Problems in Real Variables, II (Math608), Solutions
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Problem 1. Let X be a compact space and Y a metric space. Define

$$C(X,Y) = \{f : X \to Y \text{continuous}\},$$

and for $f, g \in C(X,Y)$, put $d(f,g) := \sup_{x\in X} d_Y(f(x), g(x)$.

State and prove a version of the Theorem Arzela Ascoli for $C(X,Y)$.

Proof. The Statement is:

Theorem (Generalized Theorem of Arzela Ascoli, version 1) Assume $K \subset C(X,Y)$.

Then K is totally bounded if and only if

a) $I = \bigcup_{f\in K} f(X)$ is totally bounded
b) K is equicontinous, i.e. for all $\varepsilon > 0$ and all $x \in X$ there exists an $U \in \mathcal{N}_x$, so that $d_Y(f(x), f(z)) < \varepsilon$ for all $z \in U$.

For that result we will not need completeness of Y. But if (Y,d) is moreover complete, then $C(X,Y)$ is also complete (can be shown as $Y = \mathbb{R}$) and then it follows that $K \subset C(X,Y)$ is compact if and only if K is totally bounded and closed and we derive if (Y,d) is complete that:

Theorem (Generalized Theorem of Arzela Ascoli, version 2) Assume $K \subset C(X,Y)$.

Then K is compact if and only if

a) $I = \bigcup_{f\in K} f(X)$ is totally bounded
b) K is closed
c) K is equicontinous, i.e. for all $\varepsilon > 0$ and all $x \in X$ there exists an $U \in \mathcal{N}_x$, so that $d_Y(f(x), f(z)) < \varepsilon$ for all $z \in U$.

Proof. We can follow the proof of the Theorem of Arzela Ascoli almost word by word.

Problem 2. (Old Qualifier Problem) Let $k : [0,1] \times [0,1] \to \mathbb{R}$ be continuous. For $f \in C([0,1])$ define $T(f) : [0,1] \to \mathbb{R}$ by:

$$T(f)(x) = \int_0^1 k(x,y)f(y)dy,\ x \in [0,1].$$

a) Show that $T(C[0,1]) \subset C([0,1])$.
b) Show that T maps bounded sets into compact set (bounded sets in a normed vector space space $(X, \|\cdot\|)$ are sets S for which there exists an $R > 0$ so that $\|x\| \le R$ for all $x \in S$.)

Proof. (a) let $f \in C([0,1])$ to show that g, with $g(x) = \int_0^1 k(x,y)f(y)dy$, is continuous. Therefore let $\varepsilon > 0$ and $x \in [0,1]$.

Since $k(\cdot,\cdot)$ is continuous on a compact metric space (namely $[0,1]^2$) it is uniformly continuous. We can therefore find a δ so that for all $(u,v), (\tilde{u},\tilde{v}) \in$

$[0,1]^2$ with $\sqrt{(u-\tilde{u})^2+(v-\tilde{v})^2} < \delta$ it follows that $|k(u,v)-k(\tilde{u},\tilde{v})| < \varepsilon/(1+\|f\|)$. Therefore it follows for $z \in [0,1]$, with $|x-z| < \delta$ that

$$|T(f)(x) - T(f)(z)| = \Big| \int_0^1 [k(x,y) - k(z,y)] f(y) dy \Big|$$
$$\leq \|f\|_u \int_0^1 |k(x,y) - k(z,y)| dy \leq \|f\|_u \frac{\varepsilon}{1+\|f\|_u} < \varepsilon.$$

(b) Note that in the proof of (a) the δ only depended on ε and $\|f\|_u$. This implies that $\{T(f) : f \in B\}$ is equicontinuous.

Secondly if $f \in C([0,1]$

$$\|T(f)\|_u \leq \sup_{x,y\in[0,1]} |k(x,y)| \cdot |f(x)| \leq \|f\|_u \sup_{x,y\in[0,1]} |k(x,y)|.$$

This implies that t if $B \subset C(X)$ is bounded, then $T(B)$ is bounded.

