the unit cube in L (with respect to some orthonormal basis of L) and, thus $A(\frac{1}{K}E)$ is an ellipsoid whose volume cannot exceed that of the Euclidean unit ball in L . Since $T : (X, \|\cdot\|) \rightarrow (L, \|\cdot\|_{T(B_X)})$, where $\|\cdot\|_{T(B_X)}$ is the Minkowski functional for $T(B_X)$, is an isometry it follows from Theorem 5.4 that

$$(30) \quad \begin{aligned} \text{Vol}^{1/n}(T(B_X)) &\leq dC_q(X)n^{\alpha(q)} \ln(n)\text{Vol}^{1/n}(E) \\ &\leq dcKC_q(X)n^{-\frac{1}{q}} \ln(n)\text{Vol}^{1/n}(P(Q_N)) \end{aligned}$$

(the universal constant d was introduced in Theorem 5.4).

Since the zonotope

$$P(Q_N) = T \circ S \left(\left\{ \sum_{i=1}^N a_i z_i : |a_i| \leq 1 \right\} \right) = \left\{ \sum_{i=1}^N a_i T(x_i) : |a_i| \leq 1 \right\},$$

contains at most $(1 + \frac{2C}{\delta})^N$ points from the set $D = \{ \sum \delta n_i T(x_i) : (n_i) \subset \mathbb{Z}, \max \delta |n_i| \leq C \}$ and since from our assumption that $(x_i f_i)_{i=1}^N$ satisfies the $(1, \delta, C)$ -BCNQP it follows that

$$P(Q_N) \subset \bigcup_{z \in D} z + T(B_X),$$

we deduce that

$$\left(1 + \frac{2C}{\delta}\right)^N \geq \frac{\text{Vol}(P(Q_N))}{\text{Vol}(T(B_X))} \geq \left(\frac{n^{1/q}}{KdcC_q(X)\ln(n)}\right)^n$$

and, thus,

$$N \geq \frac{n \ln(n)}{q \ln\left(1 + \frac{C}{\delta}\right)} - \frac{n \ln(\ln(n))}{\ln\left(1 + \frac{2C}{\delta}\right)} - \frac{n \ln(dcKC_q(X))}{\ln\left(1 + \frac{2C}{\delta}\right)},$$

which easily implies our claim. \square

In the next result we will show that, up to a constant factor, the result in Theorem 5.8 is sharp. We are using the simple fact that for any number $0 \leq r \leq 1$ and any $m \in \mathbb{N}$, r can be approximated by a finite sum of dyadic numbers, say $\tilde{r} = \sum_{j=1}^m \sigma_j 2^{-j}$, $\sigma_j \in \{0, 1\}$, for $j = 1, \dots, m$, so that $|r - \tilde{r}| \leq 2^{-m}$.

Proposition 5.9. *Let X be an n -dimensional space with an Auerbach basis $(e_i, e_i^*)_{i=1}^n$. Let $m \in \mathbb{N}$ and let K be the projection constant of $(e_i)_{i=1}^n$. Then there is a frame $(x_{(i,j,s)}, f_{(i,j,s)} : 1 \leq i \leq n, 1 \leq j \leq m, s = 0, 1)$ of X (ordered lexicographically) so that*

$$(31) \quad \frac{1}{2} \leq \|x_{(i,j,s)}\| \leq 2 \text{ and } \|f_{(i,j,s)}\| = 1, \text{ whenever } 1 \leq i \leq n, 1 \leq j \leq m \text{ and } s = 0, 1,$$

$$(32) \quad \forall (a_{(i,j,s)} : 1 \leq i \leq n, 1 \leq j \leq m, s = 0, 1) \subset [-1, 1]$$

$$\exists (k_{(i,j,s)} : i \leq n, j \leq m, s = 0, 1) \subset \{-3, -2, \dots, 3\}$$

$$\left\| \sum_{i=1}^n \sum_{j=1}^m \sum_{s=0}^1 a_{(i,j,s)} x_{(i,j,s)} - \sum_{i=1}^n \sum_{j=1}^m \sum_{s=0}^1 k_{(i,j,s)} x_{(i,j,s)} \right\| \leq 1 + n \frac{2^{-m}}{1 - 2^{-m}}$$

