

**Problems in Functional Analysis (Math655)**

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Bookproblems: 6.1,6.4,6.6,6.8,6.10, 6.12

- (1) Show that the summing basis  $(s_n)$  is a monotone basis for  $c_0$ . Recall

$$s_n = (\underbrace{1, 1, \dots, 1}_{n\text{-times}}, 0, 0, \dots) \text{ for } n \in \mathbb{N}.$$

- (2) Find a reordering of  $(s_n)$  which is not a basis.  
(3) Let  $\Delta = \{0, 1\}^{\mathbb{N}}$  endowed with the product of the discrete topology.  
For

$$s = (s_1, s_2, \dots, s_n) \in \bigcup_{j=0}^{\infty} \{0, 1\}^j$$

( $\{\emptyset\} = \{0, 1\}^0$ , thus  $\emptyset \in \bigcup_{j=0}^{\infty} \{0, 1\}^j$ ) and  $\sigma = (\sigma_i) \in \Delta$  we say that  $s$  is an initial segment of  $\sigma$  and write  $\sigma \succ s$  if  $\sigma_i = s_i$  for  $i = 1, 2, \dots, n$  ( $\sigma \succ \emptyset$  for all  $\sigma \in \Delta$ ).

For  $s \in \bigcup_{j=0}^{\infty} \{0, 1\}^j$  put

$$\Delta_s = \{\sigma \in \Delta : \sigma \succ s\}.$$

Show that there are sequences in the set

$$\{\chi_{\Delta_s} : s \in \bigcup_{j=0}^{\infty} \{0, 1\}^j\}$$

which are a basis of  $C(\Delta)$ .

- (4) Find a Schauder basis for the space

$$c_{00} = \{(x_i) : \{i : x_i \neq 0\} \text{ finite}\}$$

with the  $\ell_2$ -norm which is not a Schauder basis for  $\ell_2$ .

- (5) Let  $X$  be a separable Banach space. A sequence  $(x_j, f_j)_{j \in \mathbb{N}}$ , with  $(x_j)_{j \in \mathbb{N}} \subset X$ ,  $(f_j)_{j \in \mathbb{N}} \subset X^*$ , is called a (Schauder) frame of  $X$  if for every  $x \in X$

$$x = \sum_{j \in \mathbb{N}} f_j(x)x_j.$$

- a) If  $\inf_{i \in \mathbb{N}} \|x_i\| > 0$ , then  $f_i \xrightarrow{w^*} 0$  as  $i \rightarrow \infty$ .  
b) Prove that

$$K = \sup_{x \in B_X} \sup_{m \leq n} \left\| \sum_{i=m}^n f_i(x)x_i \right\| < \infty.$$

We call  $K$  the projection constant of  $(x_i, f_i)$ .

- c) For all  $f \in X^*$  and  $x \in X$  it follows that

$$f = w^* - \sum_{i=1}^{\infty} f(x_i)f_i.$$

(6) (\*) Let  $X$  be a separable Banach space and let  $(x_i)_{i \in \mathbb{N}} \subset X$  and  $(f_i)_{i \in \mathbb{N}} \subset X^*$ .  $(x_i, f_i)_{i \in \mathbb{N}}$  is a Schauder frame of  $X$  if and only if there is a Banach space  $Z$  with a Schauder basis  $(z_i)_{i \in \mathbb{N}}$  and corresponding coordinate functionals  $(z_i^*)$ , an isomorphic embedding  $T : X \rightarrow Z$  and a bounded linear surjective map  $S : Z \rightarrow X$ , so that  $S \circ T = Id_X$  (i.e.  $X$  is isomorphic to a complemented subspace of  $Z$ ), and  $S(z_i) = x_i$ , for  $i \in \mathbb{N}$ , and  $T^*(z_i^*) = f_i$ , for  $i \in \mathbb{N}$ , with  $x_i \neq 0$ .

Moreover  $S$  and  $T$  can be chosen so that  $\|S\| = 1$  and  $\|T\| \leq K$ , where  $K$  is the projection constant of  $(x_i, f_i)$ , and  $(z_i)$  can be chosen to be a monotone basis with  $\|z_i\| = \|x_i\|$  if  $i \in \mathbb{N}$ , with  $x_i \neq 0$ . Hint: assume first that  $x_1 \neq 0$  and  $f_i \neq 0$  for  $i \in \mathbb{N}$ .