

Problems to Introduction to Real Analysis, (Math447)

Due: 1/27/2005

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Problem 1. Let X be a vectorspace over \mathbb{R} . We call a map

$$\|\cdot\| : X \rightarrow [0, \infty)$$

a *norm on X* if the following properties are satisfied:

- a) For all $x \in X$: $\|x\| = 0 \iff x = 0$,
- b) for all $x \in X$ and $\alpha \in \mathbb{R}$: $\|\alpha x\| = |\alpha| \cdot \|x\|$,
- c) for all $x, y \in X$: $\|x + y\| \leq \|x\| + \|y\|$.

In that case we call $(X, \|\cdot\|)$ a *normed linear space* or *normed vectorspace*. Show that the map:

$$d : X \times X \rightarrow [0, \infty), \quad (x, y) \mapsto d(x, y) = \|x - y\|,$$

is a metric.

Problem 2. For $f, g \in C([0, 1])$:

$$d_1(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)| \text{ and } d_2(f, g) = \int_0^1 |f(x) - g(x)| dx.$$

- a) Show that d_1 and d_2 are metrics, and show that d_1 and d_2 , come from a norm as in Problem 1.
- b) Show that: If (f_n) is convergent with respect to d_1 then it is also convergent with respect to d_2 .
- c) Show that there is a sequence (f_n) which converges to 0 for d_2 but not for d_1 .

Problem 3. Let (X, d) be a metric space. For $A \subset X$ define

$$A^\circ = \bigcup \{U : U \subset A \text{ open}\} \text{ and } \bar{A} = \bigcup \{F : F \supset A \text{ closed}\}.$$

A° is called *open kernel of A* and \bar{A} is called *closure of A* .

- a) A° is open,
- b) \bar{A} is closed,
- c) $\overline{A \cup B} = \bar{A} \cup \bar{B}$,
- d) $\bar{A} = A \cup \{x : x \text{ is limit point of } A\}$.

Problem 4. Assume that (X, d) is a metric space. Denote the open sets in X by \mathcal{T}_d and the closed sets by \mathcal{F}_d . Show that

- a) \mathcal{T}_d is closed under taking arbitrary unions, and finite intersections,
- b) \mathcal{F}_d is closed under arbitrary intersections, and finite unions.

Problem 5. On a test a student was asked to write down the property of a function $d : X \times X \rightarrow \mathbb{R}$, which turns d into a metric. She wrote down the following conditions ($x, y, z \in X$ arbitrary):

- (i) $d(x, y) = 0 \iff x = y$,
- (ii) $d(x, y) = d(y, x)$,
- (iii) $d(x, z) \leq d(x, y) + d(y, z)$.

The professor deduced 2 points. Why?

After class she goes to the professor and asserts that these three properties imply that d is a metric. Is she right?

Problem 6. Assume that (X, d) is a metric space. Denote the open sets in X by \mathcal{T}_d and the closed sets by \mathcal{F}_d . Show that

- a) \mathcal{T}_d is closed under taking arbitrary unions, and finite intersections,
- b) \mathcal{F}_d is closed under arbitrary intersections, and finite unions.

Problem 7. Let $(x_i)_{i \in I}$ and $(y_i)_{i \in I}$ be two convergent nets ((I, \leq) directed partial order) in a normed linear space X . Show that:

$$\lim_{i \in I} x_i + y_i = \lim_{i \in I} x_i + \lim_{i \in I} y_i.$$

Problem 8.

- a) Let $A \subset [0, 1]$ be finite and define for $x \in [0, 1]$

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$

Show that f is Riemann integrable and compute its integral.

- b) Define for $x \in [0, 1]$

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is Riemann integrable and compute its integral.

- c) Define for $x \in [0, 1]$

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is not Riemann integrable.