(i.e. $(x_{(i,j,s)}, f_{(i,j,s)} : 1 \leq i \leq n, 1 \leq j \leq m, s = 0, 1)$ satisfies the

$$\left(1 + n \frac{2^{-m}}{1 - 2^{-m}}, 1, 3\right)\text{-BCNQP}.$$

(33) *The projection constant of $(x_{(i,j,s)}, f_{(i,j,s)} : 1 \leq i \leq n, 1 \leq j \leq m, s = 0, 1)$ does not exceed $4K$.*

Proof. For $1 \leq i \leq n$ and $1 \leq j \leq m$ define $x_{(i,j,0)} = e_i$, $x_{(i,j,1)} = e_i + \frac{2^{-j}}{1 - 2^{-m}} e_i$, $f_{(i,j,0)} = -e_i^*$ and $f_{(i,j,1)} = e_i^*$. Since for every $x \in X$

$$x = \sum_{i=1}^n e_i^*(x) e_i = \sum_{i=1}^n \sum_{j=1}^m e_i^*(x) e_i \frac{2^{-j}}{1 - 2^{-m}} = \sum_{i=1}^n \sum_{j=1}^m f_{(i,j,1)}(x) x_{(i,j,1)} + f_{(i,j,0)}(x) x_{(i,j,0)},$$

$(x_{(i,j,s)}, f_{(i,j,s)} : 1 \leq i \leq n, 1 \leq j \leq m, s = 0, 1)$ is a frame of X and it satisfies (31). In order to verify (32) let $(a_{(i,j,s)} : 1 \leq i \leq n, 1 \leq j \leq m, s = 0, 1) \subset [-1, 1]$ be given.

For $i = 1, 2, \dots, n$ it follows that $\left| \sum_{j=1}^m a_{(i,j,1)} \frac{2^{-j}}{1-2^{-m}} \right| \leq 1$, and, thus, we can choose $(k_{(i,j,1)} : j \leq m) \subset \{0, \pm 1\}$ so that for each $i \leq n$

$$(34) \quad \left| \sum_{j=1}^m a_{(i,j,1)} \frac{2^{-j}}{1-2^{-m}} - \sum_{j=1}^m k_{(i,j,1)} \frac{2^{-j}}{1-2^{-m}} \right| \leq \frac{2^{-m}}{1-2^{-m}}.$$

Since the absolute value of $M = \sum_{i=1}^n \sum_{j=1}^m a_{(i,j,1)} + a_{(i,j,0)} - k_{(i,j,1)}$ is at most $3nm$ we can choose for $1 \leq i \leq n$ and $1 \leq j \leq m$, $k_{(i,j,0)} \in \{-3, -2, \dots, 2, 3\}$ so that $a = M - \sum_{i=1}^n \sum_{j=1}^m k_{(i,j,0)}$, has absolute value at most 1. We compute

$$\begin{aligned} & \left\| \sum_{i=1}^n \sum_{j=1}^m \sum_{s=0}^1 a_{(i,j,s)} x_{(i,j,s)} - \sum_{i=1}^n \sum_{j=1}^m \sum_{s=0}^1 k_{(i,j,s)} x_{(i,j,s)} \right\| \\ & \leq \left\| \sum_{i=1}^n \sum_{j=1}^m a_{(i,j,1)} e_i \frac{2^{-j}}{1-2^{-m}} - \sum_{i=1}^n \sum_{j=1}^m k_{(i,j,1)} e_i \frac{2^{-j}}{1-2^{-m}} \right\| \\ & \quad + \left| \sum_{i=1}^n \sum_{j=1}^m a_{(i,j,1)} + a_{(i,j,0)} - k_{(i,j,0)} - k_{(i,j,1)} \right| \\ & \leq 1 + \sum_{i=1}^n \left| \sum_{j=1}^m (a_{(i,j,1)} - k_{(i,j,1)}) \frac{2^{-j}}{1-2^{-m}} \right| \leq 1 + n \frac{2^{-m}}{1-2^{-m}} \end{aligned}$$

which proves (32).

