Distortion and Asymptotic Structure

Edward Odell* Th. Schlumprecht*

The distortion problem for Hilbert space ℓ_2 may be stated as follows. Do there exist sets $A, B \subseteq S_{\ell_2}$ with $\inf\{\|a-b\| : a \in A, b \in B\} > 0$ and such that $A \cap X \neq \emptyset$, $B \cap X \neq \emptyset$ for all infinite dimensional closed subspaces $X \subseteq \ell_2$? We shall see that the answer is yes. But how does one choose such sets? What criteria can be used when sorting through the elements of S_{ℓ_2} to determine which vectors go into A or B? Any $x \in S_{\ell_2}$ can be the first element of an orthonormal basis for ℓ_2. S_{ℓ_2} viewed by standing at a point x looks no different if you move to point y. The approach we use to distort ℓ_2 is indirect. It seems impossible to distort Hilbert space by only working within the category of Hilbert space. We need to expand to the category of Banach spaces and to use some deep analysis of the structure of a certain recently discovered Banach space which has come to be called S [S1]. We then will infer the distortion of ℓ_2 via a certain non linear transfer.

The Banach space S is a descendent of Tsirelson's famous space T. The key to the marvelous properties of T (and S) is that the norm is defined implicitly as opposed to explicitly. One states a certain norm equation and shows that a solution exists. As concerns the distortion of ℓ_2 one is ultimately led to the discovery of a remarkable collection of sets $A_n \subseteq S_{\ell_2}$ which are "large" ($\inf\{\|x-a\| : a \in A,\ x \in X\} = 0$ for all infinite dimensional subspaces $X \subseteq \ell_2$) and nearly biorthogonal ($|\langle a_n, a_m \rangle| < \varepsilon_{\min(n,m)}$ for $a_n \in A_n$, $a_m \in A_m$, $m \neq n$ for some $\varepsilon_i \downarrow 0$).

In Section I we focus on the distortion problem and explain how distortion enters into Banach space theory. The distortion problem for ℓ_2 is equivalent to the following. If $f : S_{\ell_2} \to \mathbb{R}$ is Lipschitz, is f nearly constant (up to an arbitrary $\varepsilon > 0$) on some infinite dimensional subspace of any given infinite dimensional subspace of ℓ_2? We discuss both the distortion problem and the Lipschitz function stabilization problem in the broader context of general Banach spaces.

Section 2 concerns asymptotic structure. This is a notion which lies between the finite and infinite dimensional theory.

1 Distortion in Banach spaces

The distortion problem in Banach spaces arose from work of R.C. James [J1] and V.D. Milman [M] in the 1960's. James proved that every isomorph of ℓ_1 (respectively, c_0) contains a subspace almost isometric to ℓ_1 (respectively, c_0). Equivalently (see the definition below) ℓ_1 and c_0 are not

---

[*] Research supported by NSF.

distortable. Milman showed that if has X no distortable subspace then X contains either an almost isometric copy of c_0 or ℓ_p for some $1 \le p < \infty$ and asked if a distortable space could exist. A few years later B.S. Tsirelson [T] produced his famous space T which does not contain any isomorph of c_0 or ℓ_p $(1 \le p < \infty)$. Hence there exists a distortable Banach space. This left what came to be called "the distortion problem": is ℓ_p distortable for $1 < p < \infty$?

$X, Y, Z, \ldots$ shall denote separable infinite dimensional real Banach spaces. $X \subseteq Y$ shall mean that X is a closed linear subspace of Y. $F, G, \ldots$ shall denote finite dimensional real Banach spaces. S_X denotes the unit sphere of X and B_X is the closed unit ball of X.

We begin with the result of James cited above, giving the proof in the ℓ_1 case. Recall that X contains *almost isometric copies of* ℓ_1 if for all $\varepsilon > 0$ there exists $Y \subseteq X$ with $d(Y, \ell_1) < 1 + \varepsilon$. The idea behind the proof is to choose a block basis (x_i) of the unit vector basis for ℓ_1 which gives essentially the worst possible equivalence between the ℓ_1 norm w.r.t. (x_i) and the equivalent norm and then note that this block basis itself must be nearly isometric to the unit vector basis of ℓ_1.

Theorem 1.1 [J1] *If X is isomorphic to ℓ_1 (respectively, c_0) then X contains almost isometric copies of ℓ_1 (respectively, c_0).*

Proof. Let $|||\cdot|||$ be an equivalent norm on ℓ_1 and let (e_i) be the unit vector basis for ℓ_1. Let $\varepsilon > 0$. It suffices to prove that there exists a $|||\cdot|||$-normalized block basis (x_i) of (e_i) with $|||\sum a_i x_i||| \ge (1-\varepsilon)\sum |a_i|$ for all $(a_i) \in c_{00}$. Let (y_i) be any $|||\cdot|||$-normalized block basis of (e_i). For $n \in \mathbb{N}$ let

$$c_n := \inf\left\{ \frac{|||\sum_{i=n}^{\infty} a_i y_i|||}{\sum_{i=n}^{\infty} |a_i|} \quad : \quad 0 \ne (a_i) \in c_{00} \right\}.$$

Clearly $c_n \nearrow c$ for some $c > 0$. Choose δ small enough and $n_0 \in \mathbb{N}$ large enough so that $\frac{c_{n_0}}{c+\delta} > 1-\varepsilon$, and choose for $i \in \mathbb{N}$, $x_i = \sum_{j \in F_i} b_j y_j$ such that $n_0 \le F_1 < F_2 < \ldots$, $|||x_i||| = 1$, and $\sum_{j\in F_i} |b_j| > \frac{1}{c+\delta}$.

Thus we conclude for any $(a_i) \in c_{00}$ that

$$|||\sum_{i=1}^{\infty} a_i x_i||| = |||\sum_{i=1}^{\infty} a_i \sum_{j\in F_i} b_j y_j||| \ge c_{n_0} \sum_{i=1}^{\infty} |a_i| \sum_{j \in F_i} |b_j| \ge \frac{c_{n_0}}{c+\delta} \sum_{i=1}^{\infty} |a_i| \ge (1-\varepsilon) \sum |a_i| \ ,$$

which implies the claim and finishes the proof. $\diamondsuit$

The same method can be used to produce, for any equivalent norm on ℓ_p $(1 < p < \infty)$ or c_0, a normalized block basis with a very tight upper ℓ_p estimate or tight lower ℓ_p estimate but not both simultaneously. The triangle inequality in ℓ_1 gives us the upper estimate for free. If we produce a tight lower c_0 estimate in an isomorph of c_0 then a tight upper estimate automatically ensues (see e.g., [LT]) which is how Theorem 1.1 is proved for c_0. Thus ℓ_1 and c_0 are not distortable by the nature of their extreme positions (the largest and smallest norm) amongst Banach spaces. There is also the following finite dimensional version of James' blocking argument. It shows how one can improve the ℓ_1-constant of a finite basic sequence if one is willing to decrease its length.

Proposition 1.2 *Let $N, k \in \mathbb{N}$, $C > 0$ and assume that $x_1, x_2, \ldots, x_{N^k}$ is a normalized basic sequence for which*

$$\|\sum_{i=1}^{N^k} a_i x_i\| \geq C \sum_{i=1}^{N^k} |a_i|, \quad \textit{for all } a_1, a_2, \ldots, a_{N^k} \in \mathbb{R} \ .$$

Then there is a normalized block basis $(y_i)_{i=1}^N$ of (x_i) so that

$$\|\sum_{i=1}^{N} a_i y_i\| \geq C^{1/k} \sum_{i=1}^{N} |a_i|, \quad \textit{for all } a_1, a_2, \ldots, a_N \in \mathbb{R} \ .$$

Proof. We will prove the proposition by induction on $k \in \mathbb{N}$. For $k = 1$ there is nothing to proof. Assuming the claim is true for $k - 1$, and given that $(x_i)_{i=1}^{N^k}$ satisfies the assumption we are in one of the following two situations.

It may be that for all $j = 1, \ldots, N$

$$C_j = \min\{\| \sum_{i=1}^{N^{k-1}} a_i x_{i+(j-1)N^{k-1}} \| : \sum_{i=1}^{N^{k-1}} |a_i| = 1\} \leq C^{1-\frac{1}{k}} \ .$$

In this case we choose $y_j = \sum_{i=1}^{N^{k-1}} a_i^{(j)} x_{i+(j-1)N^{k-1}}$ with $\|y_j\| = 1$ and $\sum_{i=1}^{N^{k-1}} |a_i^{(j)}| \geq C^{\frac{1}{k}-1}$ and deduce from the assumptions that for $a_1, \ldots, a_n \in \mathbb{R}$

$$\|\sum_{j=1}^{N} a_j y_j\| \geq C \sum_{j=1}^{N} |a_j| \sum_{i=1}^{N^{k-1}} |a_i^{(j)}| \geq C C^{\frac{1}{k}-1} \sum_{j=1}^{N} |a_j| = C^{\frac{1}{k}} \sum_{j=1}^{N} |a_j| \ ,$$

which completes the proof.

Otherwise there is a $j \in \{1, 2, \ldots N\}$ with $C_j \geq C^{1-\frac{1}{k}}$, and we can apply the induction hypothesis to $(x_{i+(j-1)N^{k-1}})_{i=1}^{N^{k-1}}$ and $\widetilde{C} = C^{1-\frac{1}{k}}$ in order to get a normalized block $(y_i)_{i=1}^N$ for which

$$\|\sum_{i=1}^{N} a_i y_i\| \geq \left(C^{1-\frac{1}{k}}\right)^{\frac{1}{k-1}} \sum_{i=1}^{N} |a_i| = C^{\frac{1}{k}} \sum_{i=1}^{N} |a_i| \ .$$

◊

Definition 1.3 Let $\lambda > 1$. X is λ-*distortable* if there exists an equivalent norm $|\cdot|$ on X so that for all $Y \subseteq X$,

$$\sup\left\{\frac{|y_1|}{|y_2|} : y_1, y_2 \in S_{(Y,\|\cdot\|)}\right\} \geq \lambda \ .$$

X is *distortable* if X is λ-distortable for some $\lambda > 1$. X is *arbitrarily distortable* if X is λ-distortable for all $\lambda > 1$.

Definition 1.4 Let $f : S_X \to \mathbb{R}$. f *stabilizes* if for all $Y \subseteq X$ and $\varepsilon > 0$ there exists $Z \subseteq Y$ so that

$$\mathrm{osc}(f, S_Z) \equiv \sup\{f(z_1) - f(z_2) : z_1, z_2 \in S_Z\} < \varepsilon\ .$$

Clearly X does not contain a distortable subspace iff every equivalent norm on X stabilizes.

One can enlarge the question as to whether a given X contains a distortable subspace to whether every reasonable (e.g., Lipschitz or uniformly continuous, it makes no difference) $f : S_X \to \mathbb{R}$ stabilizes. Some insight to the connection between these questions is provided by the following simple proposition. A set $A \subseteq S_X$ is *asymptotic* if for all $Y \subseteq X$, $Y \cap A \neq \emptyset$. A is *nearly asymptotic* (or *large*) in X if $d(A, Y) = 0$ for all $Y \subseteq X$. Sets $A, B \subseteq S_X$ are *separated* if the minimum distance between them is positive, i.e. if $md(A, B) \equiv \inf\{\|a - b\| : a \in A\, ,\ b \in B\} > 0$.

Proposition 1.5 **a)** *There exists a Lipschitz* $f : S_X \to \mathbb{R}$ *which does not stabilize iff there exist* $Y \subseteq X$ *and separated sets* $A, B \subseteq S_Y$ *which are (nearly) asymptotic in* Y.

b) *If* X *is uniformly convex then* X *contains a distortable subspace iff there exists a Lipschitz* $f : S_X \to \mathbb{R}$ *which does not stabilize.*

c) *Let* $1 < p < \infty$. *Then* ℓ_p *is distortable iff there exist separated asymptotic sets in* S_{ℓ_p}.

