Problems in Real Variables, II (Math608), Solutions
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Problem 1. $(X, \mathcal{T})$ is a topological space. Using only the definition of the notions involved prove for some $A \subset X$:

a) A open $\iff$ $A^\circ = A$ (Recall: A^0 is the open kernel of A, i.e. the union of all open sets contained in A)
b) A close $\iff$ $\overline{A} = A$ (Recall: $\overline{A}$ is the closure of A, i.e. the intersection of closed sets containing A)
c) $\left[A^\circ\right]^c = \overline{A^c}$
d) $\overline{\bigcup_{i=1}^k A_i} = \bigcup_{i=1}^k \overline{A_i}$ if $A_1, A_2, \ldots A_k \subset X$.

Proof:
a) "$\Rightarrow$" Assume A open. Then $A^\circ = \bigcup\{U : U \subset A, U \in \mathcal{T}\} = A$, since $A \in \{U : U \subset A, U \in \mathcal{T}\}$.
"$\Leftarrow$" By definition A° is open, since it is the union of open sets.
c)"$\subset$": if $x \in X \setminus A^\circ$, we have to show for any closed $C \subset X$ with $A^c \subset C$ that $x \in C$ (then $x \in \overline{A^c} = \bigcap\{C \subset \text{ closed } : A^c \subset C\}$).

So let $C \subset X$ be closed with $A^c \subset C$. Since $C^c \subset A$ and C^c open it follows that $C^c \subset A^\circ$ and thus $x \in X \setminus A^\circ \subset C$.
"$\supset$": If $x \in \overline{A^c}$ we have to show that $x \notin A^\circ$. Assume that $x \in A^\circ$ and, thus, that there is an $U \subset A$, U open, with $x \in U$. But this means $x \notin U^c \supset \overline{A^c}$ which contradicts that $x \in \overline{A^c}$.
b) If we apply (a) and (c) (for A^c instead of A) we get

$$A \text{ closed} \iff A^c \text{ open} \iff [A^c]^\circ = A^c \iff \left[[A^c]^\circ\right]^c = A \iff \overline{[A^c]^c} = A \iff \overline{A} = A.$$

d) It is good enough to show the claim for $n = 2$ (rest follows by induction).
$\overline{A_1 \cup A_2} \subset \overline{A_1} \cup \overline{A_2}$ because $\overline{A_1} \cup \overline{A_2}$ is a closed set which contains $A_1 \cup A_2$.
Also $\overline{A_1} \subset \overline{A_1 \cup A_2}$ and $\overline{A_2} \subset \overline{A_1 \cup A_2}$ since $\overline{A_1 \cup A_2}$ is a closed set which contains A_1 and A_2 respectively.

Problem 2. Assume X and Y are topological spaces and $f : X \to Y$. The following are equivalent.

a) f is continuous.
b) $\forall A \subset X \quad f(\overline{A}) \subset \overline{f(A)}$.
c) $\forall B \subset Y \quad \overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$.

Proof. (a) $\Rightarrow$ (b) Assume $y \in f(\overline{A})$ and thus $y = f(x)$ for some $x \in \overline{A}$. To show that for all open $V \subset Y$ containing y it follows that $V \cap f(A) \neq \emptyset$.

Let $V \subset Y$ be open and containing y then $f^{-1}(V)$ is open in X and contains x and thus $A \cap f^{-1}(V) \neq \emptyset$, which implies that

$$V \cap f(A) \supset f(A \cap f^{-1}(V)) \neq \emptyset.$$

(b) $\Rightarrow$ (c) We first note that fot any $A \subset X$ it follows that $A \subset f^{-1}(f(A))$ and it follows for $B \subset Y$ that $B \supset f(f^{-1}(B))$.