To estimate the projection constant of $(x_{(i,j,s)}, f_{(i,j,s)} : 1 \leq i \leq n, 1 \leq j \leq m, s=0, 1)$ we denote by \leq_{lex} the lexicographic order on $\{(i, j, s) : i \leq n, j \leq m, s = 0, 1\}$, and let

$$x = \sum_{i=1}^n a_i e_i = \sum_{i=1}^n \sum_{j=1}^m -a_i e_1 + a_i e_1 + a_i e_i \frac{2^{-j}}{1-2^{-m}} = \sum_{i=1}^n \sum_{j=1}^m \sum_{s=0,1} f_{(i,j,s)}(x) x_{(i,j,s)}$$

and $(i_0, j_0, s_0) \leq_{\text{lex}} (i_1, j_1, s_1)$. Then, if $i_0 < i_1$,

$$\begin{aligned}
& \left\| \sum_{(i_0, j_0, s_0) \leq_{\text{lex}} (i, j, s) \leq_{\text{lex}} (i_1, j_1, s_1)} f_{(i, j, s)}(x) x_{(i, j, s)} \right\| \\
&= \left\| 1_{\{s_0=1\}} \left[a_{i_0} e_1 + a_{i_0} \frac{2^{-j_0}}{1 - 2^{-m}} e_i \right] + \sum_{j=j_0+1}^m -a_{i_0} e_1 + a_{i_0} e_1 + a_{i_0} \frac{2^{-j}}{1 - 2^{-m}} e_{i_0} \right. \\
&\quad + \sum_{i=i_0+1}^{i_1-1} \sum_{j=1}^m \left(-a_i e_1 + a_i e_1 + a_i \frac{2^{-j}}{1 - 2^{-m}} e_i \right) \\
&\quad \left. + \sum_{j=1}^{j_1} \left(-a_{i_1} e_1 + a_{i_1} e_1 + a_{i_1} \frac{2^{-j}}{1 - 2^{-m}} e_{i_1} \right) - 1_{\{s_1=0\}} a_{i_1} e_1 \right\| \\
&\leq 2|a_{i_0}| + \left\| \sum_{i=i_0+1}^{i_1-1} a_i e_i \right\| + |a_{i_1}| \leq 4K \|x\|.
\end{aligned}$$

If $i_0 = i_1$ similar estimates give the same result for the remaining cases and (33) follows. \square

Remark 5.10. If we choose in Proposition 5.9 $m = \lceil 2 \log n \rceil$ and thus $2^{-m} \simeq 1/n^2$ we obtain a frame for X of approximate size $4n \log_2(n)$ having the $(3, 1, 3)$ -BCNQP. Thus as we mentioned earlier, up to a constant Theorem 5.8 is best possible.

Remark 5.11. In Theorem 5.8 we assumed for simplicity that the x_i 's of our frame are normalized. It is easy to see that the same proof works for a general frame, in that case n_0 depends also on $a = \min\{\|x_i\| : i \leq N, x_i \neq 0\}$ and $b = \max\{\|x_i\| : i \leq N\}$.

With a similar proof to that of Theorem 5.8 we derive an upper estimate for $\min_{i \leq N} \|x_i\|$, $i \leq N$, assuming that $(x_i, f_i)_{i=1}^N$ is a frame of an n dimensional space X which satisfies the $(1, \delta, C)$ -BCNQP for some choice of $\delta > 0$ and $C < \infty$ assuming that N is proportional to n .

Theorem 5.12. *For any choice of $\delta \in (0, 1]$, and $C, K, q, c_2 \geq 1$ there is a value $h = h(\delta, C, K, q, c_2)$ so that the following holds for all $n \in \mathbb{N}$.*

If X is an n -dimensional space, $N \leq qn$ and $(x_i, f_i)_{i=1}^N$ is a frame of X with projection constant K which has the $(1, \delta, C)$ -BCNQP, then if $C_2(X) \leq c_2$,

$$\min_{i \leq N} \|x_i\| \leq \frac{h(\delta, C, K, q, c_2)}{\sqrt{n}}.$$

Sketch of proof. Let $(x_i, f_i)_{i=1}^N$ be a frame of X , $N \leq qn$, which has the $(1, \delta, C)$ -BCNQP and projection constant K . As in the proof of Theorem 5.8 we let Z be the

associated space with basis (z_i) which was constructed in Proposition 2.4, $T : X \rightarrow Z$ the associated decomposition operator, and S the associated reconstruction operator. Let $L = T(X)$, and $P = T \circ S$, and let us also assume that $Z = \mathbb{R}^N$ and $z_i = e_i$ for $i \leq N$. Note that now $\|z_i\| = \|x_i\|$ and $z_i^* = \|x_i\|^{-1}$ and we can therefore follow the proof of Theorem 5.8 replacing Q_N by the box

$$\tilde{Q}_N = \prod_{i=1}^N \left[-\frac{1}{\|x_i\|}, \frac{1}{\|x_i\|} \right].$$