Sketch of Proof. a) If A, B are separated asymptotic (or nearly asymptotic) for Y then $f(x) \equiv \inf\{\|x - a\| : a \in A\}$ is Lipschitz but does not stabilize. Conversely if f is Lipschitz and does not stabilize then there exist $Y \subseteq X$ and $c < d$ so that $\{x \in S_Y : f(x) < c\}$ and $\{x \in S_Y : f(x) > d\}$ are separated and asymptotic in Y.

b) If $Y \subseteq X$ is distorted by $|\cdot|$ then $|\cdot|$ can be extended to an equivalent norm on X and $|\cdot| : S_X \to \mathbb{R}$ is Lipschitz but does not stabilize. (This implication does not require X to be uniformly convex.) If A and B are separated asymptotic sets for $Y \subseteq X$ then $\bar{c}_0(A \cup -A)$ is the unit ball for a norm $|\cdot|$ on some $Z \subseteq Y$ and $|\cdot|$ distorts $(Z, \|\cdot\|)$.

c) This follows from a), b) and the fact that every subspace of ℓ_p contains almost isometric copies of ℓ_p. ♢

We have seen that in a special case (X isomorphic to c_0 or ℓ_1) all equivalent norms stabilize. Finite dimensionally things work out nicely; there are good stabilization results. We state two such theorems. The first was observed by V. Milman (see [MS], p.6) in connection with Dvoretsky's famous theorem that one finds almost isometric copies of ℓ_2^n for all n in any X.

Theorem 1.6 (First Stabilization principle) *For all* $C > 0$, $\varepsilon > 0$ *and* $k \in \mathbb{N}$ *there exists* $n = n(C, \varepsilon, k)$ *so that if* $\dim E = n$ *and* $f : S_E \to \mathbb{R}$ *is* C*-Lipschitz* ($|f(x) - f(y)| \leq C\|x - y\|$) *then there exists* $F \subseteq E$, $\dim F = k$ *with* $\mathrm{osc}(f, S_F) < \varepsilon$.

The second principle is a reworking of the first in the setting of finite dimensional spaces with bases and subspaces spanned by block bases.

Theorem 1.7 (Second Stabilization principle) *Given* $C > 0$, $\varepsilon > 0$ *and* $k \in \mathbb{N}$ *there exists* $n = n(C, \varepsilon, k) \in \mathbb{N}$ *so that if* $\dim E = n$ *and* E *has a basis* $(x_i)_1^n$ *whose basis constant does not exceed* C *and* $f : S_E \to \mathbb{R}$ *is* C*-Lipschitz then there is a normalized block basis* $(y_i)_1^k$ *of* $(x_i)_1^k$ *so that* $\mathrm{osc}(f, S_{\langle y_i \rangle_1^k}) < \varepsilon$.

The proof of the second stabilization principle is given in [ORS]. It relies mostly on Lemberg's [L] proof of Krivine's theorem. The exception is the case where $F = \ell_\infty^n$ (see [ORS] for a proof in this case).

Theorem 1.8 (Krivine's theorem) [K] *Given* $C > 0$, $\varepsilon > 0$ *and* $k \in \mathbb{N}$ *there exists* $n = n(C, \varepsilon, k) \in \mathbb{N}$ *so that if* $(e_i)_{i=1}^n$ *is* C*-basic then there exist* $p \in [1, \infty]$ *and a block basis* $(x_i)_1^k$ *of* $(e_i)_1^n$ *which is* $1 + \varepsilon$*-equivalent to the unit vector basis of* ℓ_p^k.

Next we present a proof of Milman's theorem [HORS].

Theorem 1.9 *If* X *does not contain almost isometric copies of* c_0 *or* ℓ_p *for some* $1 \le p < \infty$ *then* X *contains a distortable subspace.*

First we set some notation. For $x \in X$ the *type* on X generated by x is the function $\tau_x : X \to \mathbb{R}$ given by $\tau_x(y) = \|x + y\|$. The *norm on* X *generated by* x is the function $\| \cdot \|_x$ given by

$$\|y\|_x = \frac{1}{2}(\tau_{\|y\|x}(y) + \tau_{\|y\|x}(-y)) = \frac{1}{2}\left[\Big\| \|y\|x + y \Big\| + \Big\| \|y\|x - y \Big\|\right] .$$

It is not difficult to see that $\| \cdot \|_x$ is an equivalent norm on X [OS1].

A normalized block basis $(x_i) \subseteq X$ is said to *doubly generate an* ℓ_p *type over* X [R2] if for all $b \in X$ and $(\alpha, \beta) \in S_{\ell_p^2}$,

$$\lim_{i\to\infty} \lim_{j\to\infty} \|b + \alpha x_i + \beta x_j\| = \lim_{i\to\infty} \|b + x_i\| .$$

It is easy to show that if (x_i) doubly generates an ℓ_p type over X then for all $\varepsilon > 0$ there exists a subsequence of (x_i) which is $1 + \varepsilon$-equivalent to the unit vector basis of ℓ_p. A similar definition and result can be made for ℓ_p replaced by c_0.

Proof of Theorem 1.9. We shall prove that if $\| \cdot \|_x$ stabilizes on X for all $x \in X$ then there exists a sequence (x_i) in X which doubly generates an ℓ_p type over X for some $1 \le p < \infty$ or which doubly generates a c_0 type over X. We break the proof into three steps. Let (y_i) be a normalized basic sequence in X.

A normalized basic sequence (x_i) in X generates a *spreading model* $E = [(e_i)]$ if there exist $\varepsilon_n \downarrow 0$ so that for all n, $(\alpha_i)_1^n \subseteq [-1,1]^n$, $n \le k_1 < \cdots , k_n$, $n \le j_1 < \cdots < j_n$

$$\left| \Big\| \sum_1^n \alpha_i x_{k_i} \Big\| - \Big\| \sum_{i=1}^n \alpha_i x_{j_i} \Big\| \right| < \varepsilon_n \ .$$

(x_i) generates a *spreading model* $X \oplus E$ *over* X if in addition for all $x \in B_{\langle x_i \rangle_1^{n-1}}$

$$\left| \Big\| x + \sum_{i=1}^n \alpha_i x_{k_i} \Big\| - \Big\| x + \sum_{i=1}^n \alpha_i x_{j_i} \Big\| \right| < \varepsilon_n \ .$$

Under these circumstances we may define

$$\Big\| \sum_{i=1}^n \alpha_i e_i \Big\| = \lim_{k_1 \to \infty} \cdots \lim_{k_n \to \infty} \Big\| \sum_{i=1}^n \alpha_i x_{k_i} \Big\|$$

or

$$\Big\| x + \sum_{i=1}^n \alpha_i e_i \Big\| = \lim_{k_1 \to \infty} \cdots \lim_{k_n \to \infty} \Big\| x + \sum_{i=1}^n \alpha_i x_{k_i} \Big\|$$

$(e_i)_{i=1}^\infty$ then becomes a normalized monotone basis for $E \equiv [(e_i)]$. Every normalized basic sequence can be seen via Ramsey theory to yield a subsequence which generates a spreading model over X [BL]. Rosenthal [R1], cleverly using the Borsuk-Ulam theorem, showed that every basic sequence in X admits a normalized block basis (x_i) which generates a spreading model E which is 1-*unconditional over* X. This means that $\|x + e\| = \|x - e\|$ for all $x \in X$, $e \in E$.

Step 1. There exists a normalized block basis (z_i) of (y_i) which generates a 1-unconditional spreading model $E = [(e_i)]$ over X so that

$$\|x + e\| = \|x + e'\| \text{ for all } x \in X,\ e, e' \in E \text{ with } \|e\| = \|e'\| \ . \tag{1}$$

Note that a consequence of this is that for all $e \in [e_i]_{i=2}^\infty$ and $\alpha \in \mathbb{R}$

$$\|x + \alpha e_1 + e\| = \lim_{n \to \infty} \|x + \alpha z_n + e\| = \lim_{n \to \infty} \|x + \alpha z_n + \|e\| e_2\| = \|x + \alpha e_1 + \|e\| e_2\|. \tag{2}$$

More generally it follows that for any $m \in \mathbb{N}$ and $a \in \langle e_i \rangle_{i=1}^m$, $e, e' \in [e_i]_{i=m+1}^\infty$ with $\|e\| = \|e'\|$ we have

$$\|x + a + e\| = \|x + a + e'\| \ . \tag{3}$$

Proof of Step 1. Let $\varepsilon_n \downarrow 0$ and let $(d_i)_1^\infty \subseteq X$ be dense in X. Since $\| \cdot \|_{d_i}$ stabilizes for all i we can recursively choose for each $n \in \mathbb{N}$ a normalized block basis $(y_i^{(n)})$ of (y_i) so that

a) $(y_i^{(n+1)})$ is a block basis of $(y_i^{(n)})$

b) $|\,\|y\|_{d_i} - \|z\|_{d_i}| < \varepsilon_n$ if $y, z \in S_{[y_j^{(n)}]_{j=1}^\infty}$ and $i \le n$.

Set $z_n = y_n^{(n)}$. Then for all $i \le n \in \mathbb{N}$

$$\Big|\,\|y\|_{d_i} - \|z\|_{d_i}\Big| < \varepsilon_n \quad \text{whenever } y, z \in S_{[z_i]_{i=n}^\infty}.$$

In particular since (d_j) is dense in X for all normalized block bases (w_j) of (z_j) and $x \in X$, $\lim_j \|w_j\|_x$ exists and the limit depends solely upon x (not the particular sequence (w_j)). Using [R1] we may assume in addition that (z_j) generates a 1-unconditional spreading model $E = [(e_i)]$ over X.

Let (w_j) be any normalized block basis of (z_j). Then for $x \in X$

$$\|x + e_1\| = \lim_{j\to\infty} \tau_x(z_j) = \lim_{j\to\infty} \frac{1}{2}\Big(\tau_x(z_j) + \tau_x(-z_j)\Big) = \lim_j \|z_j\|_x = \lim_{j\to\infty} \|w_j\|_x = \lim_{j\to\infty} \|x + w_j\| \ .$$

and (1) follows.

Step 2. There is a subsequence (w_j) of (z_j) so that for $k, m \in \mathbb{N}$, $b \in mB_{\langle d_i, w_i\rangle_1^m}$, $m < n_1 < .. < n_k$

$$\Big|\Big\|b + \sum_{i=1}^k \alpha_i w_{n_i}\Big\| - \Big\|b + \sum_{i=1}^k \alpha_i e_i\Big\|\Big| < \varepsilon_m \text{ if } \|\sum_1^k \alpha_i e_i\| \le 1 \ . \tag{4}$$

Note that if k were replaced by m this would merely become the fact that $E = [(e_i)]$ is a spreading model of (w_j) over X. Taking $b = 0$ we see that (w_i) and (e_i) are equivalent.

(4) follows easily by a diagonal argument from the following claim. Given $x \in X$ and $\varepsilon > 0$ there exists a subsequence $(\tilde z_j)$ of (z_j) so that

$$\Big|\Big\|x + \sum_{j=1}^k \alpha_j \tilde z_j\Big\| - \Big\|x + \sum_{j=1}^k \alpha_j e_j\Big\|\Big| < \varepsilon$$

whenever $\|\sum_{j=1}^k \alpha_j e_j\| \le 1$. Moreover this holds for all subsequences of $(\tilde z_j)$.

