Problems to Introduction to Real Analysis, (Math447)

Due: 2/3/2005

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Problem 1. Show that the upper sums $U(P, f\alpha)$, (P partition of $[a,b]$ $f : [a,b] \to \mathbb{R}$, $\alpha : [a,b] \to \mathbb{R}$ increasing), are decreasing in P.

Problem 2. Show that for a continous function $f : [a,b] \to R$ and an increasing $\alpha : [a,b] \to R$

$$\int_a^b f d\alpha = \lim_{n\to\infty} U(P_n, f, \alpha),$$

where

$$P_n = \left(a, a + \frac{b-a}{n}, 2\frac{b-a}{n}, \ldots, (n-1)\frac{b-a}{n}, b\right).$$

Problem 3. Define for $t \in [0,1]$

$$\alpha(t) = \begin{cases} 0 & \text{if } t < \frac{1}{2} \\ \frac{1}{2} & \text{if } t = \frac{1}{2} \\ 1 & \text{if } t > \frac{1}{2} \end{cases}$$

Give a simple critereon for a function $f : [0,1]$ to be RS-integrable with respect to α and compute $\int_a^b f d\alpha$ in that case.

Problem 4. For a function $f : [0,1] \to \mathbb{R}$ define

$$I(f) = \lim_{c\to 0} \int_c^1 f(x)dx,$$

if that limit exist. Show that if f is Riemann integrable (with respect to dx) then

$$I(f) = \int_0^1 f(x)dx,$$

and find an example of an f which is not Riemann integrable, but so that $I(f)$ exists.

Problems 5.+6.(counts double) Let p and q be two positive numbers so that

$$\frac{1}{p} + \frac{1}{q} = 1$$

and let $\alpha : [a,b] \to \mathbb{R}$ be increasing.

a) Show that for $u, v \geq 0$

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

b) assume that f, g are non negative functions for which f^p and g^q are in $\mathcal{R}(\alpha)$. Show that $fg \in \mathcal{R}(\alpha)$, and that

$$\int_a^b fg d\alpha \leq \left[\int_a^b f^p d\alpha\right]^{1/p} \cdot \left[\int_a^b g^q d\alpha\right]^{1/q}.$$

(Special case of the *Hoelder-Inequality*)
Hint: first assume that

$$\left[\int_a^b f^p d\alpha\right] = \left[\int_a^b g^q d\alpha\right] = 1.$$