As in the proof of Theorem 5.8 it follows that $\frac{1}{K}T(B_X) \subset P(B_Z) \subset P(\tilde{Q}_N)$. For the John ellipsoid E of $T(B_X)$ it follows therefore that $\frac{1}{K}E \subset M$, where a M is a minimal volume projection of \tilde{Q}_N which is also a parallelepiped in L , and as before we deduce that $K\text{Vol}^{1/n}(P(\tilde{Q}_N)) \geq \text{Vol}^{1/n}(E)\sqrt{n}/c$. Instead of applying Theorem 5.4 we now use Theorem 5.5 and letting $\alpha = \min_{i \leq N} \|x_i\|$ we deduce that

$$\begin{aligned} \text{Vol}^{1/n}(T(B_X)) &\leq g(C_2(X))\text{Vol}^{1/n}(E) \\ &\leq \frac{g(C_2(X))cK\text{Vol}^{1/n}(P(\tilde{Q}_N))}{\sqrt{n}} \leq \frac{g(C_2(X))cK\text{Vol}^{1/n}(P(Q_N))}{\sqrt{n}\alpha}. \end{aligned}$$

We can again compare the volume of the zonotope $P(Q_N)$ with the volume of the union $\bigcup_{z \in D} z + T(B_X)$, where D is defined as in the proof of Theorem 5.4, and deduce that

$$\left(1 + \frac{2C}{\delta}\right)^{qn} \geq \left(1 + \frac{2C}{\delta}\right)^N \geq \frac{\text{Vol}(P(Q_N))}{\text{Vol}(T(B_X))} \geq \left(\frac{\sqrt{n}\alpha}{g(C_2(X))cK}\right)^n.$$

Taking the n th root on both sides yields our claim. \square

Remark 5.13. In section 6 we will recall a result of Lyubarski and Vershinin [LV] which shows that for $q > 1$ there are $\varepsilon < 1$, $\delta < 1$, $C < \infty$ so that for any $n \in \mathbb{N}$ and there is a Hilbert frame $(x_i)_{i=1}^N$ of ℓ_2^n , with $N \leq qn$, so that $(x_i)_{i=1}^N$ satisfies (ε, δ, C) -BCNQP.

As in the previous section we formulate a corollary of Theorem 5.8 for the infinite dimensional situation.

Corollary 5.14. *Assume that X has the π_λ -property and that $\bar{P} = (P_n)$ is a sequence of uniformly bounded projections approximating point-wise the identity on X . Let (x_i, f_i) be a frame of X , with (x_i) being bounded, and assume for an increasing sequence $(L_n) \subset \mathbb{N}$, $0 < r, \delta, \varepsilon < 1$ and $C < \infty$ that the following conditions are satisfied:*

a) For $n \in \mathbb{N}$ and $x \in X_n = P_n(X)$

$$\left\| x - \sum_{i=1}^{L_n} \langle f_i, x \rangle P_n(x_i) \right\| \leq (1-r)\|x\|,$$

b) $\|P_n(x_i)\| \geq r$, for all $n \in \mathbb{N}$ and $i \leq L_n$, and

c) For all $n \in \mathbb{N}$ and all $x \in \{\sum_{i=1}^{L_n} a_i x_i : |a_i| \leq 1 \text{ for } i = 1, 2, \dots, L_n\}$ there is a sequence $(k_i)_{i \leq L_n}$ so that

$$\left\| x - \sum_{i=1}^{L_n} \delta k_i x_i \right\| < \varepsilon \text{ and } \max_{i \leq L_n} |k_i| \delta \leq C$$

(in particular (x_i, f_i) satisfies the BCNQP).

Then there is either a constant $c > 0$ so that $L_n \geq c \dim P_n(X) \ln(\dim P_n(X))$ or the spaces $\{\ell_\infty^n : n \in \mathbb{N}\}$ are uniformly contained in X .