Given $x \in X$ and $\varepsilon > 0$, we inductively choose $(\tilde z_j) \subseteq (z_j)$ so that

$$\Big|\Big\|x + \sum_{j=1}^i \alpha_j \tilde z_j + \beta \tilde z_{i+1} + \gamma e_2\Big\| - \Big\|x + \sum_{j=1}^i \alpha_j \tilde z_j + \beta e_1 + \gamma e_2\Big\|\Big| < \varepsilon 2^{-(i+1)}$$

for all $(\alpha_j)_1^i \subseteq [-2, 2]$, $\gamma, \beta \in [-2, 2]$ and $i \in \mathbb{N}$. Thus if $\|\sum_{i=1}^k \alpha_i e_i\| \le 1$ (which implies that $|\alpha_i| \le 2$ for $i \le k$)

$$\begin{aligned}
&\Big|\Big\|x + \sum_{j=1}^k \alpha_j \tilde z_j\Big\| - \Big\|x + \sum_{j=1}^k \alpha_j e_j\Big\|\Big| \\
&\le \sum_{i=1}^k \Big|\Big\|x + \sum_{j=1}^i \alpha_j \tilde z_j + \sum_{j=i+1}^k \alpha_j e_j\Big\| - \Big\|x + \sum_{j=1}^{i-1} \alpha_j \tilde z_j + \sum_{j=i}^k \alpha_j e_j\Big\|\Big| \\
&= \sum_{i=1}^k \Big|\Big\|x + \sum_{j=1}^{i-1} \alpha_j \tilde z_j + \alpha_i \tilde z_i + \Big\|\sum_{j=i+1}^k \alpha_j e_j\Big\| e_2\Big\| - \Big\|x + \sum_{j=1}^{i-1} \alpha_j \tilde z_j + \alpha_i e_1 + \Big\|\sum_{j=i+1}^k \alpha_j e_j\Big\| e_2\Big\|\Big| \\
&\le \sum_{i=1}^k 2^{-i}\varepsilon < \varepsilon \ .
\end{aligned}$$

The equality in the calculation comes via (1) and (3). The same estimates remain valid for any subsequence of $(\tilde{z}_j)$. This completes step 2.

The next step combined with step 2 completes the proof.

Step 3. There exists a normalized block basis (a_i) of (e_i) which doubly generates a c_0 type or an ℓ_p type for some $1 \le p < \infty$.

Given $\delta > 0$, a finite set $S \subseteq X \oplus E$ and $k \in \mathbb{N}$ by Theorem 1.7 there exists a normalized block basis $(c_i)_1^k$ of (e_i) with $|\, \|x+c\| - \|x+c'\| \,| < \delta$ for $c, c' \in S_{\langle c_i \rangle_1^k}$ and $x \in S$. By Theorem 1.8 if k is sufficiently large, there exists $p \in [1,\infty]$ and a normalized block basis (a,b) of $(c_i)_1^k$ with

$$\Big| \|\alpha a + \beta b\| - (|\alpha|^p + |\beta|^p)^{1/p} \Big| < \delta \ \text{ if } \ |\alpha|, |\beta| \le 1 \ .$$

Using this inductively, and passing to a convergent subsequence in $[1,\infty]$ of the p's thus produced we obtain a normalized block basis $(a_1, b_1, a_2, b_2, \ldots)$ of (e_i) and $p \in [1,\infty]$ so that

$$\Big| \|x + \alpha a_m + \beta b_m\| - \|x + (|\alpha|^p + |\beta|^p)^{1/p} b_m\| \Big| < 2^{-m} \tag{5}$$

if $m \in \mathbb{N}$, $x \in mB_{\langle (d_i)_1^{m-1} \cup (a_i)_1^{m-1} \cup (b_i)_1^{m-1} \rangle}$ and $|\alpha|, |\beta| \le 1$. As usual we let $(|\alpha|^p + |\beta|^p)^{1/p} = \max(|\alpha|, |\beta|)$ if $p = \infty$.

Let $m \in \mathbb{N}$, $x \in mB_{\langle (d_i)_1^{m-1}, (a_i)_1^{m-1}, (b_i)_1^{m-1} \rangle}$, $(\alpha, \beta) \in S_{\ell_p^2}$ and $m < n_1 < n_2$. Then by (3)

$$\|x + \alpha a_{n_1} + \beta a_{n_2}\| = \|x + \alpha a_{n_1} + \beta b_{n_1}\| \ .$$

By virtue of (5) this yields that (a_i) doubly generates an ℓ_p type (c_0 type if $p = \infty$) over X. $\diamondsuit$

Tsirelson's space T was the first example of a Banach space that did not contain an isomorph of c_0 or ℓ_p for $1 \le p < \infty$. We present the description of T (actually the dual space of the example in [T]) due to Figiel and Johnson [FJ]. By virtue of Theorem 1.9, T must contain a distortable subspace. In fact T is itself distortable and we sketch this argument below. First we set some notation.

Let (e_i) be the unit vector basis of c_{00}. For $E, F \subseteq \mathbb{N}$, "$E < F$" if E or F is empty or if $\max E < \min F$. If $x, y \in c_{00}$, "$x < y$" if $\operatorname{supp} x < \operatorname{supp} y$ as subsets of $\mathbb{N}$, $(E_i)_1^n$, subsets of $\mathbb{N}$, are *admissible* if $\{n\} \le E_1 < \cdots < E_n$. For $E \subseteq \mathbb{N}$, $x \in c_{00}$ we let $Ex(i) = 0$ if $i \notin E$ and $x(i)$ otherwise. We shall choose a certain norm $\|\cdot\|$ on c_{00} and let T be the completion of c_{00} under this norm.

Lemma 1.10 *There exists a norm $\|\cdot\|$ on c_{00} satisfying*

$$\|x\| = \max\left(\|x\|_\infty, \sup\Big\{ \frac{1}{2} \sum_{i=1}^n \|E_i x\| : n \in \mathbb{N} \text{ and } (E_i)_1^n \text{ is admissible} \Big\} \right) \ . \tag{$*$}$$

This is the norm that defines T. It is easy to check that $(*)$ holds for all $x \in T$ if "max" is replaced by "sup".

T was the first "nonclassical" Banach space. Previously norms had been defined by more explicit formulas. The formula in $(*)$ is implicit. The norm is the solution of $(*)$ which of course must be shown to exist. Since [T] many variations and extensions of Tsirelson's space have been constructed. Their norms can be described by an implicit equation similar to $(*)$ and their existence follows from the following general principle. We let $\mathcal{N}$ be the class of all norms $\|\cdot\|$ on c_{00} for which (e_i) is a normalized monotone basis for $\overline{(c_{00}, \|\cdot\|)}$ satisfying $\|\sum a_i e_i\| \geq \max_i |a_i|$ for all $(a_i) \in c_{00}$. If $\|\cdot\|, |\cdot| \in \mathcal{N}$ we write $\|\cdot\| \geq |\cdot|$ if $\|x\| \geq |x|$ for all $x \in c_{00}$.

Proposition 1.11 *Let $P : \mathcal{N} \to \mathcal{N}$ satisfy $P\|\cdot\| \leq P|\cdot|$ whenever $\|\cdot\| \leq |\cdot|$. Then P admits a smallest fixed point.*

The proposition is proved in [OS2] via transfinite induction. Lemma 1.10 follows by taking $(P\|\cdot\|)(x) = \max(\|x\|_\infty, \sup\{\frac{1}{2}\sum_{i=1}^n \|E_i x\| : n \in \mathbb{N},\ (E_i)_1^n \text{ is admissible}\})$. Recall that if X has a basis (e_i) then X is *asymptotic* ℓ_1 (w.r.t. (e_i)) if there exists $\lambda > 0$ so that $\|\sum_1^n x_i\| \geq \lambda \sum_1^n \|x_i\|$ whenever $(x_i)_1^n$ is *admissible* (i.e., $(\operatorname{supp} x_i)_1^n$ is admissible).

Theorem 1.12 *T is a reflexive Banach space having a normalized 1-unconditional basis (e_i). T is asymptotic ℓ_1 (w.r.t. (e_i)) and is $2-\varepsilon$ distortable for all $\varepsilon > 0$. T does not contain any subspace isomorphic to c_0 or ℓ_p $(1 \leq p < \infty)$.*

Sketch of Proof. From $(*)$ it follows easily that (e_i) is normalized and a 1-unconditional basis for T. Also T is asymptotic ℓ_1 (with $\lambda = 1/2$). Hence it is easy to see that the only possible c_0 or ℓ_p which T could contain is ℓ_1. Once we show that T is distortable then by Theorem 1.1 it will follow that T cannot contain ℓ_1. Thus by James' theorem that a nonreflexive space with an unconditional basis must contain an isomorph of c_0 or ℓ_1 it follows that T is reflexive.

Let $\varepsilon > 0$. For $n \in \mathbb{N}$ and $x \in T$ set $\|x\|_n = \sup\{\frac{1}{2}\sum_{i=1}^n \|E_i x\| : E_1 < \cdots < E_n\}$. Let (y_i) be any block basis of (e_i). We shall show that for n sufficiently large there exist $y, z \in S_Y$ with $\|y\|_n < \frac{1}{2} + \varepsilon$ and $\|z\|_n > 1 - \varepsilon$. This results yield that T is $2-\varepsilon$ distortable for all positive ε.

From Krivine's theorem (or by Proposition 1.2) we have that for any $m \in \mathbb{N}$ there exists a normalized block basis $(x_i)_1^m$ of $(y_i)_m^\infty$ which is $1+\bar\varepsilon$ equivalent to the unit vector basis of ℓ_1^m ($\bar\varepsilon = \bar\varepsilon(\varepsilon)$ to be specified). Let $\frac{n}{m} < \bar\varepsilon$ and set $x = \frac{1}{m}\sum_1^m x_i$; x is called an ℓ_1^m*-average* with constant $1+\bar\varepsilon$. Clearly $1 \geq \|x\| \geq \frac{1}{1+\bar\varepsilon}$. Choose $E_1 < \cdots < E_n$ so that $\|x\|_n = \frac{1}{2}\sum_{j=1}^n \|E_j x\|$. Set $F = \{i \leq m : |\{j : E_j x_i \neq 0\}| > 1\}$. Then $|F| < n$ and so

$$\begin{aligned}\frac{1}{2}\sum_{j=1}^n \|E_j x\| &\leq \frac{1}{2}\sum_{j=1}^n \Big(\Big\|E_j\Big(\frac{1}{m}\sum_{i\in F} x_i\Big)\Big\| + \Big\|E_j\Big(\frac{1}{m}\sum_{i\notin F} x_i\Big)\Big\|\Big)\\ &\leq \frac{1}{m}\Big\|\sum_{i\in F} x_i\Big\| + \frac{1}{2}\,\frac{1}{m}\sum_{i\notin F}\|x_i\| < \frac{n}{m} + \frac{1}{2} < \frac{1}{2} + \bar\varepsilon\ .\end{aligned}$$

Let $y = \frac{x}{\|x\|}$. Then $y \in S_Y$ and $\|y\|_n < (\frac{1}{2} + \bar{\varepsilon})(1 + \bar{\varepsilon}) < \frac{1}{2} + \varepsilon$ for $\bar{\varepsilon}$ sufficiently small.

Next let $x = \frac{2}{n}\sum_{i=1}^n x_i$ where each x_i is an $\ell_1^{m_i}$ average with constant $\bar{\varepsilon}$ where m_i is large depending upon $\bar{\varepsilon}$ and $\max \operatorname{supp} x_{i-1}$ and $x_1 > y_n$. In [GM], $(x_i)_1^n$ is called an *RIS* for rapidly increasing sequence of ℓ_1 averages. Clearly $\|x\|_n \geq \frac{1}{1+\bar{\varepsilon}}$. Let $\|x\| = \frac{1}{2}\sum_{i=1}^k \|E_i x\|$ where $(E_i)_1^k$ is admissible. Let i_0 be minimal so that $E_i x_{i_0} \neq 0$ for some i. Then there are relatively few E_i's relative to the length m_j of the average x_j for $j > i_0$. An argument much like the one above yields

$$\|x\| \leq \frac{2}{n}\left(1 + n\Big(\frac{1}{2} + \bar{\varepsilon}\Big)\right) = 1 + \frac{2}{n} + 2\bar{\varepsilon} \ .$$

The claim follows taking $z = \frac{x}{\|x\|}$ (with $\bar{\varepsilon}$ small and n large). ♢

Distortion for T was achieved by working with two types of vectors known to exist in all block subspaces: ℓ_1^m-averages and averages of RIS sequences. This idea plays a key roll in the work of Gowers and Maurey [GM] and in our discussion of S below. The same proof yields a more general statement.

