

Problems in Real Variables, II (Math608), Solutions

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Problem 1. Assume X and Y are topological spaces, Y being Hausdorff, and $f, g : X \rightarrow Y$ are continuous.

- a) $\{x \in X : f(x) = g(x)\}$ is closed in X .
- b) The claim in (a) is not necessarily true without the assumption that Y is Hausdorff.
- c) If $f(x) = g(x)$ for all x out of a dense subset of X , then $f = g$.

Proof. (a) We have to show that $A = \{x \in X : f(x) \neq g(x)\}$ is open in X . If $x \in X$ with $f(x) \neq g(x)$ we can find two open and disjoint neighborhoods U and V of $f(x)$ and $g(x)$ respectively. Let $W = f^{-1}(U) \cap g^{-1}(V)$ which is open since f and g are continuous. Now note that for any $z \in W$ it follows that $f(z) \in U$ and $g(z) \in V$ which implies that $f(z) \neq g(z)$ and, thus, $W \subset A$, since $x \in A$ was arbitrary we proved that A is open.

(b) Example: $X = \mathbb{N}$ with the "tail-topology"

$$\mathcal{T} = \{[n, \infty) \cap \mathbb{N} : n \in \mathbb{N}\} \cup \{\emptyset\}.$$

$f, g : \mathbb{N} \rightarrow \mathbb{N}$ $f(n) = 10$ and $g(n) = n$ for all $n \in \mathbb{N}$. Then $\{n \in \mathbb{N} : f(n) = g(n)\} = \{10\}$ is not closed.

(c) Follows from (a): If the set $D = \{x \in X : f(x) = g(x)\}$ is dense and (by (a) closed) it follows that $X = \overline{D} = D$.

Problem 2. Let \mathcal{F} be a set of realvalued function on a set X and let \mathcal{T} be the weak topology on X generated by \mathcal{F} . Then

$$(X, \mathcal{T}) \text{ Hausdorff} \iff \forall x, y \in X, x \neq y, \exists f \in \mathcal{F} \quad f(x) \neq f(y).$$

Proof. Note that

$$\mathcal{B} = \left\{ \bigcap_{i=1}^n f_i^{-1}(U_i) : \begin{array}{l} n \in \mathbb{N}, \quad f_1, f_2, \dots, f_n \in \mathcal{F} \\ U_1, U_2, \dots, U_n \subset \mathbb{R} \text{ open} \end{array} \right\}$$

" \Rightarrow " Assume $x, y \in X$ and $f(x) = f(y)$. Then it follows for all $x \in B \iff y \in B$, thus there is no element of \mathcal{B} which separates x from y . Since \mathcal{T} is the set of all unions of elements in \mathcal{B} the points x and y cannot be separated by elements in \mathcal{T} .

" \Leftarrow " Let $x, y \in X$, $x \neq y$, then there is an $f \in \mathcal{F}$, so that $f(x) \neq f(y)$. Since \mathbb{R} is T_2 we find two open, disjoint sets $U, V \subset \mathbb{R}$ so that $f(x) \in U$ and $f(y) \in V$. Note that then $f^{-1}(U)$ and $f^{-1}(V)$ are two disjoint open neighborhoods of x and y respectively.

Problem 3. Only using the definition of net, convergent net and subnet, show

- (1) The subnet of a subnet is a subnet.
- (2) The subnet of a convergent net converges to the same limit.

Problem 4. If A is a directed set, a subset B of A is called *co final* if for each $\alpha \in A$ there exists $\beta \in B$ so that $\beta \geq \alpha$.

- a) If B is co final in A and $(x_\alpha)_{\alpha \in A}$ is a net, the inclusion map $B \rightarrow A$ makes $(x_\beta)_{\beta \in B}$ a subnet of $(x_\alpha)_{\alpha \in A}$.
- b) If $(x_\alpha)_{\alpha \in A}$ is a net in a topological space X and $x \in X$ then $(x_\alpha)_{\alpha \in A}$ converges to $x \iff$

$$\forall B \subset A \text{ co final } \exists C \subset B \text{ co final } \quad (x_\gamma)_{\gamma \in C} \text{ converges to } x.$$

Proof (a) clear (simply observe that the two conditions of the definition of a subnet (page 126) are satisfied).

(b) “ \Rightarrow ” Assume $x = \lim_{\alpha \in A} x_\alpha$. Let $B \subset A$ be co final. Simply choose $C = B$ (which is clearly co final in B). Now let U be a neighborhood of x . Thus there exists an $\alpha_0 \in A$ so that for all $\alpha \geq \alpha_0$ it follows that $x_\alpha \in U$ whenever $\alpha \geq \alpha_0$. Since $C = B$ is co final in A we can choose a $\gamma_0 \in C$ with $\gamma_0 \geq \alpha_0$. Now it follows that for all $\gamma \geq \gamma_0$, $\gamma \in C$ ($\subset A$) that $x_\gamma \in U$, which finishes the argument.