Proof. From assumption (a) it follows that the operators (Q_n) , with

$$Q_n : X_n \rightarrow X_n, \quad x \mapsto \sum_{i=1}^{L_n} \langle P_n^*(f_i), x \rangle P_n(x_i),$$

are uniformly bounded ($\|Q_n\| \leq 2$, for $n \in \mathbb{N}$), invertible and their inverses are uniformly bounded ($\|Q_n^{-1}\| \leq \frac{1}{r}$, for $n \in \mathbb{N}$). For $x \in X_n$ we write

$$x = Q_n^{-1} Q_n(x) = \sum_{i=1}^{L_n} \langle f_i, x \rangle (Q_n^{-1} \circ P_n)(x_i),$$

and deduce therefore that

$$(y_i^{(n)}, g_i^{(n)})_{i=1}^{L_n} := ((Q_n^{-1} \circ P_n)(x_i), f_i|_{X_n})_{i=1}^{L_n}$$

is a frame of X_n . We now verify that for $n \in \mathbb{N}$ $(y_i^{(n)}, g_i^{(n)})_{i=1}^{L_n}$, satisfies the $(\tilde{\varepsilon}, \delta, C)$ -BCNQP for some $\tilde{\varepsilon} > 0$, which is independent of n . Indeed by assumption (c) one can choose for $n \in \mathbb{N}$ and $(a_i)_{i=1}^{L_n} \in [-1, 1]$ some $(k_i)_{i=1}^{L_n} \subset \mathbb{Z}$ so that

$$\left\| \sum_{i=1}^{L_n} a_i x_i - \sum_{i=1}^{L_n} \delta k_i x_i \right\| \leq \varepsilon, \text{ and } \max_{i \leq L_n} |k_i| \leq \frac{C}{\delta}$$

and, thus,

$$\begin{aligned} \left\| \sum_{i=1}^{L_n} a_i y_i^{(n)} - \sum_{i=1}^{L_n} \delta k_i y_i^{(n)} \right\| &= \left\| \sum_{i=1}^{L_n} a_i (Q_n^{-1} \circ P_n)(x_i) - \sum_{i=1}^{L_n} \delta k_i (Q_n^{-1} \circ P_n)(x_i) \right\| \\ &\leq \varepsilon \max_{i \leq L_n} \|Q_n^{-1} \circ P_n\| \leq \frac{\varepsilon}{r} \sup_n \|P_n\| =: \tilde{\varepsilon}. \end{aligned}$$

Then for $n \in \mathbb{N}$ and $i \leq L_n$ it follows from assumption (b) that

$$\frac{r}{2} \leq \frac{r}{\|Q_n\|} \leq \|y_i^{(n)}\| \leq \|P_n\| \cdot \|Q_n^{-1}\| \cdot \|x_i\| \leq \frac{\sup_j \|P_j\| \sup_j \|x_j\|}{r} < \infty.$$

Thus Theorem 5.8, Remark 5.11 and Theorem 5.6 yield our claim. \square

By Remark 5.2, for semi-normalized bases (x_i) (together with their coordinate functionals) the properties BCNQP and NQP are equivalent. We therefore deduce from Theorem 5.8 the following corollary

Corollary 5.15. *An infinite dimensional Banach space X with non trivial cotype cannot have a semi-normalized basis having the NQP.*

In particular ℓ_1 does not have a semi-normalized basis with the NQP.

The “in particular” statement solves Problem 5.18 in [DOSZ].

Proof. Suppose (x_i) is a semi-normalized basis with the (ε, δ) -CQP. Then we can take (P_n) to be the basis projections and $L_n = n$. By Corollary 5.14 X cannot have finite cotype. \square

6. CONCLUDING REMARKS AND OPEN PROBLEMS

Kashin’s [Ka] celebrated result states that for any $\lambda > 1$ there is a $K = K_\lambda$ so that for any $n \in \mathbb{N}$ and any $N \geq \lambda n$, $N \in \mathbb{N}$, there is an orthogonal projection U from \mathbb{R}^N onto \mathbb{R}^n (i.e. U is an N by n matrix whose rows are orthonormal) so that

$$(35) \quad B_n \subset \frac{K}{\sqrt{N}} U(Q_N) \subset K B_n$$

(as before B_n is the Euclidean unit ball in \mathbb{R}^n while Q_N is the unit cube in \mathbb{R}^N).