Proposition 1.13 *Let (e_i) be a basis for a space X not containing ℓ_1. Let P_n be the basis projection onto $\langle e_i \rangle_{i=1}^n$ and for $x \in X$ set $|x| = \sup_n(\|P_n x\| + \|(I - P_n)x\|)$.*

If ℓ_1 is block finitely representable in every block basis of (e_i), then $|\cdot|$ is a distortion of some subspace of X.

Following the discovery of T many variants appeared which solved a number of problems over the next 15 years. But the distortion problem for ℓ_p, the unconditional basic sequence space and other like famous problems remained unsolved. The breakthrough came with the construction of the Tsirelson type space S in 1989 [S1]. S was the first arbitrarily distortable Banach space. Moreover it satisfies a stronger type of distortion criterion; it is biorthogonally distortable.

Definition 1.14 X is *biorthogonally distortable* if there exist sets $(A_n, A_n^*)_{n\in\mathbb{N}}$ with $A_n \subseteq S_X$, $A_n^* \subseteq B_{X^*}$ and $\lambda > 0$, $\varepsilon_i \downarrow 0$ satisfying

a) A_n is asymptotic in X for $n \in \mathbb{N}$

b) $\sup\{x^*(x) : x^* \in A_n^*\} \geq \lambda$ for each $x \in A_n$ for $n \in \mathbb{N}$

c) For $n \neq m$ and $x \in A_n$, $\sup\{|x^*(x)| : x^* \in A_m^*\} \leq \varepsilon_{\min(n,m)}$.

It is easy to see that if X is biorthogonally distortable then X is arbitrarily distortable via the collection of norms $|x|_n \equiv \frac{1}{n}\|x\| + \sup\{|x^*(x)| : x^* \in A_n^*\}$.

That the space S, which we are about to define, is biorthogonally distortable was first noted in [GM]. It was shown to be arbitrarily distortable in [S1].

Set $f(n) = \log_2(n+1)$ for $n \geq 1$. S is the completion of c_{00} under the implicit norm (whose existence follows from Proposition 1.11)

$$\|x\| = \max\biggl(\|x\|_\infty, \sup\Bigl\{\frac{1}{f(\ell)}\sum_{i=1}^{\infty}\|E_i x\| : \ell \geq 2 \text{ and } E_1 < \cdots < E_\ell\Bigr\}\biggr) .$$

The unit vector basis (e_i) is a 1-unconditional 1-subsymmetric basis for S. Thus $\|\sum a_i e_i\| = \|\sum \pm a_i e_{n(i)}\|$ for all choices of sign and $n_1 < n_2 < \cdots$. The admissibility criterion necessary in T to avoid ℓ_1 is no longer needed due to the damping factors $f(\ell)^{-1}$. As in the case of T once we show that S is (biorthogonally) distortable it follows that S is reflexive and does not contain any ℓ_p or c_0. (Further results on S can be found in [AS1] and [AS2].

Theorem 1.15 *S is biorthogonally distortable.*

Sketch of Proof. For $\ell \in \mathbb{N}$ and $x \in S$ we set $\|x\|_\ell = \sup\{\frac{1}{f(\ell)}\sum_{i=1}^{\ell}\|E_i x\| : E_1 < \cdots < E_\ell\}$.

It suffices to prove the following claim:

Given $\varepsilon_k \downarrow 0$ there is a sequence $k_n \uparrow 0$ so that for all $n \in \mathbb{N}$ and all infinte dimensional subspaces Y there is a $y \in Y$, $\|y\| = 1$ so that

$$\|y\|_{k_n} > 1 - \varepsilon_n \text{ and } \|y\|_{k_m} < \varepsilon_{\min(m,n)} \text{ for all } m \neq n . \tag{1}$$

The theorem follows taking A_n to be all such y's and

$$A_n^* = \Bigl\{\frac{1}{f(k_n)}\sum_{i=1}^{k_n} x_i^* : x_1^* < \cdots < x_\ell^*,\ x_{k_n}^* \in B_{S^*} \text{ for } i \leq \ell\Bigr\} .$$

By $x_1^* < x_2^*$ we mean w.r.t. (e_i^*), the sequence of biorthogonal functionals of (e_i).

In order to show the claim we will proceede as in the proof Theorem 3 in [S1] in which it is shown that for each subspace Z there is a $z \in S_Z$ so that $\|z\|_\ell \equiv \frac{1}{f(\ell)}$. This will follow from the fact that a sequence in S consisting of increasing ℓ_1-averages has a spreading model isometric to the basis (e_i) in S. Actually in [S2] it was shown that there are subsequences of such sequences which are isomorphically equivalent to (e_i). If $x_1 < \cdots < x_\ell$ in S then by the definition of the norm $\|\sum_{i=1}^{\ell} x_i\| \geq \frac{1}{f(\ell)}\sum_{i=1}^{\ell}\|x_i\|$. It follows then from Proposition 1.2 that ℓ_1 is block finitely representable in every block basis of (e_i). In particular if (z_i) is a block basis of (e_i) we can find ℓ_1^m averages with constant $1+\varepsilon$ for all $m \in \mathbb{N}$ and $\varepsilon > 0$. Thus we may choose a block basis (y_n) of (z_i) so that given $\varepsilon_n \downarrow 0$ and integers $m_n \uparrow \infty$ each y_n is an $\ell_1^{m_n}$-average with constant $1+\varepsilon_n$. The following two key observations appear in [S1]. The first one follows from the fact that the triangle inequality is an equality when applied to blocks in ℓ_1 and can be shown in a similar way as in the proof of the distortability of T.

$$\lim_{n\to\infty}\|y_n\|_\ell = \frac{1}{f(\ell)} \quad \text{for } \ell \in \mathbb{N} . \tag{2}$$

This in turn implies that for $x \in S$ and $\ell_0 \in \mathbb{N}$

$$\lim_{n\to\infty} \sup_{\ell\geq\ell_0} \|x+y_n\|_\ell \leq \max\Big(1, \sup_{\ell\geq\ell_0} \|x\|_\ell + \frac{1}{f(\ell_0)}\Big). \tag{3}$$

Iterating (3) we obtain by induction for every $k \in \mathbb{N}$

$$\lim_{n_1\to\infty} \cdots \lim_{n_k\to\infty} \sup_{\ell\geq\ell_0} \Big\|\frac{f(k)}{k}\sum_{i=1}^k y_{n_i}\Big\| \leq \frac{f(k)}{k} + \frac{k}{f(\ell_0)}. \tag{4}$$

The limit in (4) may be presumed to exist via Ramsey theory by first passing to a subsequence of (y_n) if necessary.

The second key observation requires a more extensive proof which we omit (see Lemmas 4 and 6 in [S1]).

$$\lim_{n_1\to\infty} \cdots \lim_{n_k\to\infty} \Big\|\sum_{i=1}^k y_{n_i}\Big\| = \lim_{n_1\to\infty} \cdots \lim_{n_k\to\infty} \Big\|\sum_{i=1}^k y_{n_i}\Big\|_k = \frac{k}{f(k)} = \Big\|\sum_{i=1}^k e_i\Big\| \text{ , for all } k \in \mathbb{N}. \tag{5}$$

For $\ell < k$ in $\mathbb{N}$ we obtain from (4),

$$\begin{aligned} &\lim_{n_1\to\infty} \cdots \lim_{n_k\to\infty} \Big\|\frac{f(k)}{k}\sum_{i=1}^k y_{n_i}\Big\|_\ell \\ &\leq \frac{1}{f(\ell)}\frac{f(k)}{k} \lim_{n_1\to\infty} \cdots \lim_{n_k\to\infty} \max\Big\{\sum_{i=1}^\ell \Big\|\sum_{j=K_{i-1}}^{K_i} y_{n_j}\Big\| : 1 = K_0 \leq K_1 \leq \cdots \leq K_\ell = k\Big\} \\ &= \frac{1}{f(\ell)}\frac{f(k)}{k} \max\Big\{\sum_{i=1}^\ell \Big\|\sum_{j=1}^{k_i} e_i\Big\| : k_1 + \cdots + k_\ell \leq \ell + k\Big\} \\ &= \frac{1}{f(\ell)}\frac{f(k)}{k} \max\Big\{\sum_{i=1}^\ell \frac{k_i}{f(k_i)} : k_1 + \cdots + k_\ell \leq \ell + k\Big\} \leq \frac{1}{f(\ell)}\frac{f(k)}{f((k+\ell)/\ell)}\frac{\ell+k}{k} \ . \end{aligned} \tag{6}$$

(Here the concavity of f is used as in the proof of Lemma 4 in [S1].) Given $\varepsilon_n \downarrow 0$ choose $k_n \uparrow \infty$ with

$$\frac{f(k_n)}{f(k_m)f(\frac{k_n+k_m}{k_m})} \cdot \frac{k_n+k_m}{k_n} < \varepsilon_m \text{ and } \frac{f(k_m)}{k_m} + \frac{f(k_m)}{f(k_n)} < \varepsilon_m \text{ whenever } m < n \ .$$

Using this choice for k_n it now follows from (4) and (6) that every infinite dimensional subspace Y contains a normalized vector y satisfying (1). ♢

Let us return to the distortion problem: is ℓ_2 (or ℓ_p) distortable? By Proposition 1.5 this is equivalent to finding asymptotic sets $A, B \subseteq S_{\ell_2}$ with $md(A,B) > 0$. Distortion can be transferred between one ℓ_p space and another via the Mazur map $M_p : S_{\ell_1} \to S_{\ell_p}$ defined by $M_p(x)(i) = \text{sign } x(i)|x(i)|^{1/p}$. M_p is a uniform homeomorphism [Ri] and moreover preserves block bases of (e_i). Thus if A is asymptotic in ℓ_1 (respectively, ℓ_p) then $M_p(A)$ satisfies $d(X, M_p(A)) = 0$ (respectively, $d(X, M_p^{-1}(A)) = 0$) for all $X \subseteq \ell_1$ (respectively, $X \subseteq \ell_p$).

Thus ℓ_p is distortable for some (or all) p iff S_{ℓ_1} admits separated asymptotic sets iff there exists a Lipschitz function $f : S_{\ell_1} \to \mathbb{R}$ which does not stabilize (Proposition 1.5). This equivalence does not solve the problem but it does suggest that one might try to find separated asymptotic sets in the sphere of some space and then find a generalized Mazur map to transfer the distortion to ℓ_p. This turns out to work.

The Generalized Mazur Map ([Lo], [Gi])

Let X have a 1-unconditional normalized basis (e_i). If $x = \sum a_i e_i$ we set $|x| = \sum |a_i| e_i$. The *entropy map*

$$E : (\ell_1 \cap c_{00}) \times X \to [-\infty, \infty)$$

is defined by

$$E(h, x) = E(|h|, |x|) = \sum_i |h_i| \, |\log x_i|$$

under the convention $0 \log 0 \equiv 0$, for $x = \sum x_i e_i$ and $h = (h_i)$. $F_X : S_{\ell_1} \cap c_{00} \to X$ is defined as follows. For $h = (h_i) \in S_{\ell_1} \cap c_{00}$, $F_X(h)$ is the unique $x = \sum x_i e_i \in X$ satisfying

i) $E(h, x) \geq E(h, y)$ for all $y \in S_X$,

ii) $\operatorname{supp} h = \operatorname{supp} x \equiv B$, and

iii) $\operatorname{sign} x_i = \operatorname{sign} h_i$ for $i \in B$.

Of course one must observe that such an element x exists, which is easy. The uniqueness of x follows from the strict convexity of the log function: if $\operatorname{supp} x = \operatorname{supp} y = B$ and $x \neq y$ then $E(h, \frac{1}{2}|x| + \frac{1}{2}|y|) > \frac{1}{2}E(h, |x|) + \frac{1}{2}E(h, |y|)$.