From the theorem of Arzela and Ascoli it follows that $\{T(f) : f \in B\}$ is totally bounded and thus $\overline{\{T(f) : f \in B\}}$ is compact

Problem 3. (Old Qualifier Problem) On the set $[0,\infty]$ consider the topology $\mathcal{T}$ generated by the open sets of $[0,\infty)$ and the sets of the form $[0,\infty] \setminus C$, with $C \subset [0,\infty)$ compact.

a) Show that $[0,\infty]$ with above defined topology is a compact space.
b) Show that $[0,\infty]$ with above defined topology is metrizable.
   Hint: consider a continuous, strictly inceasing, and bounded function $f : [0,\infty) \to [0,\infty)$.
c) Show that the linear space generated by the functions of the form e^{-nx^2}, $n = 1,2,3\ldots$, is dense (with respect to sup-norm) in the space of all continuous functions $f : [0,\infty) \to \mathbb{R}$, having the property that $\lim_{x\to\infty} f(x) = 0$.

Proof. (a) assume that $[0,\infty] \subset \bigcup_{i\in I} U_i$ is an open covering. One of the U_i's, say U_{i_0} must contain ∞ and thus a set of the form $[0,\infty]\setminus C$, C compact in $[0,\infty)$. Since every open set U in $[0,\infty]$ has the property that $U \setminus \{\infty\}$ is open in $[0,\infty)$ C is also compact in $[0,\infty]$. Now $(U_i)_{i\in I}$ is also an open covering of C therefore there must be a finite $I' \subset I$ so that $C \subset \bigcup_{i\in I'} U_i$. But this implies that $[0,\infty] = \bigcup_{i\in\{i_0\}\cup I'} U_i$.

(b) take a continuous and strictly increasing function $f : [0,\infty) \to \mathbb{R}$ with $f(\infty) = \lim_{t\to\infty} f(t)$ exists in $\mathbb{R}$. Then define for $x,y \in [0,\infty]$ $d(x,y) = |f(x) - f(y)|$. It is easy to check that $d(\cdot,\cdot)$ is a metric on $[0,\infty]$. Also note that open balls in d are either open intervals in $[0,\infty)$ or intervals of the form $(a,\infty]$. Vice versa, every intervall in $[0,\infty)$ and every set of the form $(a,\infty]$ are balls in with respect to d. Therefore d generates the topology $\mathcal{T}$ on $[0,\infty]$

(c) Follows from Stone Weierstrass (use version of Corollary 4.50 on page 141). Note the linear span of all the functions of the form e^{-nx^2} forms an algebra which is sparating points.

Problem 4. Problem 64/Page 138. Let (X, ρ) be a metric space. Call a function $f \in C(X)$ *Holder continuous of exponent* α, $\alpha > 0$ if

$$N_\alpha(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho^\alpha(x, y)} < \infty.$$

Show that if X is compact then the set

$$S = \{f \in C(X) : \|f\|_u| \leq 1 \text{ and } N_\alpha(f) \leq 1\}.$$

is compact.

Proof. We first show that S is closed. Assume that $(f_n) \subset S$ and $f \in C(X)$ so that $\|f_n - f\|_u \to 0$ for $n \to \infty$. Then

$$\begin{aligned} \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho^\alpha(x, y)} &= \sup_{x \neq y} \lim_{n \to \infty} \frac{|f_n(x) - f_n(y)|}{\rho^\alpha(x, y)} \\ &\leq \sup_{x \neq y} \sup_{n \in \mathbb{N}} \frac{|f_n(x) - f_n(y)|}{\rho^\alpha(x, y)} \\ &= \sup_{n \in \mathbb{N}} \sup_{x \neq y} \frac{|f_n(x) - f_n(y)|}{\rho^\alpha(x, y)} = \sup_{n \in \mathbb{N}} N_\alpha(f_n) \leq 1. \end{aligned}$$

Thus $f \in S$.