Lyubarskii and Vershinin observed in [LV] that the column vectors $(u_i)_{i=1}^N$ form a tight frame (with $A = B = 1$), that the first inclusion in (35) yields that every $x \in B_n$ can be written as

$$x = \sum_{i=1}^N \frac{K}{\sqrt{N}} a_i u_i \text{ with } \|(a_i)\|_{\ell_\infty} \leq 1,$$

and that the second inclusion implies that the operator $\frac{1}{\sqrt{N}} U : \ell_\infty^N \rightarrow \ell_2^n$ is of norm not greater than 1, and that therefore for given $\varepsilon > 0$ there is a sequence $(k_i)_{i=1}^N \subset \mathbb{Z} \cup [-K/\varepsilon, K/\varepsilon]$, so that $\max_{i \leq N} |a_i K - k_i \varepsilon| \leq \varepsilon$ and, thus,

$$(36) \quad \left\| x - \sum_{i=1}^N \varepsilon k_i \frac{u_i}{\sqrt{N}} \right\| \leq \varepsilon \text{ and } \max | \varepsilon k_i | \leq K.$$

Thus, Kashin's orthogonal projections (which are actually chosen randomly), lead to a frame $(x_i^{(n)})_{i \leq N} = (u_i^{(n)}/\sqrt{N})_{i \leq N}$ for ℓ_2^n , whose length is not larger than a fixed multiple of n , and, for which we can represent any element x in B_n as a quantized linear combination with bounded coefficients. Since the zonotope $\{\sum_{i=1}^N a_i x_i : |a_i| \leq 1 \text{ for } i = 1, 2, \dots, N\}$ lies in B_n , it follows that the Hilbert frame $(x_i^{(n)})_{i \leq N}$ satisfies for any $\varepsilon > 0$ the $(\varepsilon, \varepsilon, K)$ -BCNQP.

In view of the results presented in sections 4 and 5 this is the best one could do in the finite dimensional case. We are therefore interested in extensions of this result by Lyubarskii and Vershinin to other spaces as well as the infinite dimensional space,

Problem 6.1. Does the above cited result hold for other finite dimensional spaces? More precisely, assume that $0 < \delta, \varepsilon < 1$, $C \geq 1$ are fixed. For which $n \in \mathbb{N}$ and which n -dimensional spaces X can we find a frame $(x_i, f_i)_{i=1}^N$, with, say $N = 2n$, so that for any $x \in B_X$ there is a $(k_i)_{i=1}^N \subset \mathbb{Z}$ so that

$$\left\| x - \sum_{i=1}^N \delta k_i x_i \right\| \leq \varepsilon \text{ and } \max |k_i| \leq C.$$

Remark 6.2. The argument, presented above, from [LV] shows that if there is a quotient $Q : \ell_\infty^N \rightarrow X$, and a frame $(x_i, f_i)_{i=1}^N$ of X , so that $Q(e_i) = x_i$, for $i = 1, \dots, N$, and so that for some $K < \infty$ $B_X \subset Q(B_{\ell_\infty}) \subset KB_X$, then there is for all $x \in B_X$ and all $\delta > 0$ a sequence $(k_i)_{i \leq N} \subset \mathbb{Z}$, so that

$$\left\| x - \sum_{i=1}^N \delta k_i x_i \right\| \leq \|Q\| \delta / 2 \leq K \delta / 2 \text{ and } \max_{i \leq N} |k_i| \leq \frac{1}{\delta}.$$

Conversely, assume that for some $0 < \delta, \varepsilon < 1$, $C \geq 1$ we can find for all $x \in B_X$ a sequence $(k_i) \subset \mathbb{Z}$ so that

$$\left\| x - \sum_{i=1}^N \delta k_i x_i \right\| \leq \varepsilon \text{ and } \max_{i \leq N} |k_i| \leq \frac{C}{\delta}.$$

Then we can choose by induction for $x \in B_X$ a sequence (z_n) with

$$z_n = \sum_{i=1}^N k_i^{(n)} \delta x_i, (k_i) \subset \mathbb{Z} \cap [-C/\delta, C/\delta],$$

so that for all $n \in \mathbb{N}$

$$\left\| x - \sum_{i=1}^n \varepsilon^{i-1} z_i \right\| \leq \varepsilon^n.$$

Indeed assuming z_1, \dots, z_{n-1} have been chosen we apply our assumption to $y = \varepsilon^{1-n} [x - \sum_{i=1}^{n-1} \varepsilon^{1-i} z_i] \in B_X$ to find z_n .

