Problems in Functional Analysis (Math655)  
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Please attempt 6 out the following 12 problems.
From Book
page 86 ff: 3.63, 3.70, 3.83, 3.90
page 130ff: 4.4, 4.19

1. Show that the unit ball in $\ell_\infty^*$ is not sequentially compact in $w^*$, i.e. not every sequence has a converging subsequence.

2. Prove the non linear Hahn Banach Theorem:
   $X$ is a metric space $Y \subset X$ and $f : Y \to \mathbb{R}$ Lipschitz continuous.
   Then $f$ can be extended to a Lipschitz function $F : X \to Y$ having the same Lipschitz constant as $f$.

3. (*) Prove above non linear Hahn Banach Theorem without using the axiom of choice.

4. Let $X$ be a Banach space, we call a set $A \subset X$ limited in $X$ if every sequence $(x_n^*)$ in $X^*$ converging $w^*$ to 0 converges uniformly to 0 on $A$.
   Prove that relatively compact sets are limited and that if $X$ is separable every limited set is relatively compact.

5. (*) Find a Banach space $X$ which admits limited sets which are not relatively compact.

6. Let $X$ be a Banach space. Show that $(X^*, w^*)^* = (X, w)$.