Problems to Introduction to Real Analysis, (Math446)

Due: 09/23/04

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Problem 1. Let $n \in \mathbb{N}$ and let (x_k) be a sequence in $\mathbb{R}^n$. Write for each $k \in \mathbb{N}$,

$$x_k = (x_{(k,1)}, x_{(k,2)}, \dots x_{(k,n)}).$$

a) (x_k) converges in $\mathbb{R}^n$ if and only if for all $i = 1, 2 \dots n$ the sequence $(x_{(k,i)})$ converges in $\mathbb{R}$

b) (x_k) converges in $\mathbb{R}^n$ if and only if it is a Cauchy sequence.

Problem 2. A set $A \subset R^n$ is open if and only if for each $x \in A$ there is a an open rectangle R so that, $x \in R$ and $R \subset A$.

An open rectangle R is a set of the form:

$$R = (a_1, b_1) \times (a_2, b_2) \times \dots (a_n, b_n) \quad = \{(z_1, z_2, \dots z_n) : \forall i \le n \; a_i < z_i < b_i\}.$$

Problem 3. For $f : \mathbb{R}^n \to \mathbb{R}^m$ the following are equivalent:

a) f contiunous,

b) f maps convergent sequences into convergent sequences,

c) $f^{-1}(U)$ is open if $U \subset \mathbb{R}^m$ is open

d) $f^{-1}(F)$ is closed if $F \subset \mathbb{R}^m$ is closed.

Problem 4. For $F \subset R^n$ the following is equivalent.

a) F is closed (i.e., complement of an open set)

b) the limit of any convergent sequence in F is in F.

Problem 5. For $C \subset R^n$ the following is equivalent.

a) C is compact (i.e., has finite covering property),

b) C is closed and bounded,

c) Every sequence in C has a a convergent subsequence whose limit is in C.

Problem 6. Using the definition of $\mathbb{R}$ as the set of all cuts, prove:

a) $A \cdot (B + C) = A \cdot B + A \cdot C$, where you can assume that $A > 0$.

b) The existence of the neutral element with respect to multiplication.

Problem 7. *The Hotel Infinity* The *Hotel Infinity* has countable infinitely many rooms and all rooms are occupied. Please find a matematical formulation and proof of the following statements, by only using the definition of countable (but not by using the Theorem of Schroeder Bernstein).

a) One tired tourist arrives, show that he will be able to get a room.
b) A bus with countable infinitly many tourists arrives, show that all of them can get a room.
c) Countably infinitely many buses with countable infinitly many tourists each arrives, show that all of the tourists arriving tourist will find a room.