Since S is also bounded it is enough to show (by Arzela Ascoli) that S is equicontinuous. Let $x \in X$ and let $\varepsilon > 0$. Choose $\delta = \varepsilon^{1/\alpha}$ (does not depend on any $f \in S$).

Then for all $y \in X$ all $f \in S$ with $\rho(x, y) < \delta$ it follows that

$$|f(x) - f(y)| \leq \rho^\alpha(x, y) \sup_{z \in X \setminus \{x\}} \frac{|f(x) - f(y)|}{\rho^\alpha(x, z)} \leq \rho^\alpha(x, y) < \delta^\alpha = \varepsilon.$$

Problem 5. The purpose of this probelm is to derive the textbook version of the Stone-Weierstrass theorem from the version presented in class.

Using the Stone-Weierstrass theorem as stated in class prove the following:

If X is a compact space and $\mathcal{A} \subset C(X)$ is a point separating algebra (but not necessarily "$\forall x \in X \exists g \in \mathcal{A}\, g(x) \neq 0$"), then

Either $\mathcal{A}$ is dense in $C(X)$,

or there is an $x_0 \in X$ so that

$$\mathcal{A} \subset \{g \in C(X) : g(x_0) = 0\} \text{ and } \mathcal{A} \text{ dense in } \{g \in C(X) : g(x_0) = 0\}.$$

Proof. Assume $\mathcal{A} \subset C(X)$ is a point separating subalgebra.

Case 1: If $\mathcal{A}$ also has the property:

$$\forall x \in X \exists g \in \mathcal{A} \quad g(x) \neq 0,$$

then, by the version we showed in class, it follows that $\mathcal{A}$ is dense in $C(X)$.

Other wise we are in Case 2:

Case 2: There exists $x_0 \in X$ so that: $\forall g \in \mathcal{A} \quad g(x_0) = 0$.

We want to show in this case that $\mathcal{A}$ is dense in $\{f \in C(X) : f(x_0) = 0\}$.

Consider the the set $\mathcal{A}' = \{g + c : g \in \mathcal{A} \text{ and } c \in \mathbb{F}\}$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$ and by "c" we mean the constant function. Now it is easy to see that

$\mathcal{A}'$ is still an algebra, but it satisfies the conditions of the class' version of Stone-Weierstrass. Therefore $\mathcal{A}'$ is dense in $C(X)$.

In order to prove our claim we let $g \in \{f \in C(X) : f(x_0) = 0\}$ arbitrary. We first find a sequence $h_n \in \mathcal{A}'$ with $\|h_n - g\| \to 0$, for $n \to \infty$. Write h_n as $h_n = g_n + c_n$ with $g_n \in \mathcal{A}$ and $c_n \in \mathbb{F}$ for $n \in \mathbb{N}$. Now $g_n(x_0) = 0$ and secondly $0 = g(x_0)$ which implies that $lim h_n(0)$. Thus it follows that $c_n = \lim_{n\to\infty} h_n(x_0) - g_n(x_0) = 0$ therefore

$$\begin{aligned}\limsup_{n\to\infty} \|f - g_n\| &\leq \limsup_{n\to\infty} \|f - h_n\| + \limsup_{n\to\infty} \|h_n - g_n\| \\ &\leq \limsup_{n\to\infty} \|f - h_n\| + \limsup_{n\to\infty} |c_n| = 0.\end{aligned}$$

This proves the claim.

Problem 6. (Old Qualifying Problem) In $C[0,1]$, let

$$A = \text{span}\,\{x^n(1-x) : n \geq 1\}.$$

(By span$S$, for the subset S of a vector space V, we mean the subspace of V generated by S, i.e. the set of all linear combinations of elements of S.)