“ \Leftarrow ” Assume that $(x_\alpha)_{\alpha \in A}$ is **not** converging to x . We therefore can find a neighborhood U of x so that for all α we can pick a $\beta_\alpha \geq \alpha$, $\beta_\alpha \in A$ so that $x_{\beta_\alpha} \notin U$. Now let $B := \{\beta_\alpha : \alpha \in A\} \subset A$.

Note that B is co final (simply because for any α we picked $\beta_\alpha \geq \alpha$). Secondly for all $\beta \in B$ we have $x_\beta \notin U$ thus not for single $\gamma \in C$, with $C \subset A$ co final, we have $x_\gamma \in U$. Thus we proved

$$\exists B \subset A \text{ co final } \forall C \subset B \text{ co final } \quad \forall \gamma_0 \in C \exists \gamma \geq \gamma_0 \quad x_\gamma \notin U,$$

which is the negation of the “right side”.

Problem 5. If X is Hausdorff, then any net in X converges to a at most one element

Proof. Let $x \neq y$ and $x = \mathcal{T} - \lim_{i \in I} x_i$. Since X is Hausdorff there are open sets U and V so that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Since $x = \mathcal{T} - \lim_{i \in I} x_i$, there is an $i_0 \in I$ so that $x_i \in U$, for all $i \geq i_0$. But this means that $x_i \notin V$ for all, $i \geq i_0$, thus, if there existed an i_1 so that $x_i \in V$, if $i \geq i_1$, we could choose $i_2 \in I$ so that $i_2 \geq i_1$ and $i_2 \geq i_0$, and thus $x_{i_2} \in U \cap V$, which contradicts the assumption that U and V are disjoint.

Problem 6. Let $X = [0, 1]^{[0, 1]}$ and consider on X the product topology. Define

$$A := \left\{ (x_t)_{t \in [0, 1]} \in X : \{t \in [0, 1] : x_t \neq 0\} \text{ is countable} \right\}.$$

Show that every sequence in A has a convergent subsequence whose limit is still in A .

Proof. We think of the elements in X being maps $x : [0, 1] \rightarrow [0, 1]$, $t \mapsto x(t)$. Assume that $x_n \in A$ for $n \in \mathbb{N}$. For $n \in \mathbb{N}$ define the (countable) set $C_n = \{t \in [0, 1] : x_n(t) \neq 0\}$. Then also $C = \bigcup_{n \in \mathbb{N}} C_n$ is countable. Write $C = \{t_k : k \in \mathbb{N}\}$. For fixed $k \in \mathbb{N}$ the sequence $(x_n(t_k))_{n \in \mathbb{N}}$ is in the compact set $[0, 1]$ and must therefore have a convergent subsequence.

We proceed now as follows (“diagonalization argument”): First find convergent subsequence of $(x_n(t_1))_{n \in \mathbb{N}}$. Denote it by $(x_{n_i^{(1)}}(t_1))_{i \in \mathbb{N}}$ and write $x(t_1) = \lim_{i \rightarrow \infty} x_{n_i^{(1)}}(t_1)$. Then find a further subsequence $(n_i^{(2)}) \subset (n_i^{(1)})$ so that also $(x_{n_i^{(2)}}(t_2))_{i \in \mathbb{N}}$ converges, write $x(t_2) = \lim_{i \rightarrow \infty} x_{n_i^{(2)}}(t_2)$. Continue this way, finding successive subsequences of \mathbb{N} , i.e. $\mathbb{N} \supset (n_i^{(1)}) \supset (n_i^{(2)}) \supset (n_i^{(3)}) \supset \dots$, so that for every $k \in \mathbb{N}$

$$x(t_k) = \lim_{i \rightarrow \infty} x(t_{n_i^{(k)}}), \text{ exists.}$$

For $t \in [0, 1] \setminus C$ put $x(t) = 0$.

Now take the *diagonal sequence* of the sequences $(n_i^{(k)})$, $k = 1, 2, \dots$, namely $m_i = n_i^{(i)}$ and note that (m_i) is up to the first k element (which are irrelevant for taking limits) a subsequence of each $(n_i^{(k)})$. Therefore we have for each $k \in \mathbb{N}$

$$x(t_k) = \lim x_{m_i}(t_k).$$

We also have trivially for $t \in [0, 1] \setminus C$

$$x(t) = 0 = x_{m_i}(t) \text{ for all } i \in \mathbb{N}.$$

This will imply that x_{m_i} converges in the product topology to x . Indeed, take any open neighborhood U of x . We can assume that U is of the form

$$U = \prod_{t \in [0, 1]} U_t,$$

where $U_t \subset [0, 1]$ open and containing the number $x(t)$, and further more only for a finite set $F = \{t_1, t_2, \dots, t_\ell\} \subset [0, 1]$ we have $U_t \neq [0, 1]$, and other wise $U_t = [0, 1]$.