Thus, it follows that

$$x = \sum_{i=1}^{\infty} \varepsilon^{i-1} z_i = \sum_{j=1}^N x_j \sum_{i=1}^{\infty} \varepsilon^{i1} \delta k_j^{(i)},$$

which means that there is a $C_1 < \infty$ only depending on ε, δ and C so that

$$B_X \subset \left\{ \sum_{j=1}^N a_j x_j : |a_j| \leq C_1 \right\}.$$

If we define now

$$Q : \ell_{\infty}^N \rightarrow B_X, \quad z \mapsto \sum_{i=1}^N C_1 z_i x_i,$$

we deduce that Q is a quotient map and that $B_X \subset Q(B_{\ell_{\infty}})$ but we cannot deduce (at least not obviously) a bound for $\|Q\|$.

Problem 6.3. Is there an infinite dimensional version of the result of Lyubarskii and Vershinin? I.e. for which infinite dimensional Banach spaces X with a basis (e_i) does there exist $0 < \delta, \varepsilon < 1$, $C \geq 1$ and a frame $(x_i, f_i)_{i \in \mathbb{N}}$ so that for any $x \in B_X$, $n = \max \text{supp}(x) < \infty$, there is a $(k_i)_{i=1}^N$, with, say, $N \leq 2n$, so that

$$\left\| x - \sum_{i=1}^N \delta k_i x_i \right\| \leq \varepsilon \text{ and } \max |\delta k_i| \leq C?$$

REFERENCES

- [AG] A. Alroubi and K. Gröchenig, *Nonuniform sampling and reconstruction in shift invariant spaces*, SIAM rev. **43** (2001) 585–620.
- [BP] K. Ball and A. Pajor, *The entropy of convex bodies with "few" extreme points*. Geometry of Banach spaces (Strobl, 1989), 25–32, London Math. Soc. Lecture Note Ser., **158**, Cambridge Univ. Press, Cambridge, 1990.
- [Ca1] P. G. Casazza, *The art of frame theory*, Taiwanese Journal of Math. **4** No.2 (2000) 129–201.
- [Ca2] P. G. Casazza, *Approximation Properties*, in: William B. Johnson and Joram Lindenstrauss (eds.), Handbook on the Geometry of Banach Spaces Vol. 1, North Holland, Amsterdam (2001) 271–316.
- [Ch] O. Christensen, *An introduction to frames and Riesz bases*, Birkhauser (2003).
- [CHL] P. G. Casazza, D. Han, and D. R. Larson, *Frames for Banach spaces*, The functional and harmonic analysis of wavelets and frames (San Antonio, TX, 1999), Contemp. Math., **247**, (1999) 149–182.
- [DD] I. Daubechies and R. DeVore, *Approximating a bandlimited function using very coarsely quantized data: A family of stable sigma-delta modulators of arbitrary order*, Annals of Mathematics **158** (2003), 679 – 710.

