Problem 1. Assume \( X \) and \( Y \) are topological spaces, \( Y \) being Hausdorff, and \( f, g : X \to Y \) are continuous.

a) \( \{ x \in X : f(x) = g(x) \} \) is closed in \( X \).

b) The claim in (a) is not necessarily true without the assumption that \( Y \) is Hausdorff.

c) If \( f(x) = g(x) \) for all \( x \) out of a dense subset of \( X \), then \( f = g \).

Problem 2. Let \( \mathcal{F} \) be a set of real-valued function on a set \( X \) and let \( T \) be the weak topology on \( X \) generated by \( \mathcal{F} \). Then

\[(X, T) \text{ Hausdorff } \iff \forall x, y \in X, x \neq y, \exists f \in \mathcal{F} \quad f(x) \neq f(y).\]

Problem 3. Only using the definition of net, convergent net and subnet, show

a) The subnet of a subnet is a subnet.

b) The subnet of a convergent net converges to the same limit.

Problem 4. If \( A \) is a directed set, a subset \( B \) of \( A \) is called co final if for each \( \alpha \in A \) there exists \( \beta \in B \) so that \( \beta \geq \alpha \).

a) If \( B \) is co final in \( A \) and \( (x_\alpha)_{\alpha \in A} \) is a net, the inclusion map \( B \to A \) makes \( (x_\beta)_{\beta \in B} \) a subnet of \( (x_\alpha)_{\alpha \in A} \).

b) If \( (x_\alpha)_{\alpha \in A} \) is a net in a topological space \( X \) and \( x \in X \) then \( (x_\alpha)_{\alpha \in A} \) converges to \( x \iff \forall B \subset A \text{ co final } \exists C \subset B \text{ co final } (x_\gamma)_{\gamma \in C} \text{ converges to } x. \)

Remark: Note that (b) is the analogous statement of: ”a sequence in a metric space converges to \( x \) if and only if every subsequence has a further subsequence which converges to \( x \).”

Problem 5. If \( X \) is Hausdorff, then any net in \( X \) converges to a at most one element.

Problem 6. Let \( X = [0, 1]^{[0,1]} \) and consider on \( X \) the product topology. Define

\[ A := \left\{ (x_t)_{t \in [0,1]} \in X : \{t \in [0,1] : x_t \neq 0 \} \text{ is countable} \right\}. \]

Show that every sequence in \( A \) has a convergent subsequence whose limit is still in \( A \).

Problem 7. Show that the set \( A \) in Problem 6 is not compact.

Problem 8. Assume that \((\mathbb{N}, <)\) is the wellordered set from Problem 6 in first Homework. Show that every closed interval is compact in the order topology.