By above convergence it follows that there must be for each $k = 1, \dots, \ell$ an N_k so that $x_{m_i}(t_k) \in U_{t_k}$ whenever $i \geq N_k$. Take $N = \max_{k \leq \ell} N_k$ and deduce that $x_{m_i}(t_k) \in U_{t_k}$ whenever $i \geq N$ and $k \in \{1, 2, \dots, \ell\}$. But this means that $x_{m_i} \in U$ for all $i \geq N$.

Secondly, in order to show that A is not compact we will show that we can find a net in A which converges to the constant map 1 (meaning $[0, 1] \ni t \mapsto 1$) which is clearly not in A . Consider

$$\mathcal{I} = \{C \subset [0, 1] : C \text{ countable}\}.$$

on C we use the order defined by $C \geq C' \iff C \supset C'$. Secondly define for each $C \in \mathcal{I}$ the following element x_C in A .

$$x_C : [0, 1] \mapsto [0, 1], \quad t \mapsto \begin{cases} 1 & \text{if } t \in C \\ 0 & \text{if } t \notin C \end{cases}$$

We claim that the net $(x_C)_{C \in \mathcal{I}}$ converges to the constant 1. Let U be an element of the neighborhood basis of X , i.e. U is of the form

$$U = \prod_{t \in [0,1]} U_t,$$

with U_t open in $[0, 1]$ for all $t \in [0, 1]$ and $F = \{t : U_t \neq [0, 1]\}$ finite. We have to find a $C \in \mathcal{I}$ so that for all $C' \in \mathcal{I}$ with $C' \supset C$ it follows that $x_{C'} \in U$. Note that for $t \in F$ $1 \in U_t$. Well,... simply take $C = F$ and notice that for all $C' \in \mathcal{I}$ with $C' \supset C$ it follows that $x_{C'}(t) = 0 \in [0, 1] = U_t$, if $t \notin C'$, $x_{C'}(t) = 1 \in U_t$ if $t \in F$, and $x_{C'}(t) = 1 \in [0, 1] = U_t$ if $t \in C' \setminus F$, and thus $x_{C'} \in U$.

Problem 7. Show that the set A in Problem 6 is not compact.

Proof In order to show that A is not compact we will show that we can find a net in A which converges to the constant map 1 (meaning $[0, 1] \ni t \mapsto 1$) which is clearly not in A . Consider

$$\mathcal{I} = \{C \subset [0, 1] : C \text{ countable}\}.$$

on C we use the order defined by $C \geq C' \iff C \supset C'$. Secondly define for each $C \in \mathcal{I}$ the following element x_C in A .

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We claim that the net $(x_C)_{C \in \mathcal{I}}$ converges to the constant 1. Let U be an element of the neighborhood basis of X , i.e. U is of the form

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with U_t open in $[0, 1]$ for all $t \in [0, 1]$ and $F = \{t : U_t \neq [0, 1]\}$ finite. We have to find a $C \in \mathcal{I}$ so that for all $C' \in \mathcal{I}$ with $C' \supset C$ it follows that $x_{C'} \in U$. Note that for $t \in F$ $1 \in U_t$. Well,... simply take $C = F$ and notice that for all $C' \in \mathcal{I}$ with $C' \supset C$ it follows that $x_{C'}(t) = 0 \in [0, 1] = U_t$, if $t \notin C'$, $x_{C'}(t) = 1 \in U_t$ if $t \in F$, and $x_{C'}(t) = 1 \in [0, 1] = U_t$ if $t \in C' \setminus F$, and thus $x_{C'} \in U$.

Remark. By the way, if we let $\mathcal{F} = \{F \subset [0, 1] : F \text{ finite}\}$ ordered by inclusion. Then the net $(\chi_F)_{F \in \mathcal{F}}$ converges to the constant function 1. Strange, isn't it.....Well, it shows that the product topology on a "big" product is a very coarse topology.

Problem 8. Assume that $(\bar{\Omega}, <)$ is the wellordered set from Problem 6 in first Homework. Show that every closed interval is compact in the order topology.

Proof. Let $\alpha \in \bar{\Omega}$, To show that for all $\beta \in \bar{\Omega}$, $\beta > \alpha$, the interval $[\alpha, \beta]$ is compact. Assume that were not the case. Then we could take

$$\beta_0 = \min\{\beta \in \bar{\Omega} : [\alpha, \beta] \text{ is not compact}\}.$$

Let $(U_i)_{i \in I}$ be an arbitrary open cover of $[\alpha, \beta_0]$. There is some $i_0 \in I$ with $\beta_0 \in U_{i_0}$, and since U_{i_0} is open there is some $\beta \in [\alpha, \beta_0)$ so that $\beta_0 \in (\alpha, \beta] \subset U_{i_0}$. By choice of β_0 there is a finite $I_0 \subset I$ so that $(U_i)_{i \in I_0}$ covers $[\alpha, \beta]$, and thus $(U_i)_{i \in I_0 \cup \{i_0\}}$ covers $[\alpha, \beta_0]$. Since we found a finite sub cover of any open cover of $[\alpha, \beta_0]$ we derive a contradiction.