Problems in Real Variables, II (Math608) **Due: 02/03/10**
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Problem 1. Assume X and Y are topological spaces, Y being Hausdorff, and $f, g : X \to Y$ are continuous.

a) $\{x \in X : f(x) = g(x)\}$ is closed in X.
b) The claim in (a) is not necessarily true without the assumption that Y is Hausdorff.
c) If $f(x) = g(x)$ for all x out of a dense subset of X, then $f = g$.

Problem 2. Let $\mathcal{F}$ be a set of realvalued function on a set X and let $\mathcal{T}$ be the weak topology on X generated by $\mathcal{F}$. Then

$$(X, \mathcal{T}) \text{ Hausdorff} \iff \forall x, y \in X, x \neq y, \exists f \in \mathcal{F} \quad f(x) \neq f(y).$$

Problem 3. Only using the definition of net, convergent net and subnet, show

a) The subnet of a subnet is a subnet.
b) The subnet of a convergent net converges to the same limit.

Problem 4. If A is a directed set, a subset B of A is called *co final* if for each $\alpha \in A$ there exists $\beta \in B$ so that $\beta \geq \alpha$.

a) If B is co final in A and $(x_\alpha)_{\alpha \in A}$ is a net, the inclusion map $B \to A$ makes $(x_\beta)_{\beta \in B}$ a subnet of $(x_\alpha)_{\alpha \in A}$.
b) If $(x_\alpha)_{\alpha \in A}$ is a net in a topological space X and $x \in X$ then

$$(x_\alpha)_{\alpha \in A} \text{ converges to } x \iff$$
$$\forall B \subset A \text{ co final } \exists C \subset B \text{ co final} \quad (x_\gamma)_{\gamma \in C} \text{ converges to } x.$$

Remark: Note that (b) is the analogous statement of: "a sequence in a metric space converges to x if and only if every subsequence has a further subsequence which converges to x."

Problem 5. If X is Hausdorff, then any net in X converges to a at most one element

Problem 6. Let $X = [0, 1]^{[0,1]}$ and consider on X the product topology. Define

$$A := \Big\{(x_t)_{t \in [0,1]} \in X : \{t \in [0, 1] : x_t \neq 0\} \text{ is countable}\Big\}.$$

Show that every sequence in A has a convergent subsequence whose limit is still in A.

Problem 7. Show that the set A in Problem 6 is not compact.

Problem 8. Assume that $(\overline{\Omega}, <)$ is the wellordered set from Problem 6 in first Homework. Show that every closed interval is compact in the order topology.