Problem 1. Let $X$ be a compact space and $(Y,d_Y)$ a complete metric space. Define

$$C(X,Y) = \{f : X \to Y \text{continuous}\},$$

and for $f, g \in C(X,Y)$, put $d(f,g) := \sup_{x \in X} d_Y(f(x),g(x))$.

State and prove a version of the Theorem Arzela Ascoli for $C(X,Y)$ (like in the textbook only prove sufficiency for total boundedness in $C(X,Y)$).

Problem 2. (Old Qualifier Problem) Let $k : [0,1] \times [0,1] \to \mathbb{R}$ be continuous. For $f \in C([0,1])$ define $T(f) : [0,1] \to \mathbb{R}$ by:

$$T(f)(x) = \int_0^1 k(x,y)f(y)dy, \quad x \in [0,1].$$

$(T$ is called an *Integral operator with kernel $k$)

a) Show that $T(C[0,1]) \subset C([0,1])$.

b) Show that $T$ maps bounded subsets of $C([0,1])$ into compact subsets of $C([0,1])$.

Problem 3. The purpose of this problem is to derive the textbook version of the Stone-Weierstrass theorem from the version presented in class.

Using the Stone-Weierstrass theorem as proven in class prove the following:

If $X$ is a compact space and $\mathcal{A} \subset C(X)$ is a point separating algebra (but not necessarily “$\forall x \in X \exists g \in \mathcal{A} \ g(x) \neq 0$”), then

Either $\mathcal{A}$ is dense in $C(X)$,

or there is an $x_0 \in X$ so that

$$\mathcal{A} \subset \{g \in C(X) : g(x_0) = 0\} \text{ and } \mathcal{A} \text{ dense in } \{g \in C(X) : g(x_0) = 0\}.$$

Problem 4. (Old Qualifier Problem) On the set $[0,\infty]$ consider the topology $T$ generated by the open sets of $[0,\infty)$ and the sets of the form $[0,\infty) \setminus C$, with $C \subset [0,\infty)$ compact.

a) Show that $[0,\infty]$ with above defined topology is a compact space.

b) Show that $[0,\infty]$ with above defined topology is metrizable.

Hint: consider a continuous, strictly increasing, and bounded function $f : [0,\infty) \to [0,\infty)$.

c) Show that the linear space generated by the functions of the form $e^{-nx^2}$, $n = 1,2,3 \ldots$, is dense (with respect to sup-norm) in the space of all continuous functions $f : [0,\infty) \to \mathbb{R}$, having the property that $\lim_{x \to \infty} f(x) = 0$.

Problem 5. Problem 64/Page 138.

Problem 6. (Old Qualifying Problem) In $C[0,1]$, let

$$A = \text{span} \{x^n(1-x) : n \geq 1\}.$$
(By span\(S\), for the subset \(S\) of a vector space \(V\), we mean the subspace of \(V\) generated by \(S\), i.e. the set of all linear combinations of elements of \(S\).)

Prove that \(A\) is an algebra whose uniform closure is

\[\{ f \in C[0,1] : f(0) = f(1) = 0 \}\].

**Problem 7.** Problem 58/Page 138.