The map F_X is uniformly continuous if X is uniformly convex ([OS3], Proposition 2.4) and agrees with the Mazur map M_p if $X = \ell_p$ ([OS3], Proposition 2.5). When X is in addition uniformly smooth, F_X extends to a uniform homeomorphism between S_{ℓ_1} and S_X ([OS3], Proposition 2.6). Using some renorming tricks one has

Theorem 1.16 *Let X be a Banach space with an unconditional basis. Then S_X and S_{ℓ_1} are uniformly homeomorphic iff X does not contain ℓ_∞^n's uniformly in n.*

The "only if" direction is due to Enflo [E]. This theorem has been extended to Banach lattices [C] and via the complex interpolation method to other spaces ([BeL], chapter 12). Using this generalized Mazur map we ultimately obtain

Theorem 1.17 *ℓ_p is biorthogonally distortable for $1 < p < \infty$.*

We sketch the proof. Our proof that S was biorthogonally distortable yielded the following

Proposition 1.18 *Let $\varepsilon_i \downarrow 0$. There exist sets $A_n \subset S_S$ and $A_n^* \subset B_S$ satisfying the properies (a), (b) and (c). (cf. Definition 1.14)*

a) $|x_k^*(x_\ell)| < \varepsilon_{\min(k,\ell)}$ *if* $k \neq \ell$, $x_k^* \in A_k^*$, $x_\ell \in A_\ell$

b) *For* $k \in \mathbb{N}$ *and* $x \in A_k$ *there exists* $x^* \in A_k^*$ *with* $x^*(x) > 1 - \varepsilon_k$

c) A_k *is nearly asymptotic in* S*, i.e.,* $d(X, A_k) = 0$ *for all* $X \subseteq S$.

We define $(x^* \circ x)(i) = x^*(i)x(i)$ for $x^* = \sum x^*(i)e_i^* \in S^*$ and $x = \sum x(i)e_i \in S$. Set

$$B_k = \left\{ \frac{x_k^* \circ x_k}{|x_k^*|(|x_k|)} : x_k^* \in A_k^*, x_k \in A_k \text{ and } |x_k^*|(|x_k|) = \|x_k^* \circ x_k\|_{\ell_1} \geq 1 - \varepsilon_k \right\}.$$

The sets $B_k \subseteq \ell_1$ are easily seen to be *unconditional* ($x = (x_i) \in B \Leftrightarrow (\pm x_i) \in B$) and *spreading* ($x = (x_i) \in B \Rightarrow \sum x_i e_{n_i} \in B$ for all $n_1 < n_2 < \cdots$) by the proof of Theorem 1.15.

Proposition 1.19 *B_k is nearly asymptotic in ℓ_1.*

We omit the technical proof which depends on the map F_{S^*} (see [OS3], Theorem 3.4). Theorem 1.17 follows for $p = 2$ by taking $C_k = M_2(B_k)$. C_k is nearly asymptotic and moreover if $v_k \in C_k$, $v_\ell \in C_\ell$ with $k \neq \ell$ let

$$|v_k|^2 = x_k^* \circ x_k / |x_k^*|(|x_k|) \quad \text{and} \quad |v_\ell|^2 = x_\ell^* \circ x_\ell / |x_\ell^*|(|x_\ell|)$$

where x_k^*, x_k and x_ℓ^*, x_ℓ are as in the definition of B_k and B_ℓ. Letting $\lambda = (1 - \varepsilon_1)^{-1}$,

$$\begin{aligned}
(|v_k|, |v_\ell|) &\leq \lambda \sum_j |x_k^*(j)x_k(j)x_\ell^*(j)x_\ell(j)|^2 \\
&\leq \lambda \Big(\sum_j x_k^*(j)x_\ell(j)\Big)^{1/2} \Big(\sum_j x_\ell^*(j)x_k(j)\Big)^{1/2} \quad \text{(by Cauchy-Schwarz)} \\
&= \lambda \langle |x_k^*|, |x_\ell| \rangle^{1/2} \langle |x_\ell^*|, |x_k| \rangle^{1/2} \leq \lambda \varepsilon_{\min(k,\ell)} \quad \text{(by Proposition 1.18).}
\end{aligned}$$

For $p \neq 2$ a similar argument works. (C_k, D_k) biorthogonally distort ℓ_p where $C_k = M_p(B_k)$ and $D_k = M_q(B_k)$, $\frac{1}{p} + \frac{1}{q} = 1$. ♢

Combining Theorems 1.9 and 1.17 we obtain.

Theorem 1.20 *If X does not contain a distortable subspace then every subspace X contains an isomorph of c_0 or ℓ_1.*

Gowers [G] proved that every Lipschitz function $f : S_{c_0} \to \mathbb{R}$ stabilizes. Thus combining our above work with this result we have.

Theorem 1.21 *Suppose that every Lipschitz $f : S_X \to \mathbb{R}$ stabilizes. Then every subspace of X contains an isomorph of c_0.*

The fact that ℓ_2 is biorthogonally distortable leads to some interesting renormings. One can prove that given $k \in \mathbb{N}$ and $\varepsilon > 0$ there exists a renorming $|\cdot|$ of ℓ_2 so that for all $X \subseteq \ell_2$ there exists $E \subseteq X$ with $d(E, \ell_\infty^k) < 1 + \varepsilon$. Thus every infinite dimensional subspace contains k-cubes as k dimensional slices of the unit sphere (up to ε). More generally we have [OS3]

Theorem 1.22 *Let X be a biorthogonally distortable Banach space with basis (e_i). For $k \in \mathbb{N}$ and $\varepsilon > 0$ there exists an equivalent norm $|\cdot|$ on X so that if $(w_i)_1^k$ is a normalized monotone basis and $(x_i)_1^\infty$ is a block basis of (e_i) then there exists a block basis $(b_i)_1^k$ of (x_i) which is $1+\varepsilon$-equivalent to $(w_i)_1^k$.*

This is proved by a renorming trick that derives from [MR] and was exploited by Gowers and Maurey [GM] to show that for all $K < \infty$ such a space could be renormed so that no block basis of (e_i) is K-unconditional.

One can easily prove that every X contains a basic sequence (e_i) with basis projections P_n satisfying $\lim_n \|P_n\| = 1$ ((e_i) is asymptotically monotone). It was open as to whether one could also obtain $\lim \|I - P_n\| = 1$ ((e_i) is asymptotically bimonotone) Theorem 1.22 yields that this is false even for isomorphs of ℓ_2. This follows by considering the summing basis (s_i) whose norm is given by $\|\sum a_i s_i\| = \sup_k |\sum_{i=1}^k a_i|$. This monotone basis has $\|s_1 - 2s_2\| = 1$ yet $\|(I - P_1)(s_1 - 2s_2)\| = \|2s_2\| = 2$.

The relations between the notions X is distortable, X is arbitrarily distortable and X is biorthogonally distortable remain unclear. If is unknown if a distortable space contains an arbitrarily distortable subspace or if an arbitrarily distortable space contains a biorthogonally distortable subspace. It seems to be quite an interesting question as to whether there exists a distortable space of bounded distortion.

Definition 1.23 X is of λ-*bounded distortion* if for all equivalent norms $|\cdot|$ on X and all $Y \subseteq X$ there exists $Z \subseteq Y$ with

$$\sup\Big\{\frac{|y|}{|z|} : y, z \in S_{Z,\|\cdot\|}\Big\} \le \lambda\ .$$

At this point the prime candidate for such a space is T, the first distortable space. Theorem 1.12 gives the best current estimate: it is not even known if T is 2-distortable. We do know something about the structure of spaces of bounded distortion, should they exist.

Theorem 1.24 *Let X be a space of λ-bounded distortion. Then X contains a basic sequence (e_i) which is*

(a) [MT] *asymptotically c_0 or ℓ_p for some $1 \le p < \infty$ and*

(b) [TJ1] *unconditional.*

We sketch the proof following the argument in [Ma1] for (a). First by Krivine's theorem it is easy to see that if (e_i) is a basic sequence then

$$K(e_i) \equiv \{p \in [1,\infty] : \ell_p^n \text{ is block finitely representable in } (e_i) \text{ for all } n \in \mathbb{N}\}$$

is a closed nonempty subset of $[1,\infty]$. Since $K(f_i) \subseteq K(g_i)$ if (f_i) and (g_i) are block bases of (e_i) with $(f_i)_n^\infty$ a block basis of $(g_i)_1^\infty$ for some n it follows that one can find a block basis (f_i) of (e_i) with $K(f_i) = K(g_i)$ for all block bases (g_i) of (f_i). In other words, the set of Krivine p's can be stabilized by passing to a block basis.

(a) We may choose a basic sequence (e_i) in X so that the Krivine p's are stabilized. First we note that $K(e_i) = \{p\}$ for some unique p. Indeed if there exist $p < q$ in $K(e_i)$ then as in the proof of Theorem 1.12 one obtains that the class of norms $(\|\cdot\|)_{n\in\mathbb{N}}$ arbitrarily distort $[(e_i)]$ where

$$\|x\|_n = \sup\left\{\left(\sum_{i=1}^n \|E_i x\|^p\right)^{1/p} : E_1 < \cdots < E_n\right\} .$$

Secondly using that X is of λ-bounded distortion one obtains, using the same family of norms, that there exists $C = C(\lambda)$ and a block basis (x_i) of (e_i) so that if $(y_i)_1^n$ is admissible w.r.t. (x_i) then

$$C\Big\|\sum_1^n y_i\Big\| \ge \Big(\sum_1^n \|y_i\|\Big)^{1/p} .$$

It remains to produce uniform asymptotic upper ℓ_p estimates.

To accomplish this one passes to another normalized block basis, (y_i) which has essentially stabilized asymptotic upper ℓ_p estimates. Say

$$\Big\|\sum_1^n z_i\Big\| \le C_n\Big(\sum_1^n \|z_i\|^p\Big)^{1/p}$$

whenever $y_n \le z_1 < \cdots < z_n$ w.r.t. (y_i) where C_n is the essentially smallest constant that can be achieved. If (C_n) is unbounded choose m with C_m large. Choose $N \gg m$. Set

$$B = \left\{\sum_{i=1}^m z_i/C_m : z_1 < \cdots z_m \text{ w.r.t. } (y_i) \qquad \text{and } \sum_{i=1}^m \|z_i\|^p \le 1\right\} \qquad \text{and}$$

$$A_N^* = \left(N^{-1/q}\sum_{i=1}^N x_i^* : x_1^* < \cdots < x_N^* \qquad \text{are in the unit ball of } \langle y_i^*\rangle\right\} .$$

One shows (again by a similar argument) that $|x^*(b)| \le C(\lambda)/C_m$ for $x^* \in A_N^*$ and $b \in B$. Also for all block subspaces of (y_i) one can find an ℓ_p^N vector b on which $x^*(b) > \frac{1}{2}$ say for some $x^* \in A_N^*$. It follows that the norms $|x|_N = \sup\{|x^*(x)| : x^* \in A_N^*\}$ arbitrarily distort $[(y_i)]$. ◇

b) The technique used to show that X must contain an unconditional basic sequence is indirect. One shows that X contains basic sequences of unbounded order in terms of their unconditional structure. Given $K < \infty$ let $T(X, K)$ be the set of all normalized finite basic sequences $(x_i)_1^n \subseteq X$ which are K-unconditional. Then $T(X, K)$ is naturally a tree under $(x_i)_1^n \le (y_i)_1^m$ if $n \le m$ and $x_i = y_i$ for $i \le n$. If X does not contain an unconditional basic sequence then $T(X, K)$ is well founded (the tree has no infinite branches). Set $T^{(0)} = T(X, K)$, $T^{(\alpha+1)} = \{(x_i)_1^n :$ there exists x_{n+1} so that $(x_i)_1^{n+1} \in T^{(\alpha)}\}$ and $T^{(\beta)} = \bigcap_{\alpha<\beta} T^{(\alpha)}$ if β is a limit ordinal. Since X is separable and $T(X, K)$ is closed in the product topology, it follows that $o(T) \equiv \inf\{\alpha < w_1 : T^{(\alpha)} = \phi\} < w_1$ (see [Bo1]).

One way to prove b) is to use a) to produce for a certain $K = K(\lambda)$ for all $\alpha < w_1$ a normalized basic sequence (x_i^α) so that the tree of subsets of $\mathbb{N}$, $\{F \subseteq \mathbb{N} : (x_i^\alpha)_{i\in F}$ is K-unconditional$\}$, has order w^α (see [O]; for a direct proof avoiding the use of a) but following the same tree complexity idea see the original proof [TJ1]). The order for this tree of subsets is by extension.

