**Problem 1.** Assume $X$ and $Y$ are topological spaces, $Y$ being Hausdorff, and $f, g : X \to Y$ are continuous.

a) $\{x \in X : f(x) = g(x)\}$ is closed in $X$.

b) The claim in (a) is not necessarily true without the assumption that $Y$ is Hausdorff.

c) If $f(x) = g(x)$ for all $x$ out of a dense subset of $X$, then $f = g$.

**Proof.** (a) We have to show that $A = \{x \in X : f(x) \neq g(x)\}$ is open in $X$. If $x \in X$ with $f(x) \neq g(x)$ we can find two open and disjoint neighborhoods $U$ and $V$ of $f(x)$ and $g(x)$ respectively. Let $W = f^{-1}(U) \cap g^{-1}(V)$ which is open since $f$ and $g$ are continuous. Now note that for any $z \in W$ it follows that $f(z) \in U$ and $g(z) \in V$ which implies that $f(z) \neq g(z)$ and, thus, $W \subset A$, since $x \in A$ was arbitrary we proved that $A$ is open.

(b) Example: $X = \mathbb{N}$ with the "tail-topology" $\mathcal{T} = \{[n, \infty) \cap \mathbb{N} : n \in \mathbb{N}\} \cup \{\emptyset\}$.

$f, g : \mathbb{N} \to \mathbb{N}$ $f(n) = 10$ and $g(n) = n$ for all $n \in \mathbb{N}$. Then $\{n \in \mathbb{N} : f(n) = g(n)\} = \{10\}$ is not closed.

(c) Follows from (a): If the set $D = \{x \in X : f(x) = g(x)\}$ is dense and (by (a) closed) it follows that $X = \overline{D} = D$.

**Problem 2.** Let $\mathcal{F}$ be a set of realvalued function on a set $X$ and let $\mathcal{T}$ be the weak topology on $X$ generated by $\mathcal{F}$. Then

$$(X, \mathcal{T}) \text{ Hausdorff } \iff \forall x, y \in X, x \neq y, \exists f \in \mathcal{F} \ f(x) \neq f(y).$$

**Proof.** Note that

$$\mathcal{B} = \left\{ \bigcap_{i=1}^{n} f_{i}^{-1}(U_{i}) : \begin{array}{c} n \in \mathbb{N}, \\ f_{1}, f_{2}, \ldots, f_{n} \in \mathcal{F} \\ U_{1}, U_{2}, \ldots, U_{n} \subset \mathbb{R} \text{ open} \end{array} \right\}.$$

"$\Rightarrow$" Assume $x, y \in X$ and $f(x) = f(y)$. Then it follows for all $x \in \mathcal{B} \iff y \in \mathcal{B}$, thus there is no element of $\mathcal{B}$ which separates $x$ from $y$. Since $\mathcal{T}$ is the set of all unions of elements in $\mathcal{B}$ the points $x$ and $y$ cannot be separated by elements in $\mathcal{T}$.

"$\Leftarrow$" Let $x, y \in X, x \neq y$, then there is an $f \in \mathcal{F}$, so that $f(x) \neq f(y)$

Since $\mathbb{R}$ is $T_{2}$ we find two open, disjoint sets $U, V \subset \mathbb{R}$ so that $f(x) \in U$ and $f(y) \in V$. Note that then $f^{-1}(U)$ and $f^{-1}(V)$ are two disjoint open neighborhoods of $x$ and $y$ respectively.

**Problem 3.** Only using the definition of net, convergent net and subnet, show

(1) The subnet of a subnet is a subnet.

(2) The subnet of a convergent net converges to the same limit.
**Problem 4.** If $A$ is a directed set, a subset $B$ of $A$ is called *co final* if for each $\alpha \in A$ there exists $\beta \in B$ so that $\beta \geq \alpha$.

(a) If $B$ is co final in $A$ and $(x_\alpha)_{\alpha \in A}$ is a net, the inclusion map $B \to A$ makes $(x_\beta)_{\beta \in B}$ a subnet of $(x_\alpha)_{\alpha \in A}$.

