Problems in Real Variables, II (Math608) **Due: 02/10/10**

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Problem 1. Let X be a compact space and (Y, d_Y) a complete metric space. Define

$$C(X,Y) = \{f : X \to Y \text{continuous}\},$$

and for $f, g \in C(X,Y)$, put $d(f,g) := \sup_{x\in X} d_Y(f(x), g(x))$.

State and prove a version of the Theorem Arzela Ascoli for $C(X,Y)$ (like in the text book only prove sufficiency for total boundedness in $C(X,Y)$).

Problem 2. (Old Qualifier Problem) Let $k : [0,1] \times [0,1] \to \mathbb{R}$ be continuous. For $f \in C([0,1])$ define $T(f) : [0,1] \to \mathbb{R}$ by:

$$T(f)(x) = \int_0^1 k(x,y)f(y)dy, \ x \in [0,1].$$

(T is called an *Integral operator with kernel* k)

a) Show that $T(C[0,1]) \subset C([0,1])$.
b) Show that T maps bounded subsets of $C([0,1])$ into compact subsets of $C([0,1])$.

Problem 3. The purpose of this problem is to derive the textbook version of the Stone-Weierstrass theorem from the version presented in class.

Using the Stone-Weierstrass theorem as proven in class prove the following:

If X is a compact space and $\mathcal{A} \subset C(X)$ is a point separating algebra (but not necessarily "$\forall x \in X \exists g \in \mathcal{A}\, g(x) \neq 0$"), then
Either $\mathcal{A}$ is dense in $C(X)$,
or there is an $x_0 \in X$ so that

$$\mathcal{A} \subset \{g \in C(X) : g(x_0) = 0\} \text{ and } \mathcal{A} \text{ dense in } \{g \in C(X) : g(x_0) = 0\}.$$

Problem 4. (Old Qualifier Problem) On the set $[0,\infty]$ consider the topology $\mathcal{T}$ generated by the open sets of $[0,\infty)$ and the sets of the form $[0,\infty] \setminus C$, with $C \subset [0,\infty)$ compact.

a) Show that $[0,\infty]$ with above defined topology is a compact space.
b) Show that $[0,\infty]$ with above defined topology is metrizable.
   Hint: consider a continuous, strictly inceasing, and bounded function $f : [0,\infty) \to [0,\infty)$.
c) Show that the linear space generated by the functions of the form e^{-nx^2}, $n = 1, 2, 3 \ldots$, is dense (with respect to sup-norm) in the space of all continuous functions $f : [0,\infty) \to \mathbb{R}$, having the property that $\lim_{x\to\infty} f(x) = 0$.

Problem 5. Problem 64/Page 138.

Problem 6. (Old Qualifying Problem) In $C[0,1]$, let

$$A = \text{span}\,\{x^n(1-x) : n \geq 1\}.$$

(By span$S$, for the subset S of a vector space V, we mean the subspace of V generated by S, i.e. the set of all linear combinations of elements of S.)

Prove that A is an algebra whose uniform closure is

$$\{f \in C[0,1] : f(0) = f(1) = 0\}.$$

Problem 7. Problem 58/Page 138.