The Schreier classes $(S_\alpha)_{\alpha<w_1}$ [AA] are defined as follows:

$$S_0 = \{\{n\} : n \in \mathbb{N}\} \cup \phi$$

$$S_{\alpha+1} = \Big\{ \bigcup_1^n E_i : n \in \mathbb{N}\ ,\ \{n\} \le E_1 < \cdots < E_n \text{ and } E_i \in S_\alpha \text{ for } i \le n \Big\}\ .$$

If β is a limit ordinal choose $\beta_n \uparrow \beta$ and set $S_\beta = \{E :$ for some $n \in \mathbb{N}$, $\{n\} \le E \in S_{\beta_n}\}$. $(E_i)_1^n$ is *α-admissible* if $E_1 < \cdots < E_n$ and $(\min E_i)_{i=1}^n \in S_\alpha$. It is easy to see that S_α is a well founded tree of sets with $o(S_\alpha) = w^\alpha$. (x_i) is said to be *α-unconditional* with constant C if $(x_i)_{i\in F}$ is C-unconditional for all $F \in S_\alpha$.

Let $(x_i^0)_{i=1}^\infty$ be an asymptotic ℓ_p basis in X (here we have used a)) for some p. Assume (x_i^α) has been chosen to be α-unconditional. Define an equivalent norm on $\langle x_i^\alpha \rangle$ by $|x| = \sup\{\| \sum_{i=1}^n \pm E_i x\| :$ $(E_i)_1^n$ is $\alpha{+}1$-admissible and the E_i's are intervals of integers$\}$. $|\cdot|$ is an equivalent norm since (x_i^α) is asymptotic ℓ_p. One shows that if $(E_i)_1^n$ is an α-admissible family of intervals then $|\sum_1^n \pm E_i x| \le 4|x|$ for $x \in \langle x_i^\alpha \rangle$. This requires a combinatorial argument given in [TJ1]. Then, using that X is of λ-bounded distortion it follows that there exists a block basis $(x_i^{\alpha+1})$ of (x_i) so that $(x_i^{\alpha+1})$ is $\alpha + 1$-unconditional with constant $K(\lambda)$. A diagonal argument is used at limit ordinals. $\diamondsuit$

Maurey [Ma1] has extended Theorem 1.17. He has shown that the argument used to biorthogonally distort ℓ_p $(1 < p < \infty)$ can be generalized to the case of an asymptotic ℓ_p space not containing ℓ_1^n's uniformly with an unconditional basis. Thus one obtains, since every X must contain either an arbitrarily distortable subspace or a subspace of bounded distortion,

Theorem 1.25 [Ma1], [TJ1] *If X does not contain ℓ_1^n's uniformly then X contains an arbitrarily distortable subspace.*

There do exist asymptotic ℓ_1 spaces which are arbitrarily distortable (and even asymptotic ℓ_1 spaces not containing an unconditional basic sequence) [AD]. These are certain *mixed Tsirelson* *spaces*. For example (see [AO]) the space $X = T(S_n, \frac{1}{n+1})_{n\in\mathbb{N}}$ whose norm is given by the implicit equation: for $x \in c_{00}$,

$$\|x\| = \max\left(\|x\|_\infty, \sup_{n\geq 1} \sup\left\{ \frac{1}{n+1} \sum_{i=1}^{k} \|E_i x\| : (E_i)_1^k \text{ is } S_n\text{-admissible} \right\} \right)$$

is arbitrarily distortable.

It is easy to see that $T = T(S_n, 2^{-n})_{n\in\mathbb{N}}$ is also a mixed Tsirelson space. One has [AO] that if $X = T(S_n, O_n)_{n\in\mathbb{N}}$ with $O \equiv \lim O_n^{1/n}$ and $O_n/O^n \to 0$ then X is arbitrarily distortable.

These results present evidence why T is a prime candidate for a space of bounded distortion. Further evidence is given in [OT] and [OTW].

Maurey [Ma2] has shown that the sets (A_n) which yield a biorthogonal distortion of ℓ_2 can in addition to the properties of being unconditional and spreading can be taken to be *symmetric* ($x = (x(i)) \in A_n$ iff $(x_{\pi(i)}) \in A_n$ for all permutations π of $\mathbb{N}$. N. Tomczak-Jaegermann [TJ2] has shown that the Schatten classes C_p are arbitrarily distortable for $1 < p < \infty$.

2 Asymptotic structure

In 1 we saw that a Banach space X need not contain c_0 or ℓ_p and Lipschitz functions on S_X need not stabilize. Yet the finite dimensional analogues are valid. One is left with the task of defining a structural framework for an arbitrary X which bridges the finite and infinite dimensional structure. There are two such structures that have been defined. One, mentioned in the first section, is that of spreading models E or more generally spreading models over X, $X \oplus E$. While extremely useful, spreading models have certain shortfalls. For example the relationship is not transitive; if F is a spreading model of E and E is a spreading model of X then F need not be a spreading model of X [BL]. Secondly spreading models do not satisfy our desire to find an infinite dimensional extension of Krivine's theorem; indeed there exists a space X so that no spreading model is isomorphic to c_0 or ℓ_p [OS4]. Perhaps the strongest stabilization result involving spreading models is the following result. The proof uses the second stabilization principle.

Theorem 2.1 [ORS] *Let (F_n) be a sequence of finite dimensional subspaces of a space X with $\dim F_n \to \infty$. There exist $G_n \subseteq F_{k_n}$ for some $k_1 < k_2 < \cdots$ with $\dim G_n \to \infty$ so that all sequences (x_n) with $x_n \in S_{G_n}$ have the same spreading model $E = [(e_i)]$ over X. In particular (e_i) is 1-unconditional over X. Moreover G_n can be chosen so that for all $\varepsilon > 0$, $k \in \mathbb{N}$ and $x \in X$ there exists $k_0 \in \mathbb{N}$ so that if $k_0 \leq n_1 < n_2 < \cdots < n_k$ then*

$$\left| \left\| x + \sum_{i=1}^{k} \alpha_i e_i \right\| - \left\| x + \sum_{i=1}^{k} \alpha_i x_i \right\| \right| < \varepsilon$$

whenever $x_i \in S_{G_{n_i}}$ and $|\alpha_i| \leq 1$ for $i \leq k$.

Recently a second finite-infinite dimensional bridge structure was defined ([MT],[MMT]). We first examine the simplest version of this notion. Suppose that X has a basis or more generally an FDD (finite dimensional decomposition) $(E_i)_{i=1}^\infty$. For $n \in \mathbb{N}$ we say $(x_i)_1^n \in \{X, (E_i)\}_n$, the n^{th} asymptotic structure of X w.r.t. (E_i), if $(x_i)_1^n$ is a normalized basic sequence with the property that:

$$\forall\, \varepsilon > 0 \quad \forall\, k_1\ \exists\, y_1 \in S_{\langle E_i\rangle_{k_1}^\infty} \quad \forall\, k_2\ \exists\, y_2 \in S_{\langle E_i\rangle_{k_2}^\infty} \quad \cdots \quad \forall\, k_n\ \exists\, y_n \in S_{\langle E_i\rangle_{k_n}^\infty}$$

with $(x_i)_1^n$ $1+\varepsilon$-equivalent to $(y_i)_1^n$.

Thus $(x_i)_1^n$ can be found (up to ε) in X arbitrarily far out and arbitrarily separated w.r.t. (E_i). Note that if $(z_i)_1^m$ is a normalized block basis of $(x_i)_1^n \in \{X, (E_i)\}_n$ then $(z_i)_1^m \in \{X, (E_i)\}_m$. It follows by the existence of spreading models and Krivine's theorem that there exists $p \in [1, \infty]$ so that the unit vector basis of ℓ_p^n belongs to $\{X, (E_i)\}_n$ for $n \in \mathbb{N}$.

Proposition 2.2 *Let (E_i) be an FDD for X and suppose that $|\{X, (E_i)\}_2| = 1$. Then there exists $p \in [1, \infty]$ so that for all n if $(x_i)_1^n \in \{X, (E_i)\}_n$ then $(x_i)_1^n$ is 1-equivalent to the unit vector basis of ℓ_p^n. Moreover X contains almost isometric copies of ℓ_p.*

The first part is easy given our previous remarks. The "moreover" statement is also not difficult to prove directly (see [MMT]) but will in fact follows from Theorem 2.5 below. Asymptotic structure can also be understood in terms of trees on X. Let $T_k = \{(n_1, \ldots, n_k) : n_i \in \mathbb{N}\}$. $\tau \in T_k(X, (E_i))$ if $\tau = \{(x_{n_1,\ldots,n_i}) : (n_1, \ldots, n_i) \in T_k\} \subseteq S_X$ and for $(n_i)_1^k \in T_k$, $(x_{n_1,\ldots,n_i})_{i=1}^k$ is a normalized block basis of (E_i). We shall say that $\tau \in T_k(X, (E_i))$ *converges* to $(x_i)_1^k$ if $(x_i)_1^k$ is a normalized basic sequence and for some $\varepsilon_i \downarrow 0$ for all $(n_1, \ldots, n_k) \in T_k$, $(x_{n_1,\ldots,n_i})_{i=1}^k$ is $1+\varepsilon_{n_1}$-equivalent to $(x_i)_1^k$.

Proposition 2.3 *Let (E_i) be an FDD for X. Let $k \in \mathbb{N}$. Then $(x_i)_1^k \in \{X, (E_i)\}_k$ iff there exists a tree $\tau \in T_k(X, (E_i))$ which converges to $(x_i)_1^k$.*

The proposition follows easily from the relevant definitions. The asymptotic structure of X may also be characterized in terms of trees as follows.

Proposition 2.4 *$\{X, (E_i)\}_k$ is the smallest class C of normalized bases of length k having the property that if $\tau \in T_k(X, (E_i))$ and $\varepsilon > 0$ then some branch of τ is $1+\varepsilon$-equivalent to a member of C.*

This follows from the fact that if $\tau \in T_k(X, (E_i))$ then there exists a convergent subtree $\tau' \subseteq \tau$, $\tau' \in T_k(X, (E_i))$. This latter fact can be proved using Ramsey theory (see [KOS]). Another interpretation of asymptotic structure is given by the next theorem. Recall that (F_i) is a *blocking* of (E_i) if $F_n = \langle E_i\rangle_{i=k_n+1}^{k_{n+1}}$ for all n and some sequence of integers $0 = k_1 < k_2 < \cdots$. (F_i) is a *skipped blocking* of (E_i) if there exist integers $1 \le p_1 \le q_1 < q_1 + 1 < p_2 \le q_2 < q_2 + 1 < p_3 \le \cdots$ so that $F_n = \langle E_i\rangle_{i=p_n}^{q_n}$ for $n \in \mathbb{N}$.

Theorem 2.5 *Let $\varepsilon_i \downarrow 0$. There exists a blocking (H_i) of (E_i) so that for all k if $(F_i)_1^k$ is any skipped blocking of $(H_i)_k^\infty$ and $x_i \in S_{F_i}$ for $i \le k$ then $(x_i)_1^k$ is $1+\varepsilon_k$-equivalent to an element of $\{X,(E_i)\}_k$.*

Proof. By a diagonal argument it suffices to produce for a fixed $k \in \mathbb{N}$ and $\varepsilon > 0$ a blocking (H_i) of (E_i) so that any normalized block basis $(x_i)_1^k$ relative to any skipped blocking $(F_i)_1^k$ of $(H_i)_2^\infty$ is $1+\varepsilon$-equivalent to an element of $\{X,(E_i)\}_k$. By a standard compactness argument one need only show the validity of the following sentence.