(b) If $(x_\alpha)_{\alpha \in A}$ is a net in a topological space $X$ and $x \in X$ then $(x_\alpha)_{\alpha \in A}$ converges to $x$ if and only if for every open set $V \subseteq A$ there exists $\beta \in B$ such that $x_\beta \in V$ for all $\beta \geq \alpha$.

**Proof** (a) clear (simply observe that the two conditions of the definition of each $\alpha$.

Thus there exists an $\alpha \in A$ such that $x_\alpha \in U$ whenever $\alpha \geq \alpha_0$. Since $C = B$ is co final in $A$ we can choose a $\gamma_0 \in C$ with $\gamma_0 \geq \alpha_0$. Now it follows that for all $\gamma \geq \gamma_0$, $\gamma \in C \subset A$ that $x_\gamma \in U$, which finishes the argument.

(b) “⇒” Assume $x = \lim_{\alpha \in A} x_\alpha$. Let $B \subseteq A$ be co final. Simply choose $C = B$ (which is clearly co final in $B$). Now let $U$ be a neighborhood of $x$. Thus there exists an $\alpha = 0 \in A$ so that for all $\alpha \geq \alpha_0$ it follows that $x_\alpha \in U$ whenever $\alpha \geq \alpha_0$. Since $C = B$ is co final in $A$ we can choose a $\gamma_0 \in C$ with $\gamma_0 \geq \alpha_0$. Now it follows that for all $\gamma \geq \gamma_0, \gamma \in C \subset A$ that $x_\gamma \in U$, which finishes the argument.

“⇒” Assume $(x_\alpha)_{\alpha \in A}$ is not converging to $x$. We therefore can find a neighborhood $U$ of $x$ so that for all $\alpha$ we can pick a $\beta_\alpha \geq \alpha, \beta_\alpha \in A$ so that $x_\beta_\alpha \notin U$. Now let $B := \{\beta_\alpha : \alpha \in A\} \subseteq A$.

Note that $B$ is co final (simply because for any $\alpha$ we picked $\beta_\alpha \geq \alpha$). Secondly for all $\beta \in B$ we have $x_\beta \notin U$ thus not for single $\gamma \in C$, with $C \subset A$ co final, we have $x_\gamma \notin U$. Thus we proved

$$\exists B \subseteq A \text{ co final } \forall C \subseteq B \text{ co final } \forall \gamma_0 \in C \exists \gamma \geq \gamma_0 \ x_\gamma \notin U,$$

which is the negation of the “right side”.

**Problem 5.** If $X$ is Hausdorff, then any net in $X$ converges to a at most one element

**Proof.** Let $x \neq y$ and $x = T^{-\lim}_{i \in I} x_i$. Since $X$ is Hausdorff there are open sets $U$ and $V$ so that $x \in U, y \in V$ and $U \cap V = \emptyset$. Since $x = T^{-\lim}_{i \in I} x_i$, there is an $i_0 \in I$ so that $x_i \in U$, for all $i \geq i_0$. But this means that $x_i \in U$ for all $i \geq i_0$, thus, if there existed an $i_1$ so that $x_i \in V$, if $i \geq i_1$, we could choose $i_2 \in I$ so that $i_2 \geq i_1$ and $i_2 \geq i_0$, and thus $x_{i_2} \in U \cap V$, which contradicts the assumption that $U$ and $V$ are disjoint.

**Problem 6.** Let $X = [0,1]^{[0,1]}$ and consider on $X$ the product topology.

Define

$$A := \left\{ (x_t)_{t \in [0,1]} : \{ t \in [0,1] : x_t \neq 0 \} \text{ is countable} \right\}.$$

Show that every sequence in $A$ has a convergent subsequence whose limit is still in $A$.

