Problems in Functional Analysis (Math655) Due: 11/8/2007

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Bookproblems: 6.9, 6.10,6.11, 6.12, 6.16, 6.17

Problem (2) includes Problem (1), it counts as two problems.

(1) Let X be an infinite dimensional Banach space and $q < 1$. Then there is a sequence $(x_n) \subset B_X$ with $\|x_n - x_m\| \geq q$, whenever $m \neq n$.

(2) (∗) Let X be an infinite dimensional Banach space. Then there is a sequence $(x_n) \subset B_X$ with $\|x_n - x_m\| > 1$, whenever $m \neq n$.

Even more is true:

Theorem.(Elton and Odell) For any infinite dimensional Banach space X there is a $q > 1$ so that B_X contains a sequence (x_n) with $\|x_n - x_m\| \geq q$, whenever $m \neq n$.

(3) Show that above Theorem of Elton and Odell is optimal, i.e. find for any $q > 1$ a Banach space X so that X does not contain a sequence (x_n) with $\|x_n - x_m\| \geq q$, whenever $m \neq n$.

(4) Show that $C([0, 1])$ has a basis consising of elements which are infinitely often differentiable.

(5) (*) Let X be an infinite dimesional subspaces of $L_p[0, 1]$, $2 < p < \infty$. Show that

either the L_2-norm and the L_p norm are equivalent on X, and then X is isomorphic to $L_2[0, 1]$,

or the L_2-norm and the L_p norm are not equivalent on X, and then X contains a space isomorphic to ℓ_p.

(6) Assume that (x_i) is a normalized sequence in a Banach space X which converges weakly to 0.

a) Find a subsequence (y_i) of (x_i) and a bounded sequence $(f_i) \in X^*$ so that $f_i(x_j) = \delta(i, j)$ for $i, j \in \mathbb{N}$.

b) For given $\varepsilon < 1$ find a subsequence (y_i) of (x_i) which is basic with constant $\leq 1 + \varepsilon$.