> "$\exists\ N_1\ \forall\ x_1 \in S_{\langle E_i\rangle_{N_1}^\infty}\ \exists\ N_2\ \forall\ x_2 \in S_{\langle E_i\rangle_{N_2}^\infty} \cdots \exists\ N_k\ \forall\ x_i \in S_{\langle E_i\rangle_{N_k}^\infty}$, $(x_i)_1^k$ is $1+\varepsilon$-equivalent to an element in $\{X,(E_i)\}_k$."

If false by formally negating the sentence one easily constructs a tree $\tau \in T_k(X,(E_i))$ so that no branch of τ is $1+\varepsilon$-equivalent to any element of $\{X,(E_i)\}_k$. Proposition 2.4 then yields a contradiction. $\diamondsuit$

Let (E_i) be an FDD for X. We shall say that X is *Asymptotic* ℓ_p w.r.t. (E_i) if there exists $K < \infty$ so that for all k and $(x_i)_1^k \in \{X,(E_i)\}_k$, $d(\langle x_i\rangle_1^k, \ell_p^k) \le K$. This is formally weaker than assuming $(x_i)_1^k$ is K-equivalent to the unit vector basis of ℓ_p^k but as observed in [MMT] the weaker assumption implies the stronger at least if $1 < p < \infty$ (the case $p = 1$ or ∞ remains open).

Theorem 2.6 *If X is Asymptotic ℓ_p w.r.t. the* FDD *(E_i) with $1 < p < \infty$ then there exists $K < \infty$ so that for all k, $(x_i)_1^k$ is K-equivalent to the unit vector basis of ℓ_p^k for all $(x_i)_1^k \in \{X,(E_i)\}_k$.*

Also one has a nice duality result. Let us note that the asymptotic structure we have discussed w.r.t. an FDD can be and is indeed done in a more general context in [MMT], e.g., with respect to fundamental, total minimal systems and the next theorem is proved in that broader context.

Theorem 2.7 *Let $1 \le p \le \infty$ and let X be a reflexive Asymptotic ℓ_p space w.r.t. the* FDD*(E_i). Then X^* is Asymptotic ℓ_q w.r.t. (E_i^*) where $\frac{1}{p} + \frac{1}{q} = 1$.*

One can also consider infinite dimensional spaces which reflect the asymptotic structure of X. A space Y with a normalized basis (y_i) is an *asymptotic version* of X if $(y_i)_1^n \in \{X,(E_i)\}_n$ for all n. This includes the class of all spreading models of normalized block bases of (E_i). Moreover one can show [MMT] that there exists such a Y which satisfies $\{Y,(y_i)\}_k = \{X,(E_i)\}_k$ for all $k \in \mathbb{N}$. (Y is called a *universal asymptotic version* of X.)

The asymptotic structure of a space can be stabilized.

Proposition 2.8 *Let (E_i) be an* FDD *for X. There exists a normalized block basis (y_i) of (E_i) so that if $Y = [(y_i)]$ and $Z = [(H_i)]$ where (H_i) is any* FDD *obtained by blocking a block basis of (y_i) then for all k*

$$\{Y, (y_i)\}_k = \{Z, (H_i)\}_k \ .$$

One may ask, how small must this stabilized asymptotic structure be? The answer is not very.

Theorem 2.9 [OS2] *There exists a normalized monotone basis (e_i) for a reflexive space X with the following property. For all k, all normalized monotone bases $(x_i)_1^k$ and all $Y = [(y_i)]$ where $(y_i)_1^\infty$ is a block basis of (e_i),*

$$(x_i)_1^k \in \{Y, (y_i)\}_k \ .$$

In particular X cannot contain an unconditional basic sequence. The example is technically difficult. We will not present the argument but shall present the norm. This gives the flavor of both the construction and of the possibilities afforded by generalizations of conditional Tsirelson type norms. It is worth noting that a somewhat simpler example is given in [OS4]. In this case the basis (e_i) is unconditional and the unit vector basis of ℓ_p^k belongs to $\{X, (e_i)\}_k$ for all k and $p \in [1, \infty]$.

$H \subseteq c_{00} \cap B_{\ell_\infty}$ is taken to be a countable set of nonzero vectors with three properties:

i) H is dense in $B_{\ell_\infty} \cap c_{00}$ w.r.t. $\|\cdot\|_{\ell_1}$

ii) $\forall\, a \in H$ and intervals I of integers, $Ia \in H$ if $Ia \neq 0$

iii) If $a_1 < \cdots < a_n$ in H then $\sum_{i=1}^n a_i$, $\frac{1}{f(n)} \sum_{i=1}^n a_i$ and $\frac{1}{n} \sum_{i=1}^n a_i$ all belong to H. A subsequence $M = (M_n)_{n=1}^\infty \subseteq \mathbb{N}$ is taken with $M_1 = 2$ and we let

$$\sigma : \{(a_1, \dots, a_n) : n \in \mathbb{N},\ a_1 < \cdots < a_n,\ a_i \in H \text{ for } i \leq n\} \to \mathbb{N}$$

be an injection satisfying 4 more properties.

iv) If $a_1 < \cdots < a_n$ belong to H and I is an interval in $\mathbb{N}$ and $[j_1, j_2] = \{i : Ia_i \neq 0, i \neq n\} \neq \emptyset$, then $\sigma(Ia_{j_1}, \dots, Ia_{j_2}) \leq \sigma(a_1, \dots, a_n)$.

v) If $a_1 < \cdots < a_n$ belong to H then $\max \operatorname{supp} a_n < \sigma(a_1, \dots, a_n)$.

vi) If $I_m(\sigma)$ is the range of σ then $\sum_{n \in I_m(\sigma)} \frac{1}{f(n)} < \infty$ where as before $f(n) = \log_2(n+1)$.

vii) If $m \in I_m(\sigma)$ and $\bar{m}, \bar{\bar{m}}$ are the predecessor and successor of m in $I_m(\sigma)$ then for $\ell \in [1, \bar{m}] \cup [\bar{\bar{m}}, \infty)$

$$2\left[\frac{f(m)}{m} + \frac{1}{f(\ell)} + \frac{f(m)}{m}\ \frac{\min(\ell, m)}{f(\ell)}\right] \leq \begin{cases} \frac{3}{f(\ell)}\ , & \ell < m \\ \frac{3f(m)}{m}\ , & \ell > m\ . \end{cases}$$

If $\|\cdot\| \in \mathcal{N}$ and $X = (c_{00}, \|\cdot\|)$ for $m \geq 2$ we set

$$\begin{aligned} A_m^X &= \left\{ \frac{1}{m} \sum_{i=1}^m a_i^* : a_i^* \in H \cap B_{X^*} \text{ for } i \leq m \text{ and } a_1^* < \cdots < a_m^* \right\} \\ A^X &= \bigcup_{m \geq 2} A_m^X \ , \ \tilde{A}_m^X = \bigcup_{n \geq m} A_n^X \text{ for } m \geq 2 \text{ and } \tilde{\mathfrak{A}}^X = (\tilde{A}_m^X)_{m=2}^\infty \ . \end{aligned}$$

Let for $m \geq 2$, $B_m^X = \{\frac{1}{f(m)} \sum_{i=1}^m a_i^* : (a_1^*, \ldots, a_m^*) \subseteq A^X$ is $(\tilde{\mathfrak{A}}^X, M, \sigma)$-admissible$\}$.

The statement that $(a_1^*, \ldots, a_m^*)$ is $(\tilde{\mathfrak{A}}^X, M, \sigma)$-admissible means that $a_1^* < \cdots < a_m^*$, $a_1^* \in \bigcup_{i \geq M_m} \tilde{A}_i^X$ and $a_{i+1}^* \in \tilde{A}^X_{\sigma(a_1^*, \ldots, a_i^*)}$ for $1 \leq i < m$. We take $B^X = \bigcup_{n=2}^\infty B_n^X$ and $\mathfrak{B}^X = (B_n^X)_{n=2}^\infty$. For $m \geq 1$ set

$$C_m^X = \left\{ \frac{1}{f(m)} \sum_{i=1}^m a_i^* : (a_1^*, \ldots, a_m^*) \subseteq B^X \text{ is } (\mathfrak{B}^X, M, \sigma)\text{-admissible} \right\} \text{ and } \ C^X = \bigcup_{n=1}^\infty C_n^X \ .$$

Then one uses Proposition 1.11 to show that there exists $\|\cdot\| \in \mathcal{N}$ so that $X = (c_{00}, \|\cdot\|)$ satisfies for all $x \in c_{00}$,

viii) $\|x\| = \max(\|x\|_\infty, \sup\{|a^*(x)| : a^* \in C^X\}$.

This is the space which yields Theorem 2.9.

Asymptotic structure has been generalized in several ways. Milman and Wagner have extended the notion to operators [MW]. Also Wagner [W] has given a higher order ordinal notion in terms of certain α-games for $\alpha < w_1$. The definition of $\{X, (E_i)\}_k$ yields this game for $\alpha = k$.

And of course one need not assume that the space X has an FDD. Indeed [MMT] consider the broader forum where the tail spaces of an FDD are replaced by finite codimensional subspaces in any nontrivial *filtration* on X. $\Gamma \subseteq \mathrm{cof}(X)$, the set of all finite codimensional subspaces of X, is called a filtration on X if for all $Y, Z \in \Gamma$ there exists $W \in \Gamma$ with $W \subseteq Y \cap Z$. One then has $(e_i)_1^k \in \{X, \Gamma\}_k$ if

$$\forall\ \varepsilon > 0\ \forall\ Y_1 \in \Gamma\ \exists\ y_1 \in S_{Y_1} \cdots\ \forall\ Y_k \in \Gamma\ \exists\ y_k \in S_{Y_1}$$

so that $(y_i)_1^k$ is the $1 + \varepsilon$-equivalent to $(e_i)_1^k$.

As noted in [MMT] this can be expressed in terms of a game where Player I chooses $Y \in \Gamma$ and Player II chooses $y \in S_Y$ with each player making k alternate plays starting with I. Thus $(e_i)_1^k \in \{X, \Gamma\}_k$ iff for all $\varepsilon > 0$ Player II has a winning strategy for the set of all normalized bases $1 + \varepsilon$-equivalent to $(e_i)_1^k$. By regarding $X \subseteq [E_n]$, some FDD, and $\Gamma = \{X \cap [E_i]_n^\infty : n \in \mathbb{N}\}$ one obtains a relativized notion of asymptotic structure w.r.t. an FDD and the relativized versions of the previous structural results remain valid [KOS]. Working with this Γ is nice because one has a coordinate system. What happens however for $\Gamma = \mathrm{cof}(X)$? And is there an infinite version of asymptotic structure? These are addressed in [OS6] and we now discuss some of the results contained therein.

We consider the two player game where Player I chooses $Y_1 \in \mathrm{cof}(X)$ and then Player II chooses $y_1 \in S_{Y_1}$ and then Player I chooses $Y_2 \in \mathrm{cof}(X)$ and so on. Player I wins the $\mathcal{A}$-game for a given $\mathcal{A} \subseteq S_X^\omega \equiv \{(x_i)_1^\infty : x_c \in S_X \text{ for all } i\}$ if $(y_i) \in \mathcal{A}$. One can define in a natural way what it means to say that Player I has a winning strategy for $\mathcal{A}$ and we denote this by $W_I(\mathcal{A})$. For $\varepsilon > 0$ we let $\mathcal{A}_\varepsilon = \{(y_i) \subseteq S_X : \text{there exists } (x_i) \in \mathcal{A} \text{ with } \|x_i - y_i\| < \varepsilon/2^i \text{ for all } i \in \mathbb{N}\}$ and we let $\bar{\mathcal{A}}_\varepsilon$ be the closure of this set in S_X^ω, given the product topology of the discrete topology on S_X.

Theorem 2.10 *Let $\mathcal{A} \subseteq S_X^\omega$. There exists a space Z with an FDD (E_i) so that $X \subseteq Z$ and such that the following are equivalent.*

a) For all $\varepsilon > 0$, $(W_I(\bar{\mathcal{A}}_\varepsilon))$.

b) For all $\varepsilon > 0$ there exists a blocking (G_i) of (E_i) and $\delta_i \downarrow 0$ so that: if $(x_n) \in S_X^\omega$ and there exist integers $1 = k_0 < k_1 < \cdots$ with $\{(\mathrm{Id} - P_{[G_j]_{j=k_{n-1}+1}^{k_n-1}})(x_n)\| < \delta_n$ for all n then $(x_n) \in \bar{\mathcal{A}}_\varepsilon$.