**Proof.** We think of the elements in $X$ being maps $x : [0,1] \to [0,1], t \mapsto x(t)$. Assume that $x_n \in A$ for $n \in \mathbb{N}$. For $n \in \mathbb{N}$ define the (countable) set $C_n = \{ t \in [0,1] : x_n(t) \neq 0 \}$. Then also $C = \bigcup_{n \in \mathbb{N}} C_n$ is countable. Write $C = \{ t_k : k \in \mathbb{N} \}$. For fixed $k \in \mathbb{N}$ the sequence $(x_n(t_k))_{k \in \mathbb{N}}$ is in the compact set $[0,1]$ and must therefore have a convergent subsequence.
We proceed now as follows ("diagonalization argument"): First find convergent subsequence of \((x_n(t_1))_{n \in \mathbb{N}}\). Denote it by \((x_{n_1^{(1)}}(t_1))_{i \in \mathbb{N}}\) and write
\[
x(t_1) = \lim_{i \to \infty} x_{n_1^{(1)}}(t_1).
\]
Then find a further subsequence \((n_i^{(2)}) \subset (n_i^{(1)})\) so that also \((x_{n_i^{(2)}}(t_2))_{n \in \mathbb{N}}\) converges, write
\[
x(t_2) = \lim_{i \to \infty} x_{n_i^{(2)}}(t_2).
\]
Continue this way, finding successive subsequences of \(\mathbb{N}\), i.e.

\[\mathbb{N} \supset (n_i^{(1)}) \supset (n_i^{(2)}) \supset (n_i^{(3)}) \supset \ldots\]

so that for every \(k \in \mathbb{N}\)
\[
x(t_k) = \lim_{i \to \infty} x(t_{n_i^{(k)}}),\]
exists.

For \(t \in [0,1] \setminus C\) put \(x(t) = 0\).

Now take the diagonal sequence of the sequences \((n_i^{(k)})\), \(k = 1, 2, \ldots\), namely \(m_i = n_i^{(i)}\) and note that \((m_i)\) is up to the first \(k\) element (which are irrelevant for taking limits) a subsequence of each \((n_i^{(k)})\). Therefore we have for each \(k \in \mathbb{N}\)
\[
x(t_k) = \lim x_{m_i}(t_k).
\]
We also have trivially for \(t \in [0,1] \setminus C\)
\[
x(t) = 0 = x_{m_i}(t)\]
for all \(i \in \mathbb{N}\).

This will imply that \(x_{m_i}\) converges in the product topology to \(x\). Indeed, take any open neighborhood \(U\) of \(x(t)\). We can assume that \(U\) is of the form
\[
U = \prod_{t \in [0,1]} U_t,
\]
where \(U_t \subset [0,1]\) open and containing the number \(x(t)\), and further more only for a finite set \(F = \{t_1, t_2, \ldots t_\ell\} \subset [0,1]\) we have \(U_t \neq [0,1]\), and other wise \(U_t = [0,1]\).

By above convergence it follows that there must be for each \(k = 1, \ldots \ell\) an \(N_k\) so that \(x_{m_i}(t_k) \in U_{t_k}\) whenever \(i \geq N_k\). Take \(N = \max_{k \leq \ell} N_k\) and deduce that \(x_{m_i}(t_k) \in U_{t_k}\) whenever \(i \geq N\) and \(k \in \{1, 2 \ldots \ell\}\). But this means that \(x_{m_i} \in U\) for all \(i \geq N\).

Secondly, in order to show that \(A\) is not compact we will show that we can find a net in \(A\) which converges to the constant map 1 (meaning \([0,1] \ni t \mapsto 1\)) which is clearly not in \(A\). Consider
\[
\mathcal{I} = \{C \subset [0,1] : C \text{ countable }\}.
\]
on \(C\) we use the order defined by \(C \supset C' \iff C \supset C'\). Secondly define for each \(C \in \mathcal{I}\) the following element \(x_C\) in \(A\).
\[
x_C : [0,1] \to [0,1], \quad t \mapsto \begin{cases} 1 & \text{if } t \in C \\ 0 & \text{if } t \notin C\end{cases}
\]
We claim that the net \( (x_C)_{C \in I} \) converges to the constant 1. Let \( U \) be an element of the neighborhood basis of \( X \), i.e. \( U \) is of the form

\[
U = \prod_{t \in [0,1]} U_t,
\]