Therefore it follows for $B \subset Y$ from assumption (b) that

$$\overline{f^{-1}(B)} \subset f^{-1}\Big(f\big(\overline{f^{-1}(B)}\big)\Big) \subset f^{-1}\overline{f(f^{-1}(B))} \subset f^{-1}(\overline{B}).$$

(c) $\Rightarrow$ (a) Let $B \subset Y$ be closed. To show that $f^{-1}(B) \subset X$ is closed. But this follows from the following:

$$\overline{f^{-1}(B)} \subset f^{-1}(\overline{B}) = f^{-1}(B) \subset \overline{f^{-1}(B)},$$

and thus $f^{-1}(B) = \overline{f^{-1}(B)}$ which means that $f^{-1}(B)$ is closed.

Problem 3. Note (trivial) that the set

$$\mathcal{B} = \{[a, b) : a < b\},$$

is a base of a topology on $\mathbb{R}$ which we denote by $\mathcal{T}'$. The usual topology on $\mathbb{R}$ is denoted by $\mathcal{T}$.

a) $\mathcal{T}'$ is strictly finer than $\mathcal{T}$.
b) The elements of $\mathcal{B}$ are not only open but also closed.
c) $(\mathbb{R}, \mathcal{T}')$ is first countable (every point has a countable neighborhood basis) but not second countable ($\mathcal{T}'$ admits a countable basis).
d) $(\mathbb{R}, \mathcal{T}')$ is separable.

Proof.
a) $\mathcal{T}$ is generated by the open intervals and for each open interval (a, b) we have $(a, b) = \bigcup_{n\in\mathbb{N}}[a + 1/n, b)$. Therefore $\mathcal{T}'$ is finer $\mathcal{T}$. Since for example $[0, 1) \in \mathcal{T}'$ but $[0, 1) \notin \mathcal{T}$, $\mathcal{T}'$ is strictly finer.
b) for $a < b$, $[a, b) = \mathbb{R} \setminus [(-\infty, a) \cup [b, \infty)$, thus $[a, b)$ is the complement of an open set and therefore must also be closed.
c) $(\mathbb{R}, \mathcal{T}')$ is first countable: for any $a \in \mathbb{R}$, the set $\mathcal{B}_x = \{[a, a+1/n) : n \in \mathbb{N}\}$ is a neighborhood basis.

$(\mathbb{R}, \mathcal{T}')$ is not second countable: Assume that $\mathcal{B}$ is a basis for $(\mathbb{R}, \mathcal{T}')$. In particular this means that for each $a \in \mathbb{R}$ there must be a subset $\mathcal{B}_a$ of $\mathcal{B}$ which is a neighborhood basis of a. Now let $a \in \mathbb{R}$. Since $[a, a+1)$ is an $\mathcal{T}'$-open set which contains a we must find a $V_a \in \mathcal{B}_a$ so that $a \in V_a \subset [a, a+1)$ but this means that V_a has a minimum and it is a. In particular we deduce that for any two distinct a, a' in $\mathbb{R}$ $V_a \neq V_{a'}$, which means that $\mathcal{B}$ contains the uncountable family $(V_a)_{a\in\mathbb{R}}$ and, thus, must be uncountable itself.
d) We will proof that also in $\mathcal{T}'$ it follows that $\overline{\mathbb{Q}} = \mathbb{R}$. Assume not and assume that $x \in \mathbb{R} \setminus \overline{\mathbb{Q}}$. Since $\mathbb{R} \setminus \overline{\mathbb{Q}}$ is open there should be $a < b$ so that $[a, b) \subset \mathbb{R} \setminus \overline{\mathbb{Q}}$, which is a contradiction since $[a, b) \cap \mathbb{Q} \neq \emptyset$.

Problem 4. Metric spaces are normal.

