Problems in Functional Analysis (Math655)
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Bookproblems: 4.11, 4.23, 5.7, 5.52, 5.55, 5.63

(1) Let $1 < p < q < \infty$.
 a) Show that for any $\varepsilon >$ every normalized weakly null sequence (x_n) in ℓ_p has a subsequence (y_n) which is $(1+\varepsilon)$-equivalent to the unit vectors in ℓ_p, i.e. for all $n \in \mathbb{N}$ and all $(a_i)_{i=1}^n \in \mathbb{K}$

$$\frac{1}{\varepsilon+1}\Big(\sum_{i=1}^n |a_i|^p\Big)^{1/p} \le \Big\|\sum_{i=1}^n a_i y_i\Big\| \le (1+\varepsilon)\Big(\sum_{i=1}^n |a_i|^p\Big)^{1/p}.$$

 b) Show that every bounded linear operator $T : \ell_q \to \ell_p$ is compact.

(2) Let X and Y be Banach spaces show that

$$WK(X,Y) := \big\{T \in B(X,Y) : T \text{ is weakly compact}\big\}$$

is a closed subspace of $B(X,Y)$.

(3) Let (X_n) be a sequence of Banach spaces. For $1 \le p \le \infty$ we put

$$(\oplus X_n)_{\ell_p} = \Big\{(x_n) : x_n \in X_n \text{ for } n \in \mathbb{N} \text{ and } \sum \|x_n\|^p < \infty\Big\}.$$

(where in the case of $p = \infty$ the sum is replaced by $\sup_{n\in\mathbb{N}} \|x_n\|$.) we also define

$$(\oplus X_n)_{c_0} = \Big\{(x_n) : x_n \in X_n \text{ for } n \in \mathbb{N} \text{ and } \lim_{n\to\infty} \|x_n\| = 0\Big\}.$$

Show that $(\oplus X_n)_{\ell_p}$ and $(\oplus X_n)_{c_0}$ (with the appropriate norm) are Banach spaces. Show that $(\oplus X_n)^*_{\ell_p}$, $1 \le p < \infty$ is isometrically isomorphic to $(\oplus X_n^*)_{\ell_q}$, where q is the adjoint of p, and that $(\oplus X_n)^*_{c_0}$ is isometrically isomorphic to $(\oplus X_n^*)_{\ell_1}$.

(4) Let A be a bounded subset of a Banach space X. Prove that A is relatively weakly compact if and only if for all $\varepsilon > 0$ there is a relatively weakly compact set $A_\varepsilon \subset X$ so that

$$A \subset \bigcup_{x\in A_\varepsilon} x + B_\varepsilon(X).$$

(5) (*) Let $\Delta = \{0,1\}$ endowed with the product topololgy (of the discrete topology). Show that every separable Banach space X is isometrically contained in $C(\Delta)$. Hint: show that there is a continuous surjection $f : \delta \to K$, where K is an arbitrary compact metric space.

(6) (*) We call $A \subset X$ *weakly conditionally compact*, if every sequence in A contains a w-Cauch subsequence. An operator $T : X \to Y$ is called *weakly conditionally compact*, if $T(B_X)$ is weakly conditionally compact.

Show that every weakly conditionally compact operator factors thorugh a space which does not contain ℓ_1.

Hint: Use *Rosenthal's* ℓ_1 *Theorem* (which will be shown later): Every bounded sequence in Banach space has a subsequence which is either equivalent to the ℓ_1 unit vector basis or weakly Cauchy.