Problems in Topology (Math 436)
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Due: 1/31/2008

Read Chapter 2.

1. Verify that $d_1$ and $d_\infty$ in Example 2.9 are metrics.

2. Prove that for a metric space $(X, d)$ and $A \subset X$

   $$A \text{ open} \iff \exists (x_i)_{i \in I} \subset X \exists (\varepsilon_i)_{i \in I} \subset (0, 1) \quad A = \bigcup_{i \in I} B(x_i, \varepsilon_i).$$

3. Let $V$ be a vector space over $\mathbb{R}$. A function

   $$\| \cdot \| : V \to [0, \infty), \quad v \mapsto \|v\|$$

   is called a norm if

   (i) $\|x\| = 0$ if and only if $x = 0$
   (ii) $\|a \cdot x\| = |\alpha| \|x\|$, if $\alpha \in \mathbb{R}$ and $x \in V$.
   (iii) $\|x + y\| \leq \|x\| + \|y\|$, if $x, y \in V$.

   Define for $x, y \in V$ $d(x, y) = \|x - y\|$ and show that $d$ is a metric.


5. (*) For a particular signal processing problem Engineer A defines a function $d : X \times X \to \mathbb{R}$ with the following properties

   (i) $d(x, y) = 0$ if and only if $x = y$
   (ii) $d(x, y) = d(y, x)$
   (iii) $d(x, z) \leq d(x, y) + d(y, z)$

   Engineer B points out that $d$ should satisfy one more condition. Is he right? If so, which is the condition?

   Engineer A asserts that $d$ is still a metric. Why?