Proof. Assume $A, B \subset X$ are closed and disjoint (X has metric d). For each $x \in A$ we find an $\varepsilon_x > 0$ so that $B_{\varepsilon_x}(x) \subset B^c$ and for any $y \in B$ we find an $\varepsilon_y > 0$ so that $B_{\varepsilon_y}(y) \subset A^c$ (A^c and B^c are open sets, and thus unions of open balls). Now take $U = \bigcup_{x\in A} B_{\varepsilon_x/2}(x)$ and $V = \bigcup_{y\in B} B_{\varepsilon_y/2}(y)$. Of course U and V are open sets containing A and B respectively. To show that $U \cap V = \emptyset$. Assume $z \in U \cap V$, which means that there is an $x \in A$ so that $d(x, z) < \varepsilon_x/2$ and there is a $y \in B$ so that $d(y, z) < \varepsilon_y/2$. Without loss of generality, we can assume that $\varepsilon_x \geq \varepsilon_y$ (in the other case the proof is the same). Now it follows that $d(x,y) \leq d(x,z)+d(y,z) < \varepsilon_x/2+\varepsilon_y/2 \leq \varepsilon_x$. There fore $B_{\varepsilon_x}(x) \cap B \neq \emptyset$, which contradicts the definition of ε_x.

Problem 5. If X is an infinite set with the cofinite topology and (x_n) a sequence of distinct points then: $\lim_{n\to\infty} x_n = x$ for any $x \in X$.
Proof. Let $(x_n) \subset X$ with $x_n \neq x_m$ whenever $n \neq m$ and let $x \in X$.

Let U be a neighborhood of x, i.e. $U = X \setminus \{z_1, z_2, \dots z_n\}$ with $n \in \mathbb{N}$, $\{z_1, z_2, \dots z_n\} \subset X$, $z_i \neq x$, for $i = 1, 2, \dots, n$.

Since the elements of (x_n) are distinct we must find for each $i \in \{1, 2, \dots n\}$ an $N_i \in \mathbb{N}$ so that $x_n \neq z_i$ for all $n \geq N_i$. Now take $N = \max N_i$, and we deduce that for all $n \geq N$ it follows that $x_n \notin \{z_1, z_2, \dots z_n\}$ and thus $x_n \in U$.

Problem 6. Let Ω be an uncountable set. By the Well Ordering Principle (see page 5) there is a linear well order $<$ on Ω. We put $\overline{\Omega} = \Omega \cup \{\Omega\}$ and put $\alpha < \Omega$, for all $\alpha \in \Omega$. Then $(\overline{\Omega}, <)$ is a well ordered set with a maximal element (namely Ω).

We denote $0 := \min \Omega$, and for $\alpha < \beta$ in Ω we write

$$(\alpha, \beta) := \{\gamma \in \Omega : \alpha < \gamma < \beta\},$$

similarly we define closed and half open intervals (α, β), $(\alpha, \beta]$, $[\alpha, \beta)$. Define

$$\omega_1 := \min\{\alpha \in \Omega : [0, \alpha) \text{ is uncountable}\}$$

(why does ω_1 exists?)

(1) The open intervals $\{(\alpha, \beta), [0, \beta), (\alpha, \Omega] : \alpha, \beta \in \Omega\}$ are the basis of a topology $\mathcal{T}$, the *Order Topology on* Ω.
(2) Every sequence in $[0, \omega_1)$ has a convergent subsequence whose limit is also in $[0, \omega_1)$.
(3) $[0, \omega_1)$ is not closed.

Proof. First we note that ω_1 exists, since the set

$$S = \{\alpha \in \overline{\Omega}; [0, \alpha] \text{ is uncountable}\}$$

is not empty (for example $\Omega \in S$), and has therefore a minimum (well order!).
a) Follows immediately by verification that conditions (a) and (b)of Proposition 4.3 (page 115) are satisfied.
b) Let $(\alpha_n : n \in \mathbb{N}) \subset [0, \omega_1)$. Choose $n_1 \in \mathbb{N}$ so that

$$\alpha_{n_1} = \min(\alpha_n : n \in \mathbb{N}),$$

and assuming $n_1 < n_2 < \dots n_k$ has been chosen, choose n_{k+1} so that

$$\alpha_{n_{k+1}} = \min(\alpha_n : n > n_{k+1}).$$

It follows that $(\alpha_{n_j} : j \in \mathbb{N})$ is non decreasing, i.e $\alpha_{n_1} \leq \alpha_{n_2} \leq \cdot$. Thus either $(\alpha_{n_j} : j \in \mathbb{N})$ is eventually constant, or it has a strictly increasing subsequence, say $(\alpha'_j : j \in \mathbb{N})$ In the first case we are done, in the second case we define

$$\alpha : \sup_{n\in\mathbb{N}} \alpha' := \min\{\beta \in \overline{\Omega} : \forall\, j \in \mathbb{N} \quad \beta > \alpha'_j\}.$$

