Problems in Real Variables, II (Math608) **Due: 01/27/10**

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Problem 1. $(X,\mathcal{T})$ is a topological space. Using only the definition of the notions involved prove for some $A\subset X$:

a) A open $\iff$ $A^\circ = A$ (Recall: A^0 is the open kernel of A, i.e. the union of all open sets contained in A)
b) A close $\iff$ $\overline{A}=A$ (Recall: $\overline{A}$ is the closure of A, i.e. the intersection of closed sets containing A)
c) $\big[A^\circ\big]^c = \overline{A^c}$
d) $\overline{\bigcup_{i=1}^k A_i} = \bigcup_{i=1}^k \overline{A_i}$ if $A_1, A_2, \dots A_k \subset X$.

Problem 2. Assume X and Y are topological spaces and $f: X\to Y$. The following are equivalent.

a) f is continuous.
b) $\forall A\subset X \quad f(\overline{A})\subset \overline{f(A)}$.
c) $\forall B\subset Y \quad \overline{f^{-1}(B)}\subset f^{-1}(\overline{B})$.

Problem 3. Note (trivial) that the set

$$\mathcal{B} = \{[a,b) : a<b\},$$

is a base of a topology on $\mathbb{R}$ which we denote by $\mathcal{T}'$. The usual topology on $\mathbb{R}$ is denoted by $\mathcal{T}$.

a) $\mathcal{T}'$ is strictly finer than $\mathcal{T}$.
b) The elements of $\mathcal{B}$ are not only open but also closed.
c) $(\mathbb{R},\mathcal{T}')$ is first countable (every point has a countable neighborhood basis) but not second countable ($\mathcal{T}'$ admits a countable basis).
d) $(\mathbb{R},\mathcal{T}')$ is separable.

Problem 4. Metric spaces are normal.

Problem 5. If X is an infinite set with the cofinite topology and (x_n) a sequence of distinct points then: $\lim_{n\to\infty} x_n = x$ for any $x\in X$.

Problem 6. Let Ω be an uncountable set. By the Well Ordering Principle (see page 5) there is a linear well order $<$ on Ω. We put $\overline{\Omega}=\Omega\cup\{\Omega\}$ (which means we add one more element to Ω) and put $\alpha<\Omega$, for all $\alpha\in\Omega$. Then $(\overline{\Omega},<)$ is a well ordered set with a maximal element (namely Ω).

We denote $0 := \min\Omega$, and for $\alpha<\beta$ in Ω we write

$$(\alpha,\beta) := \{\gamma\in\Omega : \alpha<\gamma<\beta\},$$

similarly we define closed and half open intervals (α,β), $(\alpha,\beta]$, $[\alpha,\beta)$. Define

$$\omega_1 := \min\{\alpha\in\Omega : [0,\alpha] \text{ is uncountable}\}$$

(why does ω_1 exist?)

(1) The open intervals $\{(\alpha,\beta),[0,\beta),(\alpha,\Omega] : \alpha,\beta\in\Omega\}$ are the basis of a topology $\mathcal{T}$, the *Order Topology on* Ω.

(2) Every sequence in $[0, \omega_1)$ has a convergent subsequence whose limit is also in $[0, \omega_1)$.
(3) $[0, \omega_1)$ is not closed.

Problem 7. Consider $X = [-1, 1]^{[-1,1]}$ with its product topology. You can think of

$$X = \{f | f : [-1, 1] \to [-1, 1]\}$$

Show that $(f_n) \subset X$ converges in X if and only $(f_n(t) : n \in \mathbb{N})$ converges for all t.

Show that there is a sequence in X which does not have a convergent subsequence.