Moreover if X^* is separable, (E_n) can be chosen to be shrinking and if X is reflexive, Z can be chosen to be reflexive. In these cases a) is equivalent to

c) Every weakly null tree $\mathcal{T} \in T_\omega(X)$ has a branch in $\bar{\mathcal{A}}_\varepsilon$.

The hypothesis on $\mathcal{T}$ in c) means that $\mathcal{T} = (x_{(n_1,\dots,n_k)} : n_1 < \cdots < n_k$ are positive integers$) \subseteq S_X$ and the successors of every node, including ϕ, form a weakly null sequence.

This theorem can be applied to $\mathcal{A} = \{(x_i) \in S_X^\omega : (x_i)$ is K-equivalent to the unit vector basis of $\ell_p\}$ to yield the following.

Theorem 2.11 *Let X be reflexive and let $1 < p < \infty$. Let $C \geq 1$ be such that every weakly null tree $\mathcal{T} \in T_\omega(X)$ has a branch C-equivalent to the unit vector basis of ℓ_p. Then X embeds into the ℓ_p-sum of finite dimensional spaces. In fact given $\varepsilon > 0$ therre exists a finite codimensional subspace X_0 of X which $C^2 + \varepsilon$-embeds into $(\sum F_i)_{\ell_p}$ for some sequence (F_i) of finite dimensional spaces.*

This theorem generalizes results of [KW]. A similar theorem for the $p = \infty$ case was proved by N.J. Kalton [Ka]: If X does not contain ℓ_1 and for some $K < \infty$ every weakly null tree $\mathcal{T} \in T_\omega(X)$ admits a branch K-equivalent to the unit vector basis of c_0, then X embeds into c_0.

The proofs of these theorems use Ramsey theory and Martin's theorem that Borel games are determined [Mar]. Unlike the finite asymptotic structure case there is in general no smallest closed set $\mathcal{A}$ of normalized bases so that every weakly null tree $\mathcal{T} \in T_\omega(X)$ has a branch nearly in $\mathcal{A}$. So there is no unique notion of infinite asymptotic structure, but Theorem 2.10 does allow one to say something useful.

Finally what can be said if the asymptotic structure is as small as possible, either in the [MMT] sense or in the sense of spreading models? Proposition 2.3 yielded some information but using the above results one can say more. Recall that $\{X, \mathrm{cof}(X)\}_2$ denotes the asymptotic structure (of length 2) w.r.t. filtration of all finite codimensional subspaces of X.

Theorem 2.12 *Let X be reflexive and let $|\{X, \mathrm{cof}(X)\}_2| = 1$. Then there exists $1 < p < \infty$ so that for all $\varepsilon > 0$, some finite codimensional subspace of X $1+\varepsilon$-embeds into the ℓ_p sum of finite dimensional spaces.*

If X has a basis (x_i) and there is only one spreading model (e_i) that can be obtained as a spreading model of a block basis of (e_i) then, by the proof of Krivine's theorem, one obtains that (e_i) is 1-equivalent to the unit vector basis of c_0 or ℓ_p for some $1 \leq p < \infty$.

Theorem 2.13 *Let (x_i) be a basis for X and assume that all spreading models of a normalized block basis of (x_i) are 1-equivalent to the unit vector basis of ℓ_1 (respectively, c_0). Then X contains an isomorph of ℓ_1 (respectively, c_0).*

It is still open if the theorem extends to ℓ_p $(1 < p < \infty)$. There is no isomorphic version of this theorem. For example all spreading models of T are 2-equivalent to the unit vector basis of ℓ_1 yet T does not contain ℓ_1.

References

[AA] D. Alspach and S. Argyros, *Complexity of weakly null sequences*, Diss. Math. **321** (1992), 1–44.

[AO] G. Androulakis and E. Odell, *Distorting mixed Tsirelson spaces*, Israel J. Math. **109** (1999), 125–149.

[AS1] G. Androulakis and Th. Schlumprecht, *Block sequences in S*, preprint.

[AS2] G. Androulakis and Th. Schlumprecht, *On the subsymmetric sequences in S*, preprint.

[AD] S. Argyros and I. Deliyanni, *Examples of asymptotically ℓ^1 Banach spaces*, Trans. A.M.S. **349** (1997), 973–995.

[BL] B. Beauzamy and J.-T. Lapresté, *Modèles étalés des espace de Banach* Travaux en Cours, Herman, Paris, 1984.

[BeL] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis*, a book in preparation.

[Bo1] J. Bourgain, *On convergent sequences of continuous functions*, Bull. Soc. Math. Bel. **32** (1980), 235–249.

[Bo2] J. Bourgain, *The Szlenk index and operators on $C(K)$-spaces*, Bull. Soc. Math. de Belgique, Ser.B. **31** (1979), 87–117.

[Bo3] J. Bourgain, *On finite-dimensional homogeneous Banach spaces*, GAFA Israel Seminar 1986-97, eds. J. Lindenstrauss and V. Milman, LNM 1317 (1988), 232–239.

[C] F. Chaatit, *On the uniform homeomorphisms of the unit spheres of certain Banach lattices*, Pacific J. Math. **168** (1995), 11–31.

[E] P. Enflo, *On a problem of Smirnov*, Ark. Mat. **8** (1969), 107–109.

[FJ] T. Figiel and W.B. Johnson, *A uniformly convex Banach space which contains no ℓ_p*, Compositio Math. **29** (1974), 179–190.

[Gi] T.A. Gillespie, *Factorization in Banach function spaces*, Indag, Math. **43** (1981), 287–300.

[G] W.T. Gowers, *Lipschitz functions on classical spaces*, European J. Combin. **13** (1992), 141–151.

[GM] W.T. Gowers and B. Maurey, *The unconditional basic sequence problem*, J. Amer. Math. Soc. **6** (1993), 851–874.

[HORS] R. Haydon, E. Odell, H. Rosenthal and Th. Schlumprecht, *On distorted norms in Banach spaces and the existence of ℓ_p types*, unpublished manuscript.

[J1] R.C. James, *Uniformly nonsquare Banach spaces*, Ann. of Math. (2)**80** (1964), 542–550.

[KW] N.J. Kalton and D. Werner, *Property (M), M-ideals, and almost isometric structure of Banach spaces*, J. Reine und Agnew. Math. **461** (1995) 137–178.

[Ka] N.J. Kalton, *On subspaces of c_0 and extensions of operators into $C(K)$-spaces*, preprint.

[KOS] H. Knaust, E. Odell and Th. Schlumprecht, *On asymptotic structure, the Szlenk index and UKK properties in Banach space*, Positivity **3** (1999), 173–199.

[K] J.L. Krivine, *Sous espaces de dimension finie des espaces de Banach réticulés*, Ann. of Math. (2) **104** (1976), 1–29.

[L] H. Lemberg, *Nouvelle démonstration d'un théorème de J.L. Krivine sur la finie représentation de ℓ_p dans un espaces de Banach*, Israel J. Math. **39** (1981), 341–348.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer-Verlag, New York, 1977.

[Lo] G. Ya. Lozanovskii, *On some Banach lattices*, Siberian Math. J. **10** (1969), 584–599.

[Mar] D.A. Martin, *Borel determinancy*, Annals of Math. **102** (1975), 363–371.

[Ma1] B. Maurey, *A remark about distortion*, Oper. Theory: Adv. Appl. **77** (1995), 131–142.

[Ma2] B. Maurey, *Symmetric distortion in ℓ_2*, Oper. Theory: Adv. Appl. **77** (1995), 143–147.

[MMT] B. Maurey, V.D. Milman and N. Tomczak-Jaegermann, *Asymptotic infinite-dimensional theory of Banach spaces*, Oper. Theory: Adv. Appl. **77** (1994), 149–175.

[MR] B. Maurey and H. Rosenthal, *Normalized weakly null sequences with no unconditional subsequences*, Studia Math. **61** (1971), 77–98.

[M] V.D. Milman, *Geometric theory of Banach spaces II, geometry of the unit sphere*, Russian Math. Survey **26** (1971), 79–163, (trans. from Russian).

[MS] V.D. Milman and G. Schechtman, *Asymptotic theory of finite dimensional normed spaces*, Lecture Notes in Math., vol. 1200, Springer-Verlag, Berlin and New York, 1986, 156 pp.

[MT] V.D. Milman and N. Tomczak-Jaegermann, *Asymptotic ℓ_p spaces and bounded distortions*, (Bor-Luh Lin and W.B. Johnson, eds.), Contemp. Math. **144** (1993), 173–195.

[MW] V.D. Milman and R. Wagner, *Asymptotic versions of operators and operator ideals*, Convex geometric analysis (Berkeley, CA, 1996), 165-169, Cambridge Univ. Press (1999).

[O] E. Odell, *On subspaces, asymptotic structure and distortions of Banach spaces; connections with logic*, to appear in *Analysis and Logic*, (C. Finet and C. Michaux, eds.).

[ORS] E. Odell, H. Rosenthal and Th. Schlumprecht, *On weakly null FDD's in Banach spaces*, Israel J. Math. **84** (1993), 333–351.

[OS1] E. Odell and Th. Schlumprecht, *Asymptotic properties of Banach spaces under renormings*, J. Amer. Math. Soc. **11** (1998), 175–188.

[OS2] E. Odell and Th. Schlumprecht, *A Banach space block finitely universal for monotone bases*, Trans. Amer. Math. Soc. (to appear).

[OS3] E. Odell and Th. Schlumprecht, *The distortion problem*, Acta Math. **173** (1994), 259–281.

[OS4] E. Odell and Th. Schlumprecht, *On the richness of the set of p's in Krivine's theorem*, Oper. Theory: Adv. Appl. **77** (1995), 177–198.

[OS5] E. Odell and Th. Schlumprecht, *A problem on spreading models*, J. Funct. Anal. **153** (1998), 249–261.

[OS6] E. Odell and Th. Schlumprecht, *Trees and branches in Banach spaces*, preprint.

[OT] E. Odell and N. Tomczak-Jaegermann, *On certain equivalent norms on Tsirelson's space*, Illinois J. Math., to appear.

[OTW] E. Odell, N. Tomczak-Jaegermann and R. Wagner, *Proximity to ℓ_1 and distortion in asymptotic ℓ_1 spaces*, J. Funct. Anal. **150** (1997), 101–145.

[Ri] M. Ribe, *Existence of separable uniformly homeomorphic non isomorphic Banach spaces*, Israel J. Math. **48** (1984), 139–147.

[R1] H. Rosenthal, *Some remarks concerning unconditional basic sequences*, Longhorn Notes: Texas Functional Analysis Seminar 1982-83, University of Texas, Austin, 15–48.

[R2] H. Rosenthal, *Double dual types and the Maurey characterization of Banach spaces containing ℓ^1*, Longhorn Notes: Texas Functional Analysis Seminar 1983-84, University of Texas, Austin, 1–37.

[S1] Th. Schlumprecht, *An arbitrarily distortable Banach space*, Israel J. Math. **76** (1991), 81–95.

[S2] Th. Schlumprecht, *A complementably minimal Banach space not containing c_0 or ℓ_p*, Seminar notes in Functional Analysis and Partial Differential Equations, Baton Rouge, Louisiana, (1992).

[TJ1] N. Tomczak-Jaegermann, *Banach spaces of type p have arbitrarily distortable subspaces*, GAFA **6** (1996), 1074–1082.

[TJ2] N. Tomczak-Jaegermann, *Distortions on Schatten classes C_p*, Operator Theory: Advances and Applications **77** (1995), 327–334.

[T] B.S. Tsirelson, *Not every Banach space contains ℓ_p or c_0*, Functional Anal. Appl. **8** (1974), 138–141.

[W] R. Wagner, *Finite higher-order games and an inductive approach towards Gowers' dichotomy*, preprint.