(note that set is not empty and that therefore min exists) We claim that $\alpha = \lim_{n\to\infty} \alpha'_n$ and $\alpha < \omega_1$. Indeed, the first claim follows that for any open interval (γ_1, γ_2) or $\gamma_1, \Omega]$, with $\gamma_1 < \alpha < \gamma_2$ (note that $\alpha \neq 0$, thus the a neighborhood basis of α consists of open interval of that form), it follows that γ_1 is not in the set $\{\beta \in \overline{\Omega} : \forall\, j \in \mathbb{N} \quad \beta > \alpha'_j\}$, and thus there is a k_0 so that $\gamma_1 < \alpha_{k_0} < \alpha_k < \alpha$ for all $k \in k_0$. The second claim follows from the fact that the

$$[0, \alpha) = \bigcup_{k\in\mathbb{N}} [0, \alpha'_k),$$

is (as countable union of countable sets) countable.
c) We claim that $\omega_1 \in \overline{[0, \omega_1)}$ (actually then it follows that $\overline{[0, \omega_1)} = [0, \omega_1]$, since either $\omega_1 = \Omega$ or there is an element $\omega_1 + 1 := \min\{\alpha : \alpha > \omega_1\}$, and then $[0, \omega_1] = (\omega_1 + 1, \Omega]^c$, thus $[0, \omega_1]$ is closed).

In order to show $\omega_1 \in \overline{[0, \omega_1)}$, assume, to the contrary that ω_1 is in (the open set) $\overline{\Omega} \setminus \overline{[0, \omega_1)}$. It follows that there is a $\gamma_1 < \omega_1$ so that $(\gamma_1, \omega_1] \subset \overline{\Omega} \setminus \overline{[0, \omega_1)} \subset \overline{\Omega} \setminus [0, \omega_1)$. By definition of ω_1 it follows that $[0, \gamma_1)$ is countable, thus also $[0, \gamma_1] = [0, \gamma_1 + 1)$ (with $\gamma_1 + 1 := \min\{\alpha : \alpha > \gamma_1\}$). Since $\gamma_1 + 1 < \omega_1$, but on the other hand $\gamma_1 \in \overline{\Omega} \setminus [0, \omega_1)$ we derive a contradiction.

Problem 7. Consider $X = [-1, 1]^{[-1,1]}$ with its product topology. You can think of

$$X = \{f | f : [-1, 1] \to [-1, 1]\}$$

Show that $(f_n) \subset X$ converges in X if and only $(f_n(t) : n \in \mathbb{N})$ converges for all t.

Show that there is a sequence in X which does not have a convergent subsequence.

Proof For $n \in \mathbb{N}_0$ we define the following function $f_n : [-1, 1] \to [-1, 1]$

$$f_n(t) = \sum_{k=-2^n}^{2^n - 1} (-1)^k \chi_{[k2^{-n}, (k+1)2^{-n})}.$$

Note

$$f_0 = -\chi_{[-1,0)} + \chi_{[0,1)}$$
$$f_1 = \chi_{[-1,-2^{-1})} - \chi_{[-2^{-1},0)]} + \chi_{[0,2^{-1}]} - \chi_{[2^{-1},1)]}$$
$$\vdots$$

Now for any two natural numbers $m \neq n$ one can find a $t \in [-1, 1]$ so that $f_m(t) = 1$ and $f_n(t) = -1$. But this means there cannot exist a point wise converging subsequence of (f_n).