Prove that A is an algebra whose uniform closure is

$$\{f \in C[0,1] : f(0) = f(1) = 0\}.$$

Proof. If $p(x) = \sum_{i=1}^m a_i x^i(1-x)$ and $q(x) = \sum_{j=1}^n b_j x^j(1-x)$ are in A, we deduce that

$$\begin{aligned}p(x)q(x) &= \sum_{i=1}^m \sum_{j=1}^n a_i b_j x^i(1-x)x^j(1-x) \\ &= \sum_{i=1}^m \sum_{j=1}^n a_i b_j [x^{i+j}(1-x) - x^{i+j+1}(1-x)] \in A,\end{aligned}$$

which yields that A is an algebra. Secondly

$$\begin{aligned}\tilde{A} &= \Big\{a + bx + \sum_{i=1}^m a_i x^i(1-x) : a, b \in \mathbb{R}, m \in \mathbb{N}, a_i \in \mathbb{R}, \text{ for } i = 1, 2 \ldots m\Big\} \\ &= \Big\{a + b(1-x) + \sum_{i=1}^m a_i x^i(1-x) : a, b \in \mathbb{R}, m \in \mathbb{N}, a_i \in \mathbb{R}, \text{ for } i = 1, 2 \ldots m\Big\}\end{aligned}$$

is the algebra of **all** polynomials (by induction we can easily see that $x^n \in \tilde{A}$ for all $n \in \mathbb{N} \cup \{0\}$). Let $g \in C[0,1]$ with $g(0) = g(1) = 0$ and let $\varepsilon > 0$. By the Theorem of Stone Weierstrass, we find a polynomial $p \in \tilde{A}$, which we can write as

$$p(x) = a + bx + \sum_{i=1}^m a_i x^i(1-x),$$

so that $\|p - g\|_u \leq \varepsilon/4$. This implies that

$$|a| = |p(0)| < \varepsilon/4, \text{ and } |b| \leq |a| + |p(1)| < 2\varepsilon/4.$$

Let $q(x) = p(x) - a - bx$. It follows that $q \in A$ and that

$$\|q - g\|_u \leq \|q - p\|_u + \|p - g\|_u < \frac{3}{4}\varepsilon + \frac{1}{4}\varepsilon = \varepsilon.$$

Problem 7. Problem 58/Page 138. Assume that $(X_\alpha)_{\alpha \in A}$ is a family of topological spaces for which infinitely many are not compact. Let $K \subset \prod_{\alpha \in A} X_\alpha$ be closed and compact in the product topology. Show that K is nowhere dense.

Proof. Assume the claim is not true and $K^\circ \neq \emptyset$. Thus there is an $x \in K$ and a neighborhood U of x so that $U \subset K$.

We can assume that U is of the form $U = \prod_{\alpha \in A} U_\alpha$ with U_α open in X_α for $\alpha \in A$, and with $F = \{\alpha : U_\alpha \neq X_\alpha\}$ is finite.

Since $\{\alpha : X_\alpha \text{ not compact}\}$ is infinite, we can choose an $\alpha_0 \in A$ so that X_{α_0} is not compact and so that $U_{\alpha_0} = X_{\alpha_0}$.

Now since continuous images of compact sets are also compact the projection of K $\pi_{\alpha_0}(K)$ onto X_{α_0} is compact, and therefore cannot be equal to whole space X_{α_0} we deduce that there must be an $x_{\alpha_0} \in X_{\alpha_0} \setminus \pi_{\alpha_0}(K) = U_{\alpha_0} \setminus \pi_{\alpha_0}(K)$. Now for $\alpha \in A \setminus \{\alpha_0\}$ we pick an $x_\alpha \in U_\alpha$, Therefore the whole family $(x_\alpha)_{\alpha \in A} \in U$, but, since $x_{\alpha_0} \notin \pi_{\alpha_0}(K)$ we have that $(x_\alpha)_{\alpha \in A} \notin K$, which is a contradiction.