with \( U_t \) open in \([0,1]\) for all \( t \in [0,1] \) and \( F = \{ t : U_t \neq [0,1] \} \) finite. We have to find a \( C \in I \) so that for all \( C' \in I \) with \( C' \supset C \) it follows that \( x_{C'} \in U \). Note that for \( t \in F \) \( 1 \in U_t \). Well... simply take \( C = F \) and notice that for all \( C' \in I \) with \( C' \supset C \) it follows that \( x_{C'}(t) = 0 \in [0,1] = U_t \), if \( t \notin C' \), \( x_{C'}(t) = 1 \in U_t \) if \( t \in F \), and \( x_{C'}(t) = 1 \in [0,1] = U_t \) if \( t \in C' \setminus F \), and thus \( x_{C'} \in U \).

**Problem 7.** Show that the set \( A \) in Problem 6 is not compact.

**Proof** In order to show that \( A \) is not compact we will show that we can find a net in \( A \) which converges to the constant map 1 (meaning \([0,1] \ni t \mapsto 1 \)) which is clearly not in \( A \). Consider

\[
I = \{ C \subset [0,1] : C \text{ countable} \}.
\]

on \( C \) we use the order defined by \( C \geq C' \iff C \supset C' \). Secondly define for each \( C \in I \) the following element \( x_C \) in \( A \).

\[
x_C : [0,1] \mapsto [0,1], \quad t \mapsto \begin{cases} 1 & \text{if } t \in C \\ 0 & \text{if } t \notin C \end{cases}
\]

We claim that the net \( (x_C)_{C \in I} \) converges to the constant 1. Let \( U \) be an element of the neighborhood basis of \( X \), i.e. \( U \) is of the form

\[
U = \prod_{t \in [0,1]} U_t,
\]

with \( U_t \) open in \([0,1]\) for all \( t \in [0,1] \) and \( F = \{ t : U_t \neq [0,1] \} \) finite. We have to find a \( C \in I \) so that for all \( C' \in I \) with \( C' \supset C \) it follows that \( x_{C'} \in U \). Note that for \( t \in F \) \( 1 \in U_t \). Well... simply take \( C = F \) and notice that for all \( C' \in I \) with \( C' \supset C \) it follows that \( x_{C'}(t) = 0 \in [0,1] = U_t \), if \( t \notin C' \), \( x_{C'}(t) = 1 \in U_t \) if \( t \in F \), and \( x_{C'}(t) = 1 \in [0,1] = U_t \) if \( t \in C' \setminus F \), and thus \( x_{C'} \in U \).

**Remark.** By the way, if we let \( F = \{ F \subset [0,1] : F \text{ finite} \} \) ordered by inclusion. Then the net \( (\chi_F)_{F \in F} \) converges to the constant function 1. Strange, isn’t it.....Well, it shows that the product topology on a “big” product is a very coarse topology.

**Problem 8.** Assume that \((\overline{\Omega}, <)\) is the wellordered set from Problem 6 in first Homework. Show that every closed interval is compact in the order topology.

**Proof.** Let \( \alpha \in \overline{\Omega} \), To show that for all \( \beta \in \overline{\Omega}, \beta > \alpha \), the interval \( \alpha, \beta \) is compact. Assume that were not the case. Then we could take

\[
\beta_0 = \min\{ \beta \in \overline{\Omega} : [\alpha, \beta] \text{ is not compact} \}.
\]
Let \((U_i)_{i \in I}\) be an arbitrary open cover of \([\alpha, \beta_0]\). There is some \(i_0 \in I\) with \(\beta_0 \in U_{i_0}\), and since \(U_{i_0}\) is open there is some \(\beta \in [\alpha, \beta_0)\) so that \(\beta_0 \in (\alpha, \beta) \subset U_{i_0}\). By choice of \(\beta_0\) there is a finite \(I_0 \subset I\) so that \((U_i)_{i \in I_0}\) covers \([\alpha, \beta]\), and thus \((U_i)_{i \in I_0 \cup \{i_0\}}\) covers \([\alpha, \beta_0]\). Since we found a finite sub cover of any open cover of \([\alpha, \beta_0]\) we derive a contradiction.