**Problems to Introduction to Real Analysis, (Math446)**

Due: 09/09/04

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**Problem 1.** For two non empty sets $A, B \subset \mathbb{R}$, show that

$$\inf(A + B) = \inf A + \inf B,$$

where $A + B := \{a + b : a \in A, b \in B\}$.

**Problem 2.** Assume that $(a_n)$ and $(b_n)$ are two sequences and that

$$a = \lim_{n\to\infty} a_n \text{ and } b = \lim_{n\to\infty} b_n$$

exist. Show that

$$a \cdot b = \lim_{n\to\infty} a_n b_n.$$

**Problem 3.** Assume that $f : \mathbb{R} \to \mathbb{R}$, and $g : \mathbb{R} \to \mathbb{R}$ are continuous. Show that $f + g$ and $f \circ g$ (composition) is continuous.

**Problem 4.** For a sequence $(x_n) \subset \mathbb{R}$ we define

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup_{k\ge n} x_k,$$

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf_{k\ge n} x_k,$$

a) State the theorem (without proof) which ensures that above limits exist (possibly $\pm\infty$) , and show that this theorem is applicable.

b) For two sequences $(x_n), (y_n) \subset \mathbb{R}$ show that

$$\limsup_{n\to\infty} x_n + y_n \le \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.$$

c) Find two sequences $(x_n), (y_n) \subset \mathbb{R}$ for which

$$\limsup_{n\to\infty} x_n + y_n \ne \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.$$

**Problem 5.** Assume that for a sequence $(a_n)$ $\limsup_{n\to\infty} a_n = a$. Show that there is a subsequence of $(a_n)$ which converges to $a$.

**Problem 6.** Construction of the rational numbers $\mathbb{Q}$ starting with the integers $\mathbb{Z}$.

On $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ we define the following relation:

$$(p, q) \sim (p', q') \iff pq' = p'q.$$

Show that $\sim$ is an equivalence relation on $\mathbb{Z} \times \mathbb{Z} \setminus \{0\}$.

**Problem 7.** We define $\mathbb{Q}$ to be all the equivalence classes of $\mathbb{Z} \times \mathbb{Z} \setminus \{0\}$ with respect to $\sim$, i.e.,

$$\mathbb{Q} = \{\overline{(p, q)} : p, q \in \mathbb{Z}, q \neq 0\}, \text{ where}$$
$$\overline{(p, q)} = \{(p', q') : p', q' \in \mathbb{Z}, q' \neq 0, \text{ and } p'q = pq'\}.$$

From now on we write $\frac{p}{q}$ instead of $\overline{(p, q)}$ (which is more suggestive).

We define the following operations on $\mathbb{Q}$:

$$\frac{p}{q} + \frac{s}{t} = \frac{pt + sq}{qt} \text{ and } \frac{p}{q} \cdot \frac{s}{t} = \frac{ps}{qt}.$$

Show that these operations are welldefined, and among the axioms of a field, show that $\mathbb{Q}$ verify five of them (any five you want).

**Problem 8.** For $\frac{p}{q}, \frac{s}{t} \in \mathbb{Q}$ we define:

$$\frac{p}{q} < \frac{s}{t} \iff pt < qs.$$

Show that $\mathbb{Q}$ is an ordered field.