- [DJT] J. Diestel, H. Jarchow, A. Tonge, *Absolutely summing operators*. Cambridge Studies in Advanced Mathematics, 43. Cambridge University Press, Cambridge, 1995. xvi+474 pp.
- [DOSZ] S. J. Dilworth, E. Odell, Th. Schlumprecht and András Zsák, *Coefficient quantization in Banach Spaces*, Found. Comp. Math. (in press).
- [DE] D. L. Donoho and M. Elad, *Optimally sparse representations in general (non orthogonal dictionaries via ℓ_1 minimization*, Proc. Natl. AcadSci. USA **100**, 2197–2202.
- [EF] Y. C. Eldar and G. D. Forney, *Optimal tight frames and quantum measurement*, IEEE Trans.Information Theory **48** (2002) 599–610.
- [FHHMPZ] M. Fabian, P. Habala, P.Hajek, V. Montesinos Santalucía, J. Pelant, and V. Zizler, *Functional analysis and infinite-dimensional geometry*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 8. Springer-Verlag, New York (2001) x+451 pp.
- [GJ] Y. Gordon and M. Junge, *Volume ratios in L_p -spaces*. Studia Math. **136** no. 2 (1999) 147–182.
- [GMP] Y. Gordon, M. Meyer and A. Pajor. *Ratios of volumes and factorization through l_∞* . Illinois J. Math. 40 (1996), no. 1, 91–107.
- [GKK] V. K. Goyal, J. Kovačević and J. A. Kelner *Quantized frame expansions with erasures*, Appl. Comput. Harmon. Analysis **10** (2001) 203 – 233.
- [Gr] K. Gröchenig, *Describing functions: Atomic decompositions versus frames*. Monatshefte für Mathematik **112** (1991) 1 – 41.
- [GH] K. Gröchenig and C. Heil, *Modulation spaces and pseudodifferential operators*, Integral Equations Operator Theory, **34** (1999) 439–457.
- [HL] D. Han and D. H. Larson, *Frames, bases, and group representation*, Mem. Amer. Math. Soc. **147** no. 697 (2000) x+94 pp.
- [Jo] F. John. *Extremum problems with inequalities as subsidiary conditions*. Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, Interscience Publishers, Inc., New York, N. Y. (1948) 187–204. .
- [JRZ] W. B. Johnson, H.P. Rosenthal and M. Zippin, *On bases, finite dimensional decompositions, and weaker structures in banach spaces*, Israel J. Math. **9** (1971) 488–504.
- [Ka] *The widths of certain finite dimensional sets and classes of smooth functions*, Izv. Akad. Nauk SSSR Ser. Mat. 41 (1977), no. 2, 334–351, 478.
- [LV] Y. Lyubarskii and R. Vershynin, *Uncertainty Principles and Vector Quantization*, preprint, arXiv:math/0611343v1.
- [MP1] B. Maurey and G. Pisier, *Caractérisation d’une classe d’espaces de Banach par des propriétés de séries aléatoires vectorielles*. C. R. Acad. Sci. Paris Sér. A-B **277** (1973),
- [MP2] B. Maurey and G. Pisier, *Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach*. (French) Studia Math. 58 (1976), no. 1, 45–90.
- [MiP] V. D. Milman and G. Pisier, *Banach spaces with a weak cotype 2 property*. Israel J. Math. 54 (1986), no. 2, 139–158.
- [Os] M. I. Ostrovskii, *Minimal-volume shadows of cubes*. J. Funct. Anal. **176** no. 2, (2000) 317–330.
- [Pe] A. Pełczyński, *Any separable Banach space with the bounded approximation property is a complemented subspace of a Banach space with basis*, Studia Math. **40** (1971), 239–242.

- [Pi1] G. Pisier, *Un théorème sur les opérateurs linéaires entre espaces de Banach qui se factorisent par un espace de Hilbert*. Ann. Sci. École Norm. Sup. (4) **13** (1980), no. 1, 23–43.
- [Pi2] G. Pisier, *The volume of convex bodies and Banach space geometry*. Cambridge Tracts in Mathematics, 94. Cambridge University Press, Cambridge, 1989. xvi+250 pp.
- [Ro] M. Rogalski, *Sur le quotient volumique d'un espace de dimension finie*. (Initiation Seminar on Analysis: G. Choquet-M. Rogalski-J. Saint-Raymond, 20th Year: 1980/1981, Comm. No. C3, 31 pp., Publ. Math. Univ. Pierre et Marie Curie, **46**, Univ. Paris VI, Paris, (1981).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211 USA
E-mail address: `pete@math.missouri.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH CAROLINA, COLUMBIA, SC 29208 USA
E-mail address: `dilworth@math.sc.edu`

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TEXAS, 1 UNIVERSITY STATION C1200, AUSTIN, TX 78712 USA
E-mail address: `odell@math.utexas.edu`

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843, USA
E-mail address: `thomas.schlumprecht@math.tamu.edu`

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF NOTTINGHAM, UNIVERSITY PARK, NOTTINGHAM NG7 2RD, UNITED KINGDOM
E-mail address: `andras.zsak@maths.nottingham.ac.uk`