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Since this sheet is due in three weeks, please do at least nine problems.

Bookproblems: 6.18, 6.19, 6.32, 6.33, 6.34, 6.36

- (1) Let X be a Banach space and show that for $(x_i) \subset X$ the following are equivalent:

a) There is an $x \in X$ so that

$$\forall \varepsilon > 0 \exists F \subset \mathbb{N} \text{ finite } \forall F' \subset \mathbb{N} \text{ finite, } F \subset F' \quad \left\| x - \sum_{i \in F'} x_i \right\| < \varepsilon.$$

b) There is an $x \in X$ so that for any bijections $\pi : \mathbb{N} \rightarrow \mathbb{N}$,

$$x = \lim_{n \rightarrow \infty} \sum_{i=1}^n x_i.$$

c) For all sequences $(\sigma_i) \subset \{0, 1\}$, $\sum \sigma_i x_i$ converges.

d) For all sequences $(\sigma_i) \subset \{-1, 1\}$, $\sum \sigma_i x_i$ converges.

e) For all sequences $(\sigma_i) \subset [-1, 1]$, $\sum \sigma_i x_i$ converges.

- (2) Prove that a basic normalized sequence (x_i) in a Banach space is equivalent to the unit vector basis for c_0 if and only if:

$$\sup_{F \subset \mathbb{N} \text{ finite}} \left\| \sum_{i \in F} x_i \right\| < \infty.$$

- (3) Prove that every basic normalized sequence (x_i) in a Banach space has a subsequence (y_i) such that either (y_i) is equivalent to the unit vector basis for c_0 or

$$\sup_{n \in \mathbb{N}} \left\| \sum_{i=1}^n y_{k_i} \right\| = \infty \quad \forall (k_i) \in [\mathbb{N}]^\omega.$$

- (4) Give an example of a basic sequence (x_i) such that (x_i) is not equivalent to the unit vector basis of c_0 , but every subsequence of (x_i) has a further subsequence equivalent to the unit vector basis of c_0 .

- (5) Proof *Dvoretzki's Theorem* for ℓ_1 .

a) For every finite dimensional subspace F of ℓ_1 and any $\varepsilon > 0$ there is a finite dimensional subspace E of L_1 which contains F and is $(1 + \varepsilon)$ -isomorphic to $\ell_1^{\dim(E)}$

b) Prove that there is a constant $c > 0$ so that for any $n \in \mathbb{N}$ there is a finite dimensional subspace of ℓ_1 which is c -equivalent to ℓ_2^n . (Dvoretzki's Theorem says that ℓ_1 can be replaced by any infinite dimensional Banach space and c by a number arbitrary close to 1)

- (6) (*) Find a unconditional but not absolutely converging series in ℓ